MAT2006: Elementary Real Analysis
Homework 2

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Due date: Tomorrow

Question 4.2-1. Show that

a) if $f \in \mathcal{C}(A)$ and $B \subset A$, then $f\big|_B \in \mathcal{C}(B)$;

By Theorem 4.7 in Rudin's book, if $f_1 : X \mapsto Y$, $f_2 : Y \mapsto Z$, and $f : X \mapsto Z$ is defined by $f(x) = f_2(f_1(x))$, then f is continuous at a point $p \in X$ if f_1 is continuous at p and f_2 is continuous at $f_1(p)$. Hence, we can regard $f\big|_B$ as a composite function as follows

$$f_1 : B \mapsto A, \text{ defined as } f_1(x) = x \qquad f_2 : A \mapsto Y, \text{ defined as } f_2(x) = f(x)$$

and

$$f\big|_B : B \mapsto Y, \text{ defined as } f\big|_B(x) = f_2(f_1(x))$$

Since f_2 has already been continuous at any point in A, we only need to prove $f_1(x)$ is continuous.

Consider any open set in the codomain of f_1, i.e., $E \subset A$, $f_1^{-1}(E) = E \cap B$. By Theorem 2.30 in Rudin's book, since $B \subset A$ and $E \cap B$ is a subset of B, $E \cap B$ is open relative to B if and only if E is open relative to A. Thus, $f_1^{-1}(E)$ is open relative to B, and since E is arbitrarily chosen, f_1 is continuous on B. Then we apply Theorem 4.7, we can conclude that composite function $f\big|_B$ is continuous on B.

b) if a function $f : E_1 \cup E_2 \mapsto \mathbb{R}$ is such that $f\big|_{E_i} \in \mathcal{C}(E_i)$, $i = 1, 2$, it is not always the case that $f \in \mathcal{C}(E_1 \cup E_2)$.

Consider the function

$$f(x) = \begin{cases} x + 1 & \text{for } x \leq 0 \\ x - 1 & \text{for } x > 0 \end{cases}$$

This function is continuous on $E_1 = (-\infty, 0]$ and $E_2 = (0, +\infty)$, but it is not continuous on $E_1 \cup E_2$.

c) the Riemann function $\mathcal{R}$, and its restriction $\mathcal{R}\big|_{\mathbb{Q}}$ to the set of rational numbers are both discontinuous at each point of $\mathbb{Q}$ except 0, and all the points of discontinuity are removable.

We have proved the Riemann function $\mathcal{R}$ has removable discontinuity at all rational points in Diagnosis Test (whether 0 is removable discontinuous or not really does not matter). Now we only consider the restriction version of Riemann function $\mathcal{R}\big|_{\mathbb{Q}}$.

$\forall\, r \in \mathbb{Q}$, let k be the closest integer to r, and then $r \in (k-1, k+1)$.

$\forall\, \epsilon > 0$, take N, s.t., $\frac{1}{N} < \epsilon$. In $(k-1, k+1)$, the number of p/q s.t., $0 < q \leq N$ is finite.

For all of p/q s.t. $0 < q \leq N$ in $(k-1, k+1)$, denote the closest one to r as b, where $b \neq r$. Let $\delta = |b - a|$, then $\forall\, x \in \mathbb{Q}$ s.t. $0 < |x - r| < \delta$, $x = p/q$ where $q > N$, and therefore,

$$\left| \mathcal{R}\Big|_{\mathbb{Q}}(x) \right| = \frac{1}{q} < \frac{1}{N} < \epsilon$$

which means $\lim\limits_{x \to r} \mathcal{R}\Big|_{\mathbb{Q}}(x) = 0$.

However, $\forall r \in \mathbb{Q}$, $\mathcal{R}\Big|_{\mathbb{Q}}(r) = \dfrac{1}{q} \neq 0$, hence all rational points are removable discontinuous.

Question 4.2-2. Show that for a function $f \in \mathcal{C}[a, b]$, the functions

$$m(x) = \min_{a \leq t \leq x} f(t) \quad \text{and} \quad M(x) = \max_{a \leq t \leq x} f(t)$$

are also continuous on the closed interval $[a, b]$.

