MAT2006: Elementary Real Analysis
Homework 1

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Question 2.2-11. Let us call an irrational number $\alpha \in \mathbb{R}$ *well approximated* by rational numbers if for any natural numbers n, $N \in \mathbb{N}$ there exists a rational number p/q such that $|a - p/q| < 1/(Nq^n)$.

a) Construct an example of a well-approximated irrational number.

Let α be the Liouville's constant, which means

$$\alpha = 10^{-1!} + 10^{-2!} + 10^{-3!} + \ldots + 10^{-n!} + \ldots$$

To prove this is a well-approximated irrational number, we first show it is irrational, and then show it is well-approximated.

To show its irrationality, we can verify that it is transcendental, hence not algebraic; but any rational numbers are algebraic (of order 1), therefore the proof is finished. Assume that it is an algebraic number of order n, then we construct

$$\alpha_n = \frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \cdots + \frac{1}{10^{n!}}$$

It follows that

$$\begin{aligned} |\alpha_n - \alpha| &= \frac{1}{10^{(n+1)!}} + \frac{1}{10^{(n+2)!}} + \cdots \\ &\leq 2 \times \frac{1}{10^{(n+1)!}} \\ &= \frac{2}{(10^{n!})^{n+1}} \end{aligned}$$

Let $q_n = 10^{n!}$, this means

$$\left| \alpha - \frac{p}{q_n} \right| \leq \frac{K}{q_n^{n+1}}$$

has one solution q_n. You can easily see that $q_{n+1}, q_{n+2} \ldots$ are all solutions of the above inequality, because

$$|\alpha - \alpha_n| \geq |\alpha - \alpha_{n+1}| \geq \ldots \geq |\alpha - \alpha_{n+k}| \geq \ldots$$

Therefore, it has infinitely many solutions for order $n+1$, which means it is approximable by rational numbers to order $n+1$. Since algebraic number of order n cannot be approximable to order higher than n, α must be transcendental, hence irrational.

Define $p_{n,N}/q_{n,N}$ as the sum of the first $n+N$ terms of α, we take $q_{n+N} = 10^{(n+N)!}$,

$$\begin{aligned}\left|\alpha - \frac{p_{n,N}}{q_{n,N}}\right| &= 10^{-(n+N+1)!} + 10^{-(n+N+2)!} + \dots \\ &< 2 \cdot 10^{-(n+N+1)!} \\ &< (10^{(n+N)!})^{-(n+N)} \\ &= (10^{(n+N)!})^{-n}(10^{(n+N)!})^{-N} \\ &< (10^{(n+N)!})^{-n}10^{-N} \\ &< (10^{(n+N)!})^{-n}N^{-1} = \frac{1}{Nq_{n+N}^n}\end{aligned}$$

Therefore, α is well-approximated.

b) Prove that a well-approximated irrational number cannot be algebraic, that is, it is transcendental (*Liouville's theorem*).

Suppose a well-approximated number α is algebraic of order n, define

$$f(x) = a_n x^n + \dots + a_1 x + a_0$$

Then $f(\alpha) = 0$, and if p/q, $(q > 0)$ is not a solution of $f(x) = 0$

$$\left|f\left(\frac{p}{q}\right)\right| = \frac{|a_n p^n + a_{n-1}p^{n-1} + a_{n-1}p^{n-1} + \cdots + a_0 q^n|}{q^n} \geq \frac{1}{q^n}$$

$$\left|f\left(\frac{p}{q}\right)\right| = \left|f\left(\frac{p}{q}\right) - f(\alpha)\right| \leq |f'(\eta)|\left|\alpha - \frac{p}{q}\right| \leq M\left|\alpha - \frac{p}{q}\right|$$

We conclude that if p/q is not a solution of $f(x) = 0$,

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1/M}{q^n}$$

However, for all n and N, α also satisfies there exists p/q, such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Nq^n} < \frac{1}{N}$$

Since N can be arbitrarily large, the number of solutions for $f(x) = 0$ is finite (at most n), and α is irrational (ensuring that L.H.S. not equal to zero), we can find N such that p/q is not a solution of $f(x) = 0$ but still satisfies the above inequality, which implies, for large enough N

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Nq^n}$$

has a solution p/q which is not a solution of $f(x) = 0$. Take $N > M$, then we have

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Nq^n} < \frac{1/M}{q^n}$$

and

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1/M}{q^n}$$

which yields a contradiction. Hence a well-approximated number is transcendental.

In fact, one can easily prove that a well-approximated number cannot be rational. This could also prove the number we construct in a) must be irrational.

Question 2.2-17. Let $A + B$ be the set of numbers of the form $a + b$ and $A \cdot B$ the set of numbers of the form $a \cdot b$, where $a \in A \subset \mathbb{R}$ and $b \in B \subset \mathbb{R}$. Determine whether it is always true that

a) $\sup(A + B) = \sup A + \sup B$

This is true. Let $a^* = \sup A$ and $b^* = \sup B$. Since a^* and b^* are upper bounds, we have $\forall\ a \in A,\ a^* \geq a$ and $\forall\ b \in B,\ b^* \geq b$. Thus $\forall\ a + b \in A + B$, we have $a^* + b^* \geq a + b$, which shows $a^* + B^*$ is upper bound of $A + B$.

Also, since a^* and b^* are the least upper bound, we have

$$\forall \epsilon_1 > 0,\ \exists a_0 \in A, \text{ such that } a^* - \epsilon_1 < a_0$$

$$\forall \epsilon_2 > 0,\ \exists b_0 \in B, \text{ such that } b^* - \epsilon_2 < b_0$$

By taking $\epsilon = \epsilon_1 + \epsilon_2$, we have

$$\forall \epsilon > 0,\ \exists a_0 + b_0 \in A + B, \text{ such that } a^* + b^* - \epsilon < a_0 + b_0$$

Therefore, $a^* + b^*$ is the least upper bound of $A + B$, hence $\sup(A + B) = \sup A + \sup B$.

b) $\sup(A \cdot B) = \sup A \cdot \sup B$

This is not always true. Take $A = [-2, 2]$, and $B = [-2, -1]$, then $\sup A = 2$ and $\sup B = -1$. However, $\sup(A \cdot B) = 4$, which shows $\sup(A \cdot B) \neq \sup A \cdot \sup B$.

Question 2.3-1. Show that

a) if I is any system of nested intervals, then

$$\sup\{a \in \mathbb{R} | [a, b] \in I\} = \alpha \leq \beta = \inf\{b \in \mathbb{R} | [a, b] \in I\}$$

and

$$[\alpha, \beta] = \bigcap_{[a,b] \in I} [a, b]$$

Since I is a system of nested intervals, so it must have upper bound and lower bound. We know the real any bounded subset of real number has supremum and infimum, so α and β both exists.

Now we prove $\alpha \leq \beta$. Suppose $\alpha > \beta$, and let $L = \alpha - \beta > 0$. Since α is the least upper bound, $\forall\ \epsilon > 0,\ \exists\ a_\tau > \alpha - \epsilon$; since β is the greatest lower bound, $\forall\ \epsilon > 0,\ \exists\ b_\tau < \beta + \epsilon$. Take $\epsilon = L/2$,

$$\exists\ a_{\tau_0} > \alpha - \frac{L}{2} = \frac{\alpha + \beta}{2}$$
$$\exists\ b_{\tau_1} < \beta + \frac{L}{2} = \frac{\alpha + \beta}{2}$$

Hence $a_{\tau_0} > b_{\tau_1}$, if we denote the corresponding interval of a_{τ_0} as $[a_{\tau_0}, b_{\tau_0}]$, and denote the corresponding interval of b_{τ_1} as $[a_{\tau_1}, b_{\tau_1}]$, these two intervals have no intersection, which means they are not nested intervals. Contradiction implies $\alpha \leq \beta$.

To prove $[\alpha, \beta] = \bigcap_{[a,b]\in I} [a, b]$, we need two steps as follows

(I) Prove $[\alpha, \beta] \subset \bigcap_{[a,b]\in I} [a, b]$. Take arbitrary $x \in [\alpha, \beta]$, since α is upper bound, $x \geq \alpha \geq a_\tau$ for all a_τ. Similarly, $x \leq \beta \leq b_\tau$ for all b_τ. For any a_τ, b_τ, $a_\tau \leq x \leq b_\tau$. We can conclude that for any $[a, b] \in I$, $x \in [a, b]$, which proves $x \in \bigcap_{[a,b]\in I} [a, b]$. Hence $[\alpha, \beta] \subset \bigcap_{[a,b]\in I} [a, b]$.

(II) Prove $\bigcap_{[a,b]\in I} [a, b] \subset [\alpha, \beta]$. Take arbitrary $x \in \bigcap_{[a,b]\in I} [a, b]$, we have $x \in [a, b]$, for all $[a, b] \in I$. If $x < \alpha$, $\exists\ a_{\tau_0} > x$, which means $x \notin [a_{\tau_0}, b_{\tau_0}]$. If $x > \beta$, $\exists\ b_{\tau_1} < x$, which means $x \notin [a_{\tau_1}, b_{\tau_1}]$. These two cases both contradicts the fact that $x \in [a, b]$, for all $[a, b] \in I$, hence $\alpha \leq x \leq \beta$. Therefore, $\bigcap_{[a,b]\in I} [a, b] \subset [\alpha, \beta]$.

b) if I is a system of nested open intervals (a, b) the intersection $\bigcap_{(a,b)\in I} (a, b)$ may happen to empty.