Since $f(x)$ is continuous on $[a, b]$, it is uniformly continuous. For any $x_0, y_0 \in [a, b]$, as long as $|x_0 - y_0| < \delta$,

$$|f(x_0) - f(y_0)| < \epsilon$$

Therefore, we have

$$\sup_{\substack{x_0, y_0 \\ |x_0 - y_0| < \delta}} |f(x_0) - f(y_0)| \leq \epsilon$$

Also, consider $m(x) = \min\limits_{a \leq t \leq x} f(t)$, we have

$$|m(x_0) - m(y_0)| = \left| \min_{a \leq t \leq x_0} f(t) - \min_{a \leq t \leq y_0} f(t) \right| \leq \sup_{\substack{x_0, y_0 \\ |x_0 - y_0| < \delta}} |f(x_0) - f(y_0)| \leq \epsilon$$

Therefore, $m(x)$ is also uniformly continuous on $[a, b]$.

Similarly. consider $M(x) = \max\limits_{a \leq t \leq x} f(t)$, we have

$$|M(x_0) - M(y_0)| = \left| \max_{a \leq t \leq x_0} f(t) - \max_{a \leq t \leq y_0} f(t) \right| \leq \sup_{\substack{x_0, y_0 \\ |x_0 - y_0| < \delta}} |f(x_0) - f(y_0)| \leq \epsilon$$

Therefore, $M(x)$ is also uniformly continuous on $[a, b]$.

Question 4.2-4. Show that

a) if $f \in \mathcal{C}[a, b]$ and $g \in \mathcal{C}[a, b]$, and in addition, $f(a) < g(a)$ and $f(b) > g(b)$, then there exists a point $c \in [a, b]$ at which $f(c) = g(c)$;

Since $f \in \mathcal{C}[a, b]$ and $g \in \mathcal{C}[a, b]$, we have $h(x) \triangleq [f(x) - g(x)] \in \mathcal{C}[a, b]$. In addition, we have $h(a) = f(a) - g(a) < 0$, and $h(b) = f(b) - g(b) > 0$. We need to prove that $\exists\, c \in [a, b]$ at which $h(c) = f(c) - g(c) = 0$.

Let $S \triangleq \{x \in [a,b] \mid h(x) \leq 0\}$, since $a \in S$, S is nonempty. Since S is bounded by b, we know $c = \sup S$ exists because any nonempty bounded-above set on real line has its least upper bound. We want to verify that $h(c) = 0$, since $c \in [a,b]$.

First, since $h(x)$ is continuous, $\forall\ \epsilon > 0$, $\exists\ \delta > 0$, such that $\forall\ |x-c| < \delta$, we have $|h(x)-h(c)| < \epsilon$. This is equivalent to say

$$\forall\ x \in (-\delta + c, \delta + c), \quad h(c) - \epsilon < h(x) < h(c) + \epsilon$$

Second, consider $a \in (-\delta + c, c)$, we have $h(c) - \epsilon < h(a)$. Since c is the least upper bound of S, meaning that $\forall\ c - \delta$ (no matter what δ is), there exists $a \in S$, such that $a > c - \delta$. Thus there exists $a \in (-\delta + c, c) \cap S$ for any δ, i.e., for any $\epsilon > 0$, we have a such that

$$h(c) - \epsilon < h(a) \leq 0$$

Similarly, for any $\epsilon > 0$, there exists $a \in (c, c + \delta)$, such that

$$h(c) + \epsilon > h(a) > 0$$

In conclusion, for any $\epsilon > 0$, $|h(c)| < \epsilon$, which means $h(c) = 0$.

b) any continuous mappings $f : [0,1] \mapsto [0,1]$ of a closed interval into itself has a fixed point, that is, a point $x \in [0,1]$ such that $f(x) = x$;

We have proved a general version for any $[a,b]$ in Numerical Analysis, now let us consider the simplier case $[0,1]$. If $f(0) = 0$ or $f(1) = 1$, the existence of fixed point is obvious.