Take $(a_n, b_n) = \left(0, \frac{1}{n}\right)$, if we assume there is an element in the intersection, denote it as a. Therefore, we have for any n, $a \in (0, \frac{1}{n})$. However, if we take $n_0 = \left[\frac{1}{a}\right] + 1$, then $n_0 > \frac{1}{a}$, which means $a \notin (0, \frac{1}{n_0})$. Contradiction shows that our assumption is wrong, i.e., there is no element in the intersection.

Question 2.3-2. Show that

a) from a system of closed intervals covering a closed interval it is not always possible to choose a finite subsystem covering the interval;

Let the system of closed intervals be

$$\mathcal{G} = \left\{ [2, 3], \left[0, 2 - \frac{1}{n}\right] \middle| n \in \mathbb{N}^+ \right\}$$

$\mathcal{G}$ is a cover of closed interval $[0, 3]$, but it's impossible to find finite many subcovers to cover $[0, 3]$, because if we choose N subcovers, then $\left(2 - \frac{1}{N}, 2\right)$ is not covered.

b) from a system of open intervals covering an open interval it is not always possible to choose a finite subsystem covering the interval;

Let the system of open intervals be

$$\mathcal{G} = \left\{ \left(0, 2 - \frac{1}{n}\right) \middle| n \in \mathbb{N}^+ \right\}$$

$\mathcal{G}$ is a cover of open interval $(0, 2)$, but it's impossible to find finite many subcovers to cover $(0, 2)$, because if we choose N subcovers, then $\left(2 - \frac{1}{N}, 2\right)$ is not covered.

c) from a system of closed intervals covering an open interval it is not always possible to choose a finite subsystem covering the interval.

Let the system of closed intervals be

$$\mathcal{G} = \left\{ \left[\frac{1}{n}, 3 - \frac{1}{n} \right] \middle| n \in \mathbb{N}^+ \right\}$$

$\mathcal{G}$ is a cover of open interval $(0,3)$, but it's impossible to find finite many subcovers to cover $(0,3)$, because if we choose N subcovers, then $\left(0, \frac{1}{N}\right)$ and $\left(2 - \frac{1}{N}, 3\right)$ are not covered.

Question 2.3-3. Show that if we take only the set $\mathbb{Q}$ of rational numbers instead of the complete set $\mathbb{R}$ of real numbers, taking a closed interval, open interval, and neighborhood of a point $r \in \mathbb{Q}$ to mean respectively the corresponding subsets of $\mathbb{Q}$, then none of the three lemmas proved above remains true.

(Note: The three lemmas are Cauchy-Cantor, Heine-Borel, and Bolzano-Weierstrass, which are above this chapter in textbook, not the three statements in the above Question 2.3-2.)

First, on the set $\mathbb{Q}$, the Cauchy-Cantor lemma is wrong, which means it is possible that for a system of nested closed interval $I_1 \supset I_2 \cdots I_n \supset \cdots$, whose intersection is empty.

Take $I_n = [\sqrt{2}, \sqrt{2} + n^{-1}] \cap \mathbb{Q}$, then we have

$$\bigcap_{n=1}^{\infty} I_n = \{\sqrt{2}\} \cap \mathbb{Q} = \varnothing$$

Thus the intersection of such nested closed interval is empty.

Second, on the set $\mathbb{Q}$, the Heine-Borel lemma is wrong, which means there exists a system of open intervals covering a closed interval contains no finite subcover of that closed interval.

Take $I_n = (-1, \sqrt{2} - n^{-1}] \cap \mathbb{Q}$, this system covers $[0, \sqrt{2}] \cap \mathbb{Q}$, because

$$\bigcup_{n=1}^{\infty} I_n = (-1, \sqrt{2}) \cap \mathbb{Q}$$

Also, if we take away any one of them, say I_n, the union will not contain all rational number in $(\sqrt{2} - (n-1)^{-1}, \sqrt{2} - n^{-1})$, which means not cover $[0, \sqrt{2}] \cap \mathbb{Q}$.

Third, on the set $\mathbb{Q}$, the Bolzano-Weierstrass lemma is wrong, which means some bounded infinite set of rational numbers has no limit point.

We only need to take an infinite sequence of rational number that converges to irrational number. Define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right), \qquad x_1 = 2$$

It's easy to show all terms in this sequence is rational number by induction. You will see in **Question 3.1-7** that this sequence x_n converges to $\sqrt{2}$, and actually, its explicit formula is as follows

$$x_n = \sqrt{2} + \frac{2\sqrt{2}}{(3 + 2\sqrt{2})^{2^{n-1}} - 1}$$

Question 2.4-1. Show that the set of real numbers has the same cardinality as the points of the interval $(-1, 1)$.

Construct a bijective mapping between $(-1,1)$ and $\mathbb{R}$, which is denoted as

$$f(x) = \frac{x}{1-|x|} \quad x \in (-1,1)$$

First, this mapping is injective, because for any

$$\frac{x_1}{1-|x_1|} = \frac{x_2}{1-|x_2|}$$

we have

$$x_2 - x_2|x_1| = x_1 - x_1|x_2|$$

Since $1-|x_1|$ and $1-|x_2|$ are both positive, x_1 and x_2 must be both positive or both negative. Therefore, $x_2 - x_2x_1 = x_1 - x_1x_2$ or $x_2 + x_2x_1 = x_1 + x_1x_2$, which means $x_1 = x_2$.

Second, this mapping is surjective, because for any real number c, Let

$$\frac{x}{1-|x|} = c$$

If $c \geq 0$, then $x \geq 0$, and $x = \frac{c}{1+c} \in [0,1)$; if $c < 0$, then $x < 0$, and $x = \frac{c}{1-c} \in (-1,0)$.

Therefore, these two sets have the same cardinality.

Question 2.4-2. Give an explicit one-to-one correspondence between

a) the points of two open intervals;

We don't consider interval containing infinity. We want to construct a bijective mapping between (a,b) and (c,d). The mapping $f:(a,b)\mapsto(c,d)$ is as follows

$$f(x) = c + \frac{d-c}{b-a}(x-a)$$

b) the points of two closed intervals;

We don't consider interval containing infinity. We want to construct a bijective mapping between $[a,b]$ and $[c,d]$. The mapping $f:[a,b]\mapsto[c,d]$ is as follows

$$f(x) = c + \frac{d-c}{b-a}(x-a)$$

c) the points of a closed interval and the points of an open interval;

We don't consider interval containing infinity. We only need to construct a bijective mapping between $(0,1)$ and $[0,1]$. The mapping $F:(0,1)\mapsto[0,1]$ is as follows (by Schröder–Bernstein theorem)

$$F(x) = \begin{cases} x & x \neq \frac{1}{2} \pm \frac{1}{2\cdot 3^n} \\ 3x-1 & x = \frac{1}{2} \pm \frac{1}{2\cdot 3^n} \end{cases}, \quad n \in \mathbb{N}^+$$

To generalize the result, we can apply what we construct in a) and b), which means

$$(a,b) \mapsto (0,1) \mapsto [0,1] \mapsto [c,d]$$

d) the points of the closed interval $[0,1]$ and the set $\mathbb{R}$.

We only need to construct $(0,1) \mapsto \mathbb{R}$. The mapping $F:(0,1) \mapsto \mathbb{R}$ is as follows

$$F(x) = \begin{cases} \frac{1}{2x} - 1 & x \in \left(0, \frac{1}{2}\right) \\ \frac{1}{2x-2} + 1 & x \in \left[\frac{1}{2}, 1\right) \end{cases}$$

To obtain the mapping from $[0,1]$ to $\mathbb{R}$, we can apply the inverse of the mapping we construct in c), which means

$$[0,1] \mapsto (0,1) \mapsto \mathbb{R}$$

The explicit correspondence between $[0,1]$ and $\mathbb{R}$ is

$$f(x) = \begin{cases} \dfrac{1}{2x} - 1 & x \neq \dfrac{1}{2} - \dfrac{1}{2 \cdot 3^n}, n \in \mathbb{N}^+,\ x \in \left(0, \dfrac{1}{2}\right) \\ \dfrac{1}{2x-2} + 1 & x \neq \dfrac{1}{2} + \dfrac{1}{2 \cdot 3^n}, n \in \mathbb{N}^+,\ x \in \left[\dfrac{1}{2}, 1\right) \\ \dfrac{3}{2x+2} - 1 & x = \dfrac{1}{2} - \dfrac{1}{2 \cdot 3^n}, n \in \mathbb{N} \\ \dfrac{3}{2x-4} + 1 & x = \dfrac{1}{2} + \dfrac{1}{2 \cdot 3^n}, n \in \mathbb{N} \end{cases}$$

Question 2.4-3. Show that

a) every infinite set contains a countable subset;

For an infinite set A, we pick arbitrary element in A, denote as a_1, and let $A = A/\{a_1\}$. Since A is infinite, we could pick element for any times, and A will still be infinite. Put all a_i together as a set B, then $B \subset A$, and it's obvious that each element in B has a bijective mapping to natural number $\{1, 2, 3, \ldots\}$. Hence, the B is a countable subset. (Note that the above procedure is based on the Axiom of Choice).

b) the set of even integers has the same cardinality as the set of all natural numbers.

We only need to construct a bijective mapping from $f : \mathbb{Z}_2 \mapsto \mathbb{N}$, where $\mathbb{Z}_2$ denote the set of all even integers.

$$f(x) = \begin{cases} x & x \geq 0,\ x \in \mathbb{Z}_2 \\ -(x-1) & x < 0,\ x \in \mathbb{Z}_2 \end{cases}$$

This f is injective, because for $f(x_1) = f(x_2)$, if $x_1, x_2 \geq 0$, then by definition $x_1 = x_2$; if $x_1, x_2 < 0$, $-(x_1 - 1) = -(x_2 - 1) \implies x_1 = x_2$; if $x_1 \geq 0, x_2 < 0$, then contradiction, because

$f(x_1) = x_1 \geq 0, f(x_2) = -x_2 + 1 < 0$, they cannot be equal; simlar argument can be applied for $x_1 < 0, x_2 \geq 0$. Hence $x_1 = x_2$.

This f is surjective, because for any natural number n, if it is even (including 0), then its preimage is n; if it is odd, then its preimage is $-(n+1) \in \mathbb{Z}_2$. Hence f is surjective.

Therefore, the set of even integers has the same cardinality as the set of all natural numbers.

c) the union of an infinite set and an at most countable set has the same cardinality as the original infinite set.

Let A denote the infinite set, and B denote the at most countable set. Since we have proved that infinite set has a countable subset, A has a countable subset A'. Therefore, if B is finite, then a finite set union with a countable set is still countable, hence there exists a bijective mapping $f : B \cup A' \mapsto A'$; if B infinite, then B and A' are both countable, and the union of two countable sets is still countable, hence there exists a bijective mapping $f : B \cup A' \mapsto A'$.

Now we construct an bijective mapping from $A \cup B \mapsto A$,

$$g(x) = \begin{cases} x & x \in A \backslash A' \\ f(x) & x \in B \cup A' \end{cases}$$

because $f(x)$ and x are both bijective mapping. Hence, the union of an infinite set and an at most countable set has the same cardinality as the original infinite set.

d) the set of irrational numbers has the cardinality of the continuum (same cardinality as $\mathbb{R}$).

First, it is trivial to prove the set of rational number is at most countable. We also know that irrational number set is infinite, because if it is finite, $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$ will be countable, which by definition is wrong. By part c), irrational number set union with an at most countable set $\mathbb{Q}$ will have the same cardinality as the original set $\mathbb{Q}^c$, which means $\mathbb{R}$ has the same cardinality as $\mathbb{Q}^c$.

e) the set of transcendental numbers has the cardinality of the continuum.

Using the same argument as part d), we only need to prove algebraic number is countable, then substitute the algebraic to rational, and transcendental to irrational, the logics are exactly the same. However, the proof of algebraic number being countable has been done in the lecture, so we finish the proof.

Question 2.4-7. On the closed interval $[0, 1] \subset \mathbb{R}$ describe the sets of numbers $x \in [0, 1]$ whose ternary representation $x = 0.\alpha_1\alpha_2\alpha_3 \ldots,\ \alpha_i \in \{0, 1, 2\}$, has the property:

a) $\alpha_1 \neq 1$;

This set of number represents exactly all numbers in

$$[0, 1] \Big\backslash \left(\frac{1}{3}, \frac{2}{3}\right)$$

Note that

$$\frac{1}{3} = 0.0222\ldots, \quad \frac{2}{3} = 0.2000\ldots$$

b) $(\alpha_1 \neq 1) \wedge (\alpha_2 \neq 1)$;

This set of number represents exactly all numbers in

$$[0,1] \Big\backslash \left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)$$

c) $\forall\, i \in \mathbb{N}\ (\alpha_i \neq 1)$ (the *Cantor set*).

This set of number represents exactly all numbers in

$$\mathcal{C} = [0,1] \Big\backslash \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{2^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right)$$

Question 2.4-8. (Continuation of Question 2.4-7.) Show that

a) the set of numbers $x \in [0,1]$ whose ternary representation does not contain 1 has the same cardinality as the set of all numbers whose binary representation has the form $0.\beta_1\beta_2\ldots$;

Consider the following mapping

$$f\left(\sum_{k=1}^{\infty} \alpha_k 3^{-k}\right) = \sum_{k=1}^{\infty} \frac{\alpha_k}{2} 2^{-k}$$

Ternary representation with only digits 0 and 2 was mapped to a binary representation by replacing all 2 with 1.

One can prove that f is surjective but not injective, because every element of binary representation only contains 0 and 1, if we replace all 1 by 2, the element only contains 0 and 2, which is exactly a ternary representation with only digits 0 and 2. Hence, it is surjective.

However, it is not injective because for two distinct element in Cantor set $(0.2022\ldots)_3$ and $(0.2200\ldots)_3$, their image under f is $(0.1011\ldots)_2$ and $(0.1100\ldots)_2$, but $(0.1011\ldots)_2 = (0.1100\ldots)_2$.

Also we need to know that every real number in $[0,1]$ can be represented as a binary number of the form $0.\beta_1\beta_2\ldots$ where $\beta_i \in \{0,1\}$. Thus, the Cantor set is a subset of the set of all numbers whose binary representation has the form $0.\beta_1\beta_2\ldots$.

Therefore, the set of numbers $x \in [0,1]$ whose ternary representation does not contain 1 has the same cardinality as the set of all numbers whose binary representation has the form $0.\beta_1\beta_2\ldots$.

However, if we want to avoid the use of the statement "every real number in $[0,1]$ can be represented as a binary number", we can consider what type of element in ternary representation with only 0 and 2 may be mapped to the same element in binary representation. Notice that this is caused by $0.\alpha_1\alpha_2\dots\alpha_N02222\dots$, which means 0 does not appear infinitely often. Such number will have the same image under f as the number $0.\alpha_1\alpha_2\dots\alpha_N20000\dots$.

Let a set C' contains all numbers of the form $0.\alpha_1\alpha_2\dots\alpha_N20000\dots$, this set must be countable, since for each fixed N, there are 2^N such number in total. If we denote the Cantor set as $\mathcal{C}$, the set of all numbers whose binary representation has the form $0.\beta_1\beta_2\dots$ as $\mathcal{B}$, and consider the same mapping f from $\mathcal{C}\backslash C'$ to $\mathcal{B}$, this mapping is bijective. Thus $\mathcal{C}\backslash C'$ has the same cardinality as $\mathcal{B}$.

However, we have proved in **Question 2.4-3.** that the union of an infinite set and an at most countable set has the same cardinality as the original infinite set, so $\mathcal{C}$ has the same cardinality as $\mathcal{C}\backslash C'$, hence the same cardinality as $\mathcal{B}$.

b) the Cantor set has the same cardinality as the closed interval $[0,1]$.

Since Cantor set is obviously a subset of closed interval $[0,1]$, hence the identity mapping is a injective mapping from Cantor set to $[0,1]$. We only need to show that there exists a surjective mapping from Cantor set to $[0,1]$. However, for any number $y\in[0,1]$, its binary representation can be translated into a ternary representation of a number x in Cantor set by replacing all the 1s by 2s. Therefore, such surjective mapping exists, which means Cantor set and the closed interval $[0,1]$ have the same cardinality.

Question 3.1-2. A ball has fallen from height h bounces to height qh, where q is a constant coefficient between $0<q<1$. Find the time that elapses until it comes to rest and the distance it travels through the air during the time.

We assume it is rectilinear motion with constant acceleration g, then the time the ball elapses until rest is

$$\begin{aligned}
t &= \sqrt{\frac{2h}{g}} + 2\sqrt{\frac{2qh}{g}} + 2\sqrt{\frac{2q^2h}{g}} + \dots \\
&= \sqrt{\frac{2h}{g}}(1 + 2\sqrt{q} + 2\sqrt{q^2} + \dots) \\
&= 2\sqrt{\frac{2h}{g}}(\frac{1}{2} + q^{1/2} + q^1 + q^{3/2} + \dots) \\
&= \frac{1+\sqrt{q}}{1-\sqrt{q}}\sqrt{\frac{2h}{g}}
\end{aligned}$$

The total distance is

$$
\begin{aligned}
D &= h + 2qh + 2q^2h + \dots \\
&= h(1 + 2q + 2q^2 + \dots) \\
&= 2h(\frac{1}{2} + q + q^2 + \dots) \\
&= \frac{1+q}{1-q}h
\end{aligned}
$$

Question 3.1-3. We mark all the points on a circle obtained from a fixed point by rotations of the circle through angles of n radians, where $n \in \mathbb{Z}$ ranges over all integers. Describe all the limit points of the set so constructed.

The limit points of the set are all points on the circle. To prove this, we need to prove positive integer n is dense in $[0, 2\pi]$ in radian measure, i.e., $\{\frac{n}{2\pi}\}2\pi$ is dense in $[0, 2\pi]$. Thus we only need to show that $\{\frac{n}{2\pi}\}$ is dense in $[0, 1]$.

First prove for any given real number x and positive integer $N > 1$, there exists integer p, q, $0 < q < N$, such that $|qx - p| < 1/N$.