Suppose not, since $f(0), f(1) \in [0,1]$, we know $f(0) > 0$ and $f(1) < 1$.

Define $g(x) = f(x) - x$, $g(x)$ is continuous on $[0,1]$. Moreover,

$$g(0) = f(0) - 0 = f(0) > 0, \quad g(1) = f(1) - 1 < 1 - 1 = 0$$

By part a), there exists $x_0 \in (0,1)$ for which $g(x_0) = 0$, i.e., $f(x_0) = x_0$.

Thus, in any cases, the fixed point would exist.

c) if two continuous mappings f and g of an interval into itself commute, that is, $f \circ g = g \circ f$, then they have a common fixed point;

According to Brouwer's fixed-point theorem, we know that continuous function from a convex compact set to itself can guarantee the existence of fixed point in this set. Therefore, to avoid some meaningless cases, we assume here this interval is compact, i.e., it is closed and bounded interval, denoted as $[a,b]$.

Unfortunately, William M. Boyce and other mathematicians have found an counterexample to this, see the following two references

(a) *ON COMMON FIXED POINTS OF COMMUTING CONTINUOUS FUNCTIONS ON AN INTERVAL*

(b) *COMMUTING FUNCTIONS WITH NO COMMON FIXED POINTS)*

d) a continuous mapping $f : \mathbb{R} \mapsto \mathbb{R}$ may fail to have a fixed point;

Consider $f(x) = x - 1$, which is a continuous from $\mathbb{R}$ to $\mathbb{R}$, but $f(x) = x$ has no solution, which means it has no fixed point.

e) a continuous mapping $f : (0,1) \mapsto (0,1)$ may fail to have a fixed point;

Consider $f(x) = x^2$ from $(0,1)$ to $(0,1)$, it is continuous but $f(x) = x$ gives you $x_1 = 0, x_2 = 1$ which are both outside of domain $(0,1)$. Hence, $f(x)$ has no fixed point.

f) if a mapping $f : [0,1] \mapsto [0,1]$ is continuous, $f(0) = 0$, $f(1) = 1$, and $(f \circ f)(x) \equiv x$ on $[0.1]$, then $f(x) \equiv x$.

First we need to prove that $f(x)$ is injective on $[0,1]$. If there exists $f(x_0) = f(x_0')$ but $x_0 \neq x_0'$, then since $(f \circ f)(x) = x$, we have $(f \circ f)(x_0) \neq (f \circ f)(x_0')$. This shows that

$$f(f(x_0)) \neq f(f(x_0'))$$

Let $c = f(x_0) = f(x_0')$, and this means $f(c) \neq f(c)$, which is impossible. Hence, $f(x)$ is injective.

Next we prove that an injective and continuous function on $[0,1]$ is strictly monotonic. Since it is injective, if $f(x)$ is monotonic, then it must be strictly monotonic. Continuous function on closed interval can obtain its maximum and minimum values, and its maximum and minimum values are either obtained at boundary point or extreme point. If $f(x)$ has extreme point x_0, suppose it is maximum, then $x_0 \in (0,1)$, so we can take $x_1 < x_0 < x_2$, and suppose $f(x_1) < f(x_2)$ (the other direction is similar), then apply Intermediate Value Theorem to x_0 and x_1. Since $f(x_1) < f(x_2) < f(x_0)$, we can find $x^* \in (x_1, x_0)$ such that $f(x^*) = f(x_2)$, but $x^* < x_0 < x_2$, which gives a contradiction to injective property. Hence, there is no maximum point in $(0,1)$. Similarly, there is no minimum point in $(0,1)$.