Consider $mx - [mx]$, where $m = 1, 2, \dots, N + 1$. Since $0 \le mx - [mx] < 1$, there must be m_1, m_2 such that

$$
|m_2x - [m_2x] - (m_1x - [m_1x])| < \frac{1}{N}, \qquad 0 < m_1 < m_2 \le N
$$

Let $q = m_2 - m_1$, $p = [m_2x] - [m_1x]$, then $0 < q < N$, and $|qx - p| < \frac{1}{N}$.

Since we can easily see $\frac{1}{2\pi}$ is irrational number, next we prove that $\{n\alpha\}$ is dense in $[0, 1]$ for any irrational number α.

For any $s, t \in [0, 1]$ (W.O.L.G, we suppose $0 \le t < s \le 1$), let $L = s - t$, take sufficiently large n such that $\frac{1}{n} < L < 1$. From what we proved just now, there exists integer m, w, such that $0 < m < n$ and $0 < |m\alpha - w| < \frac{1}{n}$. Let $\beta = m\alpha - w$, then $m\alpha = w + \beta$, $0 < |\beta| < \frac{1}{n}$.

(I) If $0 < \beta < \frac{1}{n}$, let $k = 0, 1, \dots, [1/\beta]$, then at least one $\{km\alpha\}$ is in $[0, 1]$. This is because $km\alpha = kw + k\beta$, so $\{km\alpha\} = \{k\beta\}$. Since $k < \frac{1}{\beta}$, we have $\{km\alpha\} = k\beta$, which corresponds to $0, \beta, 2\beta, \dots, [1/\beta]\beta$. We can observe that the distance between any two number is $\beta < \frac{1}{n} < L$, and the distance between the last number $[1/\beta]\beta$ and boundary value 1 is no more than β, because

$$
\frac{1}{\beta} \ge \left[\frac{1}{\beta}\right] \ge \frac{1}{\beta} - 1 \Longrightarrow 1 \ge \beta\left[\frac{1}{\beta}\right] \ge 1 - \beta \Longrightarrow 0 \le 1 - \beta\left[\frac{1}{\beta}\right] \le \beta
$$

Thus, at least one of $0, \beta, 2\beta, \dots, [1/\beta]\beta$ will be in (t, s), which means $\{km\alpha\}$ is dense in $[0, 1]$.

(II) If $-\frac{1}{n} < \beta < 0$, let $k = 0, 1, \dots, [1/|\beta|]$, then at least one $\{km\alpha\}$ is in $[0, 1]$. This is because $km\alpha = kw + k\beta$, but since β is negative, $\{km\alpha\} = 1 - k|\beta|$, which corresponds to $1, 1 - |\beta|, 1 - 2|\beta|, \dots, 1 - [1/|\beta|]|\beta|$. We also observe that the distance between any two number is $|\beta| < \frac{1}{n} < L$, and the distance between the last number $1 - [1/|\beta|]|\beta|$ and the boundary value 0 is also no more than $|\beta|$, because

$$0 \leq \frac{1}{|\beta|} - \left[\frac{1}{|\beta|}\right] \leq 1 \Longrightarrow 0 \leq 1 - \left[\frac{1}{|\beta|}\right]|\beta| \leq |\beta|$$

Thus, at least one of $1, 1-|\beta|, 1-2|\beta|, \ldots, 1-[1/|\beta|]|\beta|$ will be in (t,s), which means $\{km\alpha\}$ is dense in $[0,1]$.

Therefore, for any $s, t \in [0,1]$, $0 \leq t < s \leq 1$, there exists $\{n\alpha\}$ in (t,s), which shows it is dense in $[0,1]$. Take $\alpha = \frac{1}{2\pi}$, and the proof is finished.

Question 3.1-6. If a and b are positive numbers and p an arbitrary nonzero real number, then the *mean of order* p of the numbers a and b is the quantity

$$S_p(a,b) = \left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}}$$

In particular for $p=1$ we obtain the arithmetic mean of a and b, for $p=2$ their square-mean, and for $p=-1$ their harmonic mean.

a) Show that the mean $S_p(a,b)$ of any order lies between the numbers a and b.

Let $c = S_p(a,b) > 0$, then we have $c^p = (a^p + b^p)/2$ which shows c^p is between a^p and b^p.
If $p > 0$, x^p is increasing when $x > 0$, so that c^p is between a^p, b^p yields c is between a, b.
If $p < 0$, x^p is decreasing when $x > 0$, also that c^p is between a^p, b^p yields c is between a, b.

b) Find the limits of the sequences

$$\{S_n(a,b)\}, \qquad \{S_{-n}(a,b)\}$$

To find the limit of $\{S_n(a,b)\}$, using L'Hôpital's rule

$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{a^n+b^n}{2}\right)^{\frac{1}{n}} &= \lim_{n\to\infty}\exp\left\{\frac{1}{n}\ln\frac{a^n+b^n}{2}\right\} \\
&= \exp\left\{\lim_{n\to\infty}\frac{1}{n}\ln\frac{a^n+b^n}{2}\right\} \\
&= \exp\left\{\lim_{n\to\infty}\frac{a^n\ln a + b^n\ln b}{a^n+b^n}\right\} \\
&= \exp\left\{\lim_{n\to\infty}\left[\ln a + \frac{(b/a)^n\ln(b/a)}{1+(b/a)^n}\right]\right\} \\
&= \exp\left\{\lim_{n\to\infty}\left[\ln a + \frac{\ln(b/a)}{(a/b)^n+1}\right]\right\}
\end{aligned}$$

If $a \geq b$, the limit is $\exp\{\ln a\} = a$; if $a < b$, the limit is $\exp\{\ln b\} = b$.

To find the limit of $\{S_{-n}(a,b)\}$, using L'Hôpital's rule

$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{a^{-n}+b^{-n}}{2}\right)^{\frac{1}{-n}} &= \lim_{n\to\infty}\exp\left\{\frac{1}{-n}\ln\frac{a^{-n}+b^{-n}}{2}\right\} \\
&= \exp\left\{\lim_{n\to\infty}\frac{1}{-n}\ln\frac{a^{-n}+b^{-n}}{2}\right\} \\
&= \exp\left\{\lim_{n\to\infty}\frac{a^{-n}\ln a+b^{-n}\ln b}{a^{-n}+b^{-n}}\right\} \\
&= \exp\left\{\lim_{n\to\infty}\left[\ln a+\frac{(b/a)^{-n}\ln(b/a)}{1+(b/a)^{-n}}\right]\right\} \\
&= \exp\left\{\lim_{n\to\infty}\left[\ln a+\frac{\ln(b/a)}{(a/b)^{-n}+1}\right]\right\} \\
&= \exp\left\{\lim_{n\to\infty}\left[\ln a+\frac{\ln(b/a)}{(b/a)^{n}+1}\right]\right\}
\end{aligned}$$

If $a \geq b$, the limit is $\exp\{\ln b\} = b$; if $a < b$, the limit is $\exp\{\ln a\} = a$.

Question 3.1-7. Show that if $a > 0$, the sequence $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ converges to the square root of a for any $x_1 > 0$.

Estimate the rate of convergence, that is, the magnitude of the absolute error $|x_n - \sqrt{a}| = |\Delta_n|$ as a function of n.

We need to prove x_n is a decreasing and bounded below sequence when $n \geq 2$, so that we could say it is convergent.

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \geq \frac{1}{2}\cdot 2\sqrt{x_n \cdot \frac{a}{x_n}} = \sqrt{a}$$

Therefore, for any $n \geq 2$, x_n is bounded below by $\sqrt{a}$.

$$x_{n+1} - x_n = \frac{1}{2}\left(\frac{a}{x_n} - x_n\right) = \frac{a - x_n^2}{2x_n} \leq 0$$

Hence x_n is decreasing when $n \geq 2$. Therefore, x_n converges.

Take the limit as $n \to \infty$ on both sides, denote the limit as λ, and we have

$$\lambda = \frac{1}{2}\left(\lambda + \frac{a}{\lambda}\right)$$

Since all terms are no less than $\sqrt{a}$, we obtain $\lambda = \sqrt{a}$ (negative solution is impossible).

There are many ways to estimate the rate of convergence, since I'm not so smart, I just use brutal force, i.e., calculate the explicit formula of x_n.

$$x_{n+1} + \sqrt{a} = \frac{x_n^2 + 2\sqrt{a}x_n + a}{2x_n} = \frac{(x_n + \sqrt{a})^2}{2x_n}$$

$$x_{n+1} - \sqrt{a} = \frac{x_n^2 - 2\sqrt{a}x_n + a}{2x_n} = \frac{(x_n - \sqrt{a})^2}{2x_n}$$

Take the quotient of them, we obtain

$$\frac{x_{n+1} + \sqrt{a}}{x_{n+1} - \sqrt{a}} = \left(\frac{x_n + \sqrt{a}}{x_n - \sqrt{a}}\right)^2$$

Define

$$b_{n+1} = \frac{x_{n+1} + \sqrt{a}}{x_{n+1} - \sqrt{a}}$$

We conclude that $b_{n+1} = b_n^2$, it yields that $b_n = b_1{}^{2^{n-1}}$ for any $n \geq 2$, where

$$b_1 = \frac{x_1 + \sqrt{a}}{x_1 - \sqrt{a}}$$

Finally, we can solve x_n explicitly by knowing x_1,

$$x_n = \sqrt{a} + \frac{2\sqrt{a}}{\left(\frac{x_1+\sqrt{a}}{x_1-\sqrt{a}}\right)^{2^{n-1}} - 1}, \quad n \geq 2$$

Therefore, it's easy to see the rate of convergence is

$$|x_n - \sqrt{a}| = |\Delta_n| = \frac{2\sqrt{a}}{\left(\frac{x_1+\sqrt{a}}{x_1-\sqrt{a}}\right)^{2^{n-1}} - 1}, \quad n \geq 2$$

If $x_1 = \sqrt{a}$, $x_n = \sqrt{a}$ for all $n \geq 2$, $|\Delta_n| = 0$.

Question 3.1-8. Show that

a) $S_0(n) = 1^0 + \cdots + n^0 = n$,

$$S_1(n) = 1^1 + \cdots + n^1 = \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n,$$

$$S_2(n) = 1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n,$$

$$S_3(n) = \frac{n^2(n+1)^2}{4} = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2,$$

First, it's easy to calculate that $S_0(n) = 1^0 + \cdots + n^0 = n$, and consider the following binomial expansion,

$$\begin{aligned}
1^k &= (2-1)^k = \binom{k}{0} 2^k (-1)^0 + \binom{k}{1} 2^{k-1} (-1)^1 + \cdots + \binom{k}{k} 2^0 (-1)^k && (1)\\
2^k &= (3-1)^k = \binom{k}{0} 3^k (-1)^0 + \binom{k}{1} 3^{k-1} (-1)^1 + \cdots + \binom{k}{k} 3^0 (-1)^k && (2)\\
\vdots &= \quad \vdots \quad = \quad \vdots \quad + \quad \vdots \quad + \cdots + \quad \vdots \\
(n-1)^k &= (n-1)^k = \binom{k}{0} n^k (-1)^0 + \binom{k}{1} n^{k-1} (-1)^1 + \cdots + \binom{k}{k} n^0 (-1)^k && \text{(n-1)}
\end{aligned}$$

Take the summation of (1) to $(n-1)$, we have

$$S_{k-1}(n) = \frac{1}{k}n^k + 1 - \frac{1}{k} + \frac{1}{k}\sum_{i=2}^{k} \binom{k}{i} (S_{k-i}(n) - 1)(-1)^i$$

Therefore, let $k = 2, 3, 4$, we could have

$$S_1(n) = \frac{1}{2}n^2 + 1 - \frac{1}{2} + \frac{1}{2}(n-1)(-1)^2 = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$S_2(n) = \frac{1}{3}n^3 + 1 - \frac{1}{3} + \frac{1}{3}\left[3\left(\frac{1}{2}n^2 + \frac{1}{2}n - 1\right)(-1)^2 + (n-1)(-1)^3\right] = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

$$S_3(n) = \frac{1}{4}n^4 + 1 - \frac{1}{4} + \frac{1}{4}\sum_{i=2}^{4}\binom{4}{i}(S_{4-i}(n) - 1)(-1)^i = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$$

Therefore, since $S_0(n)$ is a polynomial in n of degree 1, and $S_1(n)$ is a polynomial in n of degree 2, we suppose $S_i(n)$ is a polynomial in n of degree $i+1$ for all $i \leq k$, by induction (Strong induction) we need to prove $S_{k+1}(n)$ is a polynomial in n of degree $k+2$. Since

$$S_{k+1}(n) = \frac{1}{k+2}n^{k+2} + 1 - \frac{1}{k+2} + \frac{1}{k+2}\sum_{i=2}^{k+2}\binom{k+2}{i}(S_{k+2-i}(n) - 1)(-1)^i$$

We know that the highest order term in $S_{k+1}(n)$ is $\frac{1}{k+2}n^{k+2}$, the middle term is a constant term with respect to n, and the summation term is a linear combination of $S_i(n)$, which is at most of order $n+1$. Therefore, we prove that $S_{k+1}(n)$ is a polynomial in n of degree $k+2$.

b) $\lim\limits_{n\to\infty} \frac{S_k(n)}{n^{k+1}} = \frac{1}{k+1}$.

From the above formula of $S_k(n)$ we can see that the highest order term is $\frac{1}{k+1}n^{k+1}$, and other term are $o(n^{k+1})$ terms, therefore,

$$\lim_{n\to\infty} \frac{S_k(n)}{n^{k+1}} = \lim_{n\to\infty} \frac{1}{k+1} + \lim_{n\to\infty} \frac{o(n^{k+1})}{n^{k+1}} = \frac{1}{k+1}$$

Question 3.2-1.

a) Prove that there exists a unique function defined on $\mathbb{R}$ and satisfying the following conditions:

$$f(1) = a \quad (a > 0,\ a \neq 1),$$
$$f(x_1) \cdot f(x_2) = f(x_1 + x_2),$$
$$f(x) \mapsto f(x_0) \text{ as } x \to x_0.$$

Since in the textbook we have already define the exponential function, we can easily clarify the existence of a function satisfying the above conditions, i.e., $f(x) = a^x$. Now we need to check the uniqueness.

For any function g that satisfies the above conditions, take $x_1 = 0, x_2 = 1$, we have

$$g(0) \cdot g(1) = g(1) = a \Longrightarrow g(0) = 1$$

For $x_2 \in \mathbb{N}$, take $x_1 = 1$

$$g(1) \cdot g(x_2) = g(1 + x_2) \Longrightarrow a \cdot g(x_2) = g(1 + x_2)$$

Therefore, $g(x)$ is recursively defined by $g(2) = ag(1)$, $g(3) = a^2g(1)$, $\ldots$ $g(n) = a^{n-1}g(1) = a^n$. These values are uniquely defined, because $g(1)$ is fixed. If some $g(n)$ not equal to a^n, Property 2 will not hold for such function g.

Similarly, $x_2 \in \mathbb{Z}, x_2 < 0$, take $x_1 = -x_2 \in \mathbb{N}$,

$$g(-x_2) \cdot g(x_2) = g(0) \Longrightarrow a^{-x_2} \cdot g(x_2) = 1$$

Hence, $g(x_2) = a^{x_2}$ for $x_2 \in \mathbb{Z}, x_2 < 0$. Now we obtain $g(x) = a^x$ for all $x \in \mathbb{Z}$, and they are uniquely defined. If some $g(n)$ not equal to a^n, Property 2 will not hold for such function g.

Let $x = m/n$ where $n \neq 0$, $m, n \in \mathbb{Z}$, then

$$g\left(\frac{m}{n}\right) \cdot g\left(\frac{m}{n}\right) = g\left(2\frac{m}{n}\right)$$

By induction, we have

$$\left[g\left(\frac{m}{n}\right)\right]^n = g\left(n\frac{m}{n}\right) = g(m) = a^m$$

which shows for all rational number r, $g(r) = a^r$. These values are uniquely defined. If some $g(r)$ not equal to a^r, Property 2 will not hold for such function g.

Consider $x \in \mathbb{R}$, we need to use property 3 to prove the uniqueness of it. For $r_n \in \mathbb{Q}$,

$$g(x) = \lim_{r_n \to x} a^r$$

However for any x, the limit value on the right hand side exists and is unique by the uniqueness of limit. This is because for exponential function we know it is continuous on $\mathbb{R}$, so for any convergent sequence x_n, and any real number x,

$$\lim_{x_n \to x} a^{x_n} = a^{\lim_{x_n \to x} x_n} = a^x$$

Since we could find rational number sequence r_n such that $r_n \to x$, $g(x) = a^x$ for all real number x is uniquely defined.

b) Prove that there exists a unique function defined on $\mathbb{R}_+$ and satisfying the following conditions:

$$f(a) = 1 \quad (a > 0,\ a \neq 1),$$

$$f(x_1) + f(x_2) = f(x_1 \cdot x_2),$$

$$f(x) \mapsto f(x_0) \text{ for } x, x_0 \in \mathbb{R}_+, \text{ as } x \to x_0.$$

This is simlar to part a), the existence is easy to see, just take $f(x) = \log_a x$. Now we need to check the uniqueness.

For any function g that satisfies the above conditions, take $x_1 = x_2 = 1$, we have

$$g(1) + g(1) = g(1 \cdot 1) = g(1) \Longrightarrow g(1) = 0$$

For $x_2 \in \mathbb{N}^+$, take $x_1 = x_2 = a$,

$$g(a) + g(a) = g(a^2) = 2$$

By induction, we have $g(a^x) = x$ for all positive integer x.

Similarly, take $x_1 = a, x_2 = a^{-1}$,

$$g(a) + g(a^{-1}) = g(a \cdot a^{-1}) = g(1) = 0 \Longrightarrow g(a^{-1}) = -g(a) = -1$$

By induction, we have $g(a^{-n}) = -n$, which means $g(a^x) = x$ for all integer x.

Let $x = m/n$ where $m, n \in \mathbb{N}$, $n \neq 0$, then

$$g\left(a^{\frac{m}{n}}\right) + g\left(a^{\frac{m}{n}}\right) = g\left(a^{2\frac{m}{n}}\right)$$

By induction,

$$n\left[g\left(a^{\frac{m}{n}}\right)\right] = g\left(a^{n\frac{m}{n}}\right) = g(a^m) = m \Longrightarrow g\left(a^{\frac{m}{n}}\right) = \frac{m}{n}$$

which shows for all rational number r, $g(a^r) = r$. These values are uniquely defined. If some $g(a^r)$ not equal to r, Property 2 will not hold for such function g.

Consider $x \in \mathbb{R}$, we need to use property 3 to prove the uniqueness of it. For $r_n \in \mathbb{Q}$,

$$g(a^x) = \lim_{r_n \to x} g(a^{r_n}) = \lim_{r_n \to x} r_n = x$$

Thus $g(a^x)$ is uniquely defined as x for any real number x. Since we know exponential function is nonnegative, $a^x > 0$, we can take $t = a^x$, and $t > 0$. We have $g(t) = \log_a t$. For each x we have a unique t, and also unique $\log_a t$. Hence $g(t) = \log_a t$ for all real number $t > 0$ is uniquely defined.

Question 3.2-2.

a) Establish a one-to-one correspondence $\varphi : \mathbb{R} \mapsto \mathbb{R}_+$ such that $\varphi(x+y) = \varphi(x) \cdot \varphi(y)$ for any $x, y \in \mathbb{R}$, that is so that the operation of multiplication in the image $(\mathbb{R}_+)$ corresponds to the operation of addition in the pre-image $(\mathbb{R})$. The existence of such a mapping means that the groups $(\mathbb{R}, +)$ and $(\mathbb{R}_+, \cdot)$ are identical as algebraic objects, or, as we say, they are *isomorphic*.

It's easy to think of the function $\varphi(x) = e^x$. e^x is a bijective mapping from the whole real number set to positive real number set.

b) Prove that the groups $(\mathbb{R}, +)$ and $(\mathbb{R}\backslash 0, \cdot)$ are not isomorphic.

Using what we learn in abstract algebra, we need to find a structural property that is not shared by these two binary structure. Consider the equation $x * x * x = x$, for element in $(\mathbb{R}, +)$, i.e., $x + x + x = x$ which has a unique solution $x = 0$; for element in $(\mathbb{R}\backslash 0, \cdot)$, i.e., $x \cdot x \cdot x = x$ which has two solution ± 1 ($x = 0$ is not in $\mathbb{R}\backslash 0$).

This means the groups $(\mathbb{R}, +)$ and $(\mathbb{R}\backslash 0, \cdot)$ have different structural property, which implies they are not isomorphic.

If you don't want to use the knowledge of abstract algebra, we may assume there exists bijective mapping g from $\mathbb{R}$ and $\mathbb{R}\backslash 0$. Then for any $x, y \in \mathbb{R}$, we have

$$g(x + y) = g(x) \cdot g(y)$$

Take $x = y$, we have $g(2x) = [g(x)]^2 > 0$ for all $x \in \mathbb{R}$. Since $g(x)$ is bijective, $-1 \in \mathbb{R}\backslash 0$, there must exists $x_0 \in \mathbb{R}$ such that $g(x_0) = -1$. We know $\frac{x_0}{2} \in \mathbb{R}$, therefore

$$g(x_0) = g\left(2 \cdot \frac{x_0}{2}\right) = \left[g\left(\frac{x_0}{2}\right)\right]^2 > 0$$

This is a contradiction, which means there does not exist any bijective mapping g. Hence, the groups $(\mathbb{R}, +)$ and $(\mathbb{R}\backslash 0, \cdot)$ are not isomorphic.

Warning! One-to-one correspondence means bijective mapping!

Question 3.2-3. Find the following limits.

a) $\lim_{x \to +0} x^x$

By using L'Hôpital's rule, we have

$$\begin{aligned}
\lim_{x \to +0} x^x &= \lim_{x \to +0} e^{x \ln x} \\
&= e^{\lim_{x \to +0} x \ln x} \\
&= e^{\lim_{x \to +0} \frac{\ln x}{1/x}} \\
&= e^{\lim_{x \to +0} \frac{1/x}{-1/x^2}} \\
&= e^{\lim_{x \to +0} -x} \\
&= e^0 = 1
\end{aligned}$$

b) $\lim_{x \to +\infty} x^{1/x}$

By using L'Hôpital's rule, we have

$$\begin{aligned}
\lim_{x \to +\infty} x^{1/x} &= \lim_{x \to +\infty} e^{\frac{\ln x}{x}} \\
&= e^{\lim_{x \to +\infty} \frac{\ln x}{x}} \\
&= e^{\lim_{x \to +\infty} \frac{1/x}{1}} \\
&= e^0 = 1
\end{aligned}$$

c) $\lim_{x \to 0} \frac{\log_a(1+x)}{x}$

By using L'Hôpital's rule, we have

$$\begin{aligned}
\lim_{x \to +0} \frac{\log_a(1+x)}{x} &= \lim_{x \to +0} \frac{\ln(1+x)}{x \ln a} \\
&= \lim_{x \to +0} \frac{1/(x+1)}{\ln a} \\
&= \frac{1}{\ln a}
\end{aligned}$$

d) $\lim_{x \to 0} \frac{a^x - 1}{x}$

By using L'Hôpital's rule, we have

$$\begin{aligned}\lim_{x \to 0} \frac{a^x - 1}{x} &= \lim_{x \to 0} \frac{a^x \ln a}{1} \\ &= \frac{a^0 \ln a}{1} \\ &= \ln a\end{aligned}$$

Question 3.2-4. Show that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \ln n + c + o(1) \text{ as } n \to \infty$$

where c is a constant. (The number $c = 0.57721\ldots$ is called *Euler's constant.*)

Let partial sum S_n be defined as follows

$$S_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n$$

We need to prove

$$S_n = c + o(1) \iff \lim_{n \to \infty} S_n = c$$

First we check S_n is decreasing.

$$S_n - S_{n+1} = \ln \frac{n+1}{n} - \frac{1}{n+1} = -\ln\left(1 - \frac{1}{n+1}\right) - \frac{1}{n+1} > -\left(-\frac{1}{n+1}\right) - \frac{1}{n+1} = 0$$

Second we prove S_n is bounded below.

$$S_n > \sum_{k=1}^{n} \ln\left(1 + \frac{1}{k}\right) - \ln n = \ln(n+1) - \ln n > 0$$

Therefore, S_n is decreasing and bounded below, which means it is convergent to a constant, denoted as c. Note that we don't need to calculate the value of c in this question.

Question 3.2-5. Show that

a) if two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ with positive terms are such that $a_n \sim b_n$ as $n \to \infty$, then the two series either both converge or both diverge.

First, suppose $\sum_{n=1}^{\infty} a_n$ converges and $a_n \sim b_n$ as $n \to \infty$, we have

$$\forall\, \epsilon_1 > 0,\ \exists N_1,\ \forall\, n > m \geq N_1,\ \left|\sum_{k=m}^{n} a_k\right| < \epsilon_1$$

$$\forall\, \epsilon_2 > 0,\ \exists N_2,\ \forall\, n \geq N,\ \left|\frac{b_n}{a_n} - 1\right| < \epsilon_2$$

Since a_n and b_n are positive, we have $\forall\ \epsilon_2 > 0,\ \exists N_2,\ \forall\ n \geq N$

$$|b_n - a_n| < \epsilon_2 a_n \iff (1-\epsilon_2)a_n < b_n < (1+\epsilon_2)a_n$$

Take $N = \max\{N_1, N_2\}$, and $\epsilon = (1+\epsilon_2)\epsilon_1$, for any $n > m \geq N$, we have

$$\begin{aligned}\left|\sum_{k=m}^{n} b_k\right| &\leq \sum_{k=m}^{n} |b_k| \\ &= \sum_{k=m}^{n} b_k \\ &= \sum_{k=m}^{n} (1+\epsilon_2)a_k \\ &\leq (1+\epsilon_2)\left|\sum_{k=m}^{n} a_k\right| \leq (1+\epsilon_2)\epsilon_1 = \epsilon\end{aligned}$$

Thus $\sum\limits_{n=1}^{\infty} b_n$ converges.

Second, suppose $\sum\limits_{n=1}^{\infty} a_n$ diverges and $a_n \sim b_n$ as $n \to \infty$, we have

$$\exists\ \epsilon_1 > 0,\ \forall\ N_1,\ \exists\ n > m \geq N_1,\ \left|\sum_{k=m}^{n} a_k\right| \geq \epsilon_1$$

$$\forall\ \epsilon_2 > 0,\ \exists N_2,\ \forall\ n \geq N,\ \left|\frac{b_n}{a_n} - 1\right| < \epsilon_2$$

Since a_n and b_n are positive, we have $\forall\ \epsilon_2 > 0,\ \exists N_2,\ \forall\ n \geq N$

$$|b_n - a_n| < \epsilon_2 a_n \iff (1-\epsilon_2)a_n < b_n < (1+\epsilon_2)a_n$$

Take $\epsilon_2 = \frac{1}{2}$, we have $b_n > \frac{1}{2}a_n$.

Take $N = \max\{N_1, N_2\}$. There exists $\epsilon = \frac{1}{2}\epsilon_1$, $\forall\ N_0$, $\exists\ n > m \geq \max\{N, N_0\}$, such that

$$\begin{aligned}\left|\sum_{k=m}^{n} b_k\right| &= \sum_{k=m}^{n} b_k \\ &> \sum_{k=m}^{n} \frac{1}{2}a_k \\ &= \frac{1}{2}\left|\sum_{k=m}^{n} a_k\right| \geq \frac{1}{2}\epsilon_1 = \epsilon\end{aligned}$$

b) the series $\sum\limits_{n=1}^{\infty} \sin\frac{1}{n^p}$ converges only for $p > 1$.

If $p \leq 0$, then $\frac{1}{n^p}$ is increasing to infinity as $n \to \infty$. Since sine function is a periodic function on $\mathbb{R}$, it's easy to see $\lim\limits_{n\to\infty} \sin\frac{1}{n^p} \neq 0$, thus the series cannot converge.

If $p > 0$, $\frac{1}{n^p}$ is decreasing to 0 as $n \to \infty$. Since x is equivalent to $\sin x$ as $x \to 0$, we conclude that $\frac{1}{n^p} \sim \sin\frac{1}{n^p}$. By what we proved in part a), we know $\sum\limits_{n=1}^{\infty} \sin\frac{1}{n^p}$ converges when $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$

converges, i.e., $p > 1$ (proved in Rudin's book). On the contrary, it diverges if $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ diverges, i.e., $0 < p \leq 1$.

Question 3.2-6. Show that

a) if $a_n \geq a_{n+1} > 0$ for all $n \in \mathbb{N}$ and the series $\sum\limits_{n=1}^{\infty} a_n$ converges, then $a_n = o\left(\frac{1}{n}\right)$ as $n \to \infty$;

To prove $a_n = o\left(\frac{1}{n}\right)$, we need to prove $\lim\limits_{n\to\infty} na_n = 0$.

Since $\sum\limits_{n=1}^{\infty} a_n$ converges, a_n must converges to zero, for any $\epsilon > 0$, there exists N_0, for all $n > N_0$, $a_n < \epsilon$.

Also, for any $\epsilon > 0$, there exists N for any $n > m \geq N$,

$$\sum_{i=m+1}^{n} a_i < \epsilon$$

Take $n = 2m$, we have

$$ma_{2m} < \sum_{i=m+1}^{2m} a_i < \epsilon$$

Since for any $\epsilon > 0$, there exists N, such that for any $m \geq N$,

$$ma_{2m} < \epsilon \Longrightarrow 2ma_{2m} < 2\epsilon$$

For any $\epsilon > 0$, take $n \geq 2N + 2N_0$, if $n = 2m$, from the above equation we have

$$na_n < 2\epsilon$$

If $n = 2m + 1$, we need to show

$$(2m + 1)a_{2m+1} < 3\epsilon$$

And since

$$(2m + 1)a_{2m+1} \leq (2m + 1)a_{2m} = 2ma_{2m} + a_{2m}$$

we have

$$(2m + 1)a_{2m+1} \leq 2ma_{2m} + a_{2m} < 2\epsilon + \epsilon = 3\epsilon$$

Therefore, $\lim\limits_{n\to\infty} na_n = 0$.

b) if $b_n = o\left(\frac{1}{n}\right)$, one can always construct a convergent series $\sum\limits_{n=1}^{\infty} a_n$ such that $b_n = o(a_n)$ as $n \to \infty$;

For any b_n satisfies $\lim\limits_{n\to\infty} nb_n = 0$, we can construct the same $\sum\limits_{n=1}^{\infty} a_n$,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

such that $b_n = o(a_n)$, because

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} (-1)^n nb_n = 0$$

The above statement can be easily proved by the definition of limit.

c) if a series $\sum\limits_{n=1}^{\infty} a_n$ with positive terms converges, then the series $\sum\limits_{n=1}^{\infty} A_n$, where $A_n = \sqrt{\sum\limits_{k=n}^{\infty} a_k} - \sqrt{\sum\limits_{k=n+1}^{\infty} a_k}$ also converges, and $A_n = o(a_n)$ as $n \to \infty$;

The partial sum of $\sum\limits_{n=1}^{\infty} A_n$ is

$$\begin{aligned}
\sum_{m=1}^{n} A_m &= \sqrt{\sum_{k=1}^{\infty} a_k} - \sqrt{\sum_{k=2}^{\infty} a_k} + \sqrt{\sum_{k=2}^{\infty} a_k} - \sqrt{\sum_{k=3}^{\infty} a_k} \\
&\quad + \sqrt{\sum_{k=3}^{\infty} a_k} - \sqrt{\sum_{k=4}^{\infty} a_k} + \ldots + \sqrt{\sum_{k=n}^{\infty} a_k} - \sqrt{\sum_{k=n+1}^{\infty} a_k} \\
&= \sqrt{\sum_{k=1}^{\infty} a_k} - \sqrt{\sum_{k=n+1}^{\infty} a_k}
\end{aligned}$$

Since $\sum\limits_{n=1}^{\infty} a_n$ converges to a constant, say b, and a_n is positive,

$$\lim_{n\to\infty} \sum_{k=1}^{n} A_k = \sqrt{b} - 0 = \sqrt{b}$$

Thus the series $\sum\limits_{n=1}^{\infty} A_n$ is convergent.

Consider rationalizing the numerator

$$A_n = \sqrt{\sum_{k=n}^{\infty} a_k} - \sqrt{\sum_{k=n+1}^{\infty} a_k} = \frac{a_n}{\sqrt{\sum\limits_{k=n}^{\infty} a_k} + \sqrt{\sum\limits_{k=n+1}^{\infty} a_k}}$$

We can see

$$\lim_{n\to\infty} \frac{a_n}{A_n} = \lim_{n\to\infty} \left(\sqrt{\sum_{k=n}^{\infty} a_k} + \sqrt{\sum_{k=n+1}^{\infty} a_k} \right) = 0 + 0 = 0$$

Thus, $a_n = o(A_n)$ as $n \to \infty$.

d) if a series $\sum\limits_{n=1}^{\infty} a_n$ with positive terms converges, then the series $\sum\limits_{n=1}^{\infty} A_n$, where $A_n = \sqrt{\sum\limits_{k=n}^{\infty} a_k} - \sqrt{\sum\limits_{k=n+1}^{\infty} a_k}$ also converges, and $A_n = o(a_n)$ as $n \to \infty$.

It follows from c) and d) that no convergent (resp. divergent) series can serve as a universal standard of comparison to establish the convergence (resp. divergence) of other series.

The partial sum of $\sum\limits_{n=2}^{\infty} A_n$ is

$$\begin{aligned}\sum_{m=2}^{n} A_m &= \sqrt{\sum_{k=1}^{2} a_k} - \sqrt{\sum_{k=1}^{1} a_k} + \sqrt{\sum_{k=1}^{3} a_k} - \sqrt{\sum_{k=1}^{2} a_k} \\ &\quad + \sqrt{\sum_{k=1}^{4} a_k} - \sqrt{\sum_{k=1}^{3} a_k} + \ldots + \sqrt{\sum_{k=1}^{n} a_k} - \sqrt{\sum_{k=1}^{n-1} a_k} \\ &= \sqrt{\sum_{k=1}^{n} a_k} - \sqrt{\sum_{k=1}^{1} a_k}\end{aligned}$$

Since $\sum\limits_{n=1}^{\infty} a_n$ diverge to infinity (positive terms),

$$\lim_{n\to\infty} \sum_{k=2}^{n} A_k = +\infty + \sqrt{a_1} = +\infty$$

Thus the series $\sum\limits_{n=2}^{\infty} A_n$ is divergent.

Consider rationalizing the numerator

$$A_n = \sqrt{\sum_{k=1}^{n} a_k} - \sqrt{\sum_{k=1}^{n-1} a_k} = \frac{a_n}{\sqrt{\sum\limits_{k=1}^{n} a_k} + \sqrt{\sum\limits_{k=1}^{n-1} a_k}}$$

We can see

$$\lim_{n\to\infty} \frac{A_n}{a_n} = \lim_{n\to\infty} \frac{1}{\sqrt{\sum\limits_{k=n}^{\infty} a_k} + \sqrt{\sum\limits_{k=n+1}^{\infty} a_k}} = \frac{1}{+\infty + \infty} = 0$$

Thus, $A_n = o(a_n)$ as $n \to \infty$.

Question 3.2-10. Show that

a) if $\frac{b_n}{b_{n+1}} = 1 + \beta_n$, $n = 1, 2, \ldots$, and the series $\sum\limits_{n=1}^{\infty} \beta_n$ converges absolutely, then the limit $\lim\limits_{n\to\infty} b_n = b \in \mathbb{R}$ exists;

Since $\sum\limits_{n=1}^{\infty} \beta_n$ converges, we have $\lim\limits_{n\to\infty} \beta_n = 0$. Therefore, there exists N such that $\forall\ n > N$, $|\beta_n| < \frac{1}{3}$, ensuring $1 + \beta_n > 0$. For $n > N$,

$$\frac{b_{N+1}}{b_{n+1}} = \frac{b_{N+1}}{b_{N+2}} \frac{b_{N+2}}{b_{N+3}} \cdots \frac{b_n}{b_{n+1}}$$

which gives us

$$\frac{b_{N+1}}{b_{n+1}} = \prod_{k=N+1}^{n} (1 + \beta_k) > 0$$

The corresponding infinite product of the partial product on the right hand side converges if

$$\sum_{n=N+1}^{\infty} \ln(1 + |\beta_n|) \quad \text{converges}$$

Since $|\beta_n| \to 0$ as $n \to \infty$, we have $\ln(1+|\beta_n|) \sim |\beta_n|$. By what we proved in **Question 3.2-5.**, we conclude $\sum_{n=1}^{\infty} \ln(1+|\beta_n|)$ and $\sum_{n=1}^{\infty} |\beta_n|$ both converge or diverge.

Since $\sum\limits_{n=1}^{\infty} \beta_n$ converges absolutely, we know

$$\sum_{n=1}^{\infty} \ln(1+|\beta_n|)$$

converges, and so does

$$\sum_{n=N+1}^{\infty} \ln(1+|\beta_n|)$$

Thus, as $n \to \infty$, $\frac{b_{N+1}}{b_{n+1}}$ converges to some nonzero constant value. Since b_{N+1} is also a constant value, the limit of b_{n+1} must exist.

b) if $\frac{a_n}{a_{n+1}} = 1 + \frac{p}{n} + \alpha_n$, $n = 1, 2, \ldots$, and the series $\sum\limits_{n=1}^{\infty} \alpha_n$ converges absolutely, then $a_n \sim \frac{c}{n^p}$ as $n \to \infty$;

Let $b_n = \frac{a_n}{1/n^p}$, and consider

$$\frac{b_n}{b_{n+1}} = \frac{a_n n^p}{a_{n+1}(n+1)^p} = \frac{a_n}{a_{n+1}} \frac{1}{(1+\frac{1}{n})^p} = \frac{1+\frac{p}{n}+\alpha_n}{(1+\frac{1}{n})^p}$$

For any real number p, we have binomial expansion

$$\frac{b_n}{b_{n+1}} = \frac{1+\frac{p}{n}+\alpha_n}{(1+\frac{1}{n})^p} = \frac{1+\frac{p}{n}+\alpha_n}{1+\frac{p}{n}+O(\frac{1}{n^2})} = 1 + \frac{\alpha_n - O(\frac{1}{n^2})}{(1+\frac{1}{n})^p}$$

Let

$$\beta_n = \frac{\alpha_n - O(\frac{1}{n^2})}{(1+\frac{1}{n})^p}$$

We need to prove $\sum \beta_n$ converges absolutely, then we can apply the result in part a). To achieve this,

$$|\beta_n| \le \frac{|\alpha_n|}{(1+\frac{1}{n})^p} + \frac{|O(\frac{1}{n^2})|}{(1+\frac{1}{n})^p}$$

For any p, $(1+\frac{1}{n})^{-p}$ is either decreasing or increasing, and it is bounded. Also, $\sum |\alpha_n|$ and $\sum |O(\frac{1}{n^2})|$ both converge. By Abel' Test, we conclude that

$$\sum_{n=1}^{\infty} \left(\frac{|\alpha_n|}{(1+\frac{1}{n})^p} + \frac{|O(\frac{1}{n^2})|}{(1+\frac{1}{n})^p} \right)$$

converges, and by comparison test, $\sum |\beta_n|$ converges, i.e., $\sum \beta_n$ converges absolutely.

Applying the result in part a),

$$\lim_{n\to\infty} b_n = c \ne 0 \Longleftrightarrow \lim_{n\to\infty} \frac{a_n}{\frac{1}{n^p}} = c$$

This shows for constant c,

$$a_n \sim \frac{c}{n^p}$$

Note: Newton's generalized binomial theorem ($\forall\, r \in \mathbb{R}$)

$$
\begin{aligned}
(x+y)^r &= \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k} y^k \\
&= x^r + r x^{r-1} y + \frac{r(r-1)}{2!} x^{r-2} y^2 + \frac{r(r-1)(r-2)}{3!} x^{r-3} y^3 + \cdots
\end{aligned}
$$

c) if the series $\sum_{n=1}^{\infty} a_n$ is such that $\frac{a_n}{a_{n+1}} = 1 + \frac{p}{n} + \alpha_n$ and the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges absolutely for $p > 1$ and diverges for $p \le 1$ (Gauss' test for absolute convergence of a series).

From part b), we have known that $a_n \sim \frac{c}{n^p}$, which means $|a_n| \sim \frac{|c|}{n^p}$. From **Question 3.2-5.**, we conclude that both $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} \frac{|c|}{n^p}$ converge or diverge. By Cauchy condensation law, it's easy to see $\sum_{n=1}^{\infty} \frac{|c|}{n^p}$ converges if $p > 1$, and diverges if $p \le 1$ (See Rudin's book). Therefore, we can conclude that $\sum_{n=1}^{\infty} |a_n|$ converges if $p > 1$, and diverges if $p \le 1$. This is exactly the same as $\sum_{n=1}^{\infty} a_n$ converges absolutely for $p > 1$ and diverges for $p \le 1$.

(Waring! Since $a_n \sim \frac{c}{n^p}$, there exists N such that for all $n > N$, all a_n have the same sign, which means if $\sum_{n=1}^{\infty} |a_n|$ diverges, then $\sum_{n=1}^{\infty} a_n$ must also diverge.)

Question 3.2-11. Show that

$$\overline{\lim_{n\to\infty}} \left(\frac{1 + a_{n+1}}{a_n} \right)^n \ge e$$

for any sequence $\{a_n\}$ with positive terms, and that this estimate cannot be improved.

We first prove that there

$$\frac{1 + a_{n+1}}{a_n} > 1 + \frac{1}{n}, \ \textit{i.o.}$$

Suppose it's not true, then there exists N, such that $\forall\, n > N$,

$$\frac{1 + a_{n+1}}{a_n} \le 1 + \frac{1}{n}$$

It's easy to see

$$\frac{a_{n+1}}{n+1} \le \frac{a_n}{n} - \frac{1}{n+1}$$

which means for all m

$$\frac{a_{n+m}}{n+m} \le \frac{a_n}{n} - \sum_{k=n+1}^{n+m} \frac{1}{k}$$

However, since the series (harmonic series) diverge to infinity, for sufficiently large m,

$$\frac{a_{n+m}}{n+m} \le \frac{a_n}{n} - \sum_{k=n+1}^{n+m} \frac{1}{k} < 0 \Longrightarrow a_{n+m} < 0$$

which contradicts the fact that a_n is positive. Hence we finish the proof of infinitely often.

Since there are infinitely many a_n which satisfy

$$\frac{1+a_{n+1}}{a_n} > 1 + \frac{1}{n}$$

we take a subsequence of a_n, denoted as a_{n_k}, which satisfies the above condition,

$$\varlimsup_{n\to\infty} \left(\frac{1+a_{n+1}}{a_n}\right)^n \geq \varlimsup_{k\to\infty} \left(\frac{1+a_{n_k+1}}{a_{n_k}}\right)^{n_k} \geq \varlimsup_{k\to\infty} \left(1+\frac{1}{n_k}\right)^{n_k} = e$$

Now we obtain a lower bound of this upper limit, if we can prove it is the greatest lower bound, then it cannot be improved. If we take $a_1 = 1,\ a_n = n\ln n,\ n \geq 2$, then we can prove that

$$\lim_{n\to\infty} \left[\frac{1+(n+1)\ln(n+1)}{n\ln n}\right]^n = e$$

which is given by

$$\lim_{n\to\infty} \left[1 + \frac{1+(n+1)\ln(n+1) - n\ln n}{n\ln n}\right]^{\frac{n\ln n}{1+(n+1)\ln(n+1)-n\ln n}\cdot\frac{1+(n+1)\ln(n+1)-n\ln n}{\ln n}} = e$$

We first consider the limit of $(n+1)\ln(n+1) - n\ln n$ as $n \to \infty$, it's easy to see

$$(n+1)\ln(n+1) - n\ln n = n\ln\left(1+\frac{1}{n}\right) + \ln(n+1)$$

Since $\ln(1+x) \sim x$ as $x \to 0$, if we take $n \to \infty$,

$$n\ln\left(1+\frac{1}{n}\right) \sim n\cdot\frac{1}{n} = 1$$

Combined with the fact that $\ln(n+1)$ tends to infinity, it shows that the limit of $(n+1)\ln(n+1) - n\ln n$ is infinity. Therefore, we let

$$N(n) = \frac{1+(n+1)\ln(n+1) - n\ln n}{n\ln n}$$

Since both denominator and numerator tend to infinity, we can apply L'Hôpital's rule, which shows

$$\lim_{n\to\infty} N(n) = \lim_{n\to\infty} \frac{\ln(n+1) - \ln n}{1+\ln n} = \lim_{n\to\infty} \left|-\frac{1}{n+1}\right| = 0$$

If we take $M(n) = 1/N(n)$, then $M(n)$ tends to positive infinity as $n \to \infty$. Therefore, the original problem becomes

$$\lim_{n\to\infty} \left[\frac{1+(n+1)\ln(n+1)}{n\ln n}\right]^n = \lim_{n\to\infty} \left[1+\frac{1}{M(n)}\right]^{M(n)\frac{n}{M(n)}}$$

We finally check the limit of $\frac{n}{M(n)}$,

$$\lim_{n\to\infty} \frac{1+(n+1)\ln(n+1) - n\ln n}{\ln n} = \lim_{n\to\infty} \frac{\ln(n+1) - \ln n}{\frac{1}{n}} = \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = 1$$

Therefore,

$$\lim_{n\to\infty} \left[1+\frac{1}{M(n)}\right]^{M(n)\frac{n}{M(n)}} = \left\{\lim_{n\to\infty} \left[1+\frac{1}{M(n)}\right]^{M(n)}\right\}^{\lim\limits_{n\to\infty}\frac{n}{M(n)}} = e^1$$

This means the lower bound is reached by some a_n, so it cannot be improved.