Since $f(0) = 0$, $f(1) = 1$, we conclude that $x = 0$ must be minimum point and $x = 1$ must be maximum point. If $f(x)$ is not strictly increasing, then by definition there exists two points $x_1, x_2 \in [0,1]$, such that $x_1 < x_2$ but $f(x_1) > f(x_2)$. If $x_1 = 0$, it is impossible because $f(0)$ is the minimum, you cannot find even smaller value $f(x_2)$. Similarly, $x_2 = 1$ is impossible because you cannot find even larger value $f(x_1)$. Hence, we have $0 < x_1 < x_2 < 1$, and $f(0) < f(x_2) < f(x_1) < f(1)$. Apply IVT to $0, x_1, x_2$, there exists $x^* \in (0, x_1)$ such that $f(x^*) = f(x_2)$, and since $x^* \neq x_2$, it contradicts the injective property. In conclusion, $f(x)$ is strictly increasing in $[0,1]$.

Finally, assume there exists $a \in (0,1)$ such that $f(a) \neq a$, denote $f(a) = b \in (0,1)$. Since $f(f(a)) = a$, we have $f(b) = a$ and $a \neq b$. Thus, we have $f(a) = b$ and $f(b) = a$. Since $a \neq b$, if $a > b$, then $f(f(a)) > f(f(b))$, which means $f(b) > f(a)$, but $f(x)$ is strictly increasing, so $f(a) > f(b)$ gives contradiction. Similarly, $a < b$ implies $f(f(a)) < f(f(b))$, meaning

$f(b) < f(a)$, which also contradicts strictly increasing property. Hence, such a, b does not exist, and $f(x) = x$ for all $x \in [0, 1]$.

Question 4.2-5. Show that the set of values of any function that is continuous on a closed interval is a closed interval.

First we need to know here the closed interval canno be unbounded, otherwise this theorem is wrong. For example, e^x is a continuous function on $(-\infty, \infty)$ (which is an unbounded closed interval), but its value is $(0, \infty)$ (which is not closed); $(x+1)^{-1}$ is continuous on $[0, \infty)$ (which is an unbounded closed interval), but its value is $(0, 1]$ (which is not closed). Thus, we only consider function continuous on $[a, b]$ and aim at proving its value is also an closed interval.

Theorem 4.15 in Rudin's book tells us if $f(x)$ is continuous on a compact set, then the set of values of $f(x)$ is closed and bounded. Denote the range of $f(x)$ as R, and R has least upper bound and greatest lower bound because of the completeness of real number. Since R is closed, by Theorem 2.28 in Rudin's book, we know $c = \inf f(x)$ and $d = \sup f(x)$ are both in R.

Next, we can find the pre-image of c and d, denoted as c_p and d_p. Since $c_p, d_p \in [a, b]$, W.L.O.G., we can assume $c_p < d_p$, then $[c_p, d_p] \subset [a, b]$. From **Question 4.2-1(a)**, we know that the restriction of f on $[c_p, d_p]$ is continuous. By the Intermediate Value Theorem, for any $y \in [c, d]$, there exists x, such that $f(x) = y$. Hence all value in $[c, d]$ can be obtained under the restriction of f on $[c_p, d_p]$. However, the range of f is bounded by c and d, so the range of f, i.e., $R = [c, d]$. Thus, the set of values of f (continuous) on $[a, b]$ (closed interval) is $[c, d]$ (closed interval).

Question 4.2-6. Prove the following statements.

a) If a mapping $f : [0, 1] \mapsto [0, 1]$ is continuous, $f(0) = 0$, $f(1) = 1$, and $f^n(x) \triangleq \underbrace{f \circ \cdots \circ f}_{n \text{ factors}}(x) \equiv x$ on $[0.1]$, then $f(x) \equiv x$.

Similar to **Question 4.2-4(f)**, we denote $g(x) = f^{n-1}(x)$, we first prove that $f(x)$ and $g(x)$ are both injective. $f^n(x) = f(g(x)) = g(f(x))$, suppose there exists $g(x_0) = g(x_0')$ but $x_0 \neq x_0'$, then we have

$$f^n(x_0) = f(g(x_0)) \neq f(g(x_0')) = f^n(x_0')$$

Let $c = g(x_0) = g(x_0')$, we have $f(c) \neq f(c)$, which is impossible.

Similarly, suppose there exists $f(x_0) = f(x_0')$ but $x_0 \neq x_0'$, then we have

$$f^n(x_0) = g(f(x_0)) \neq g(f(x_0')) = f^n(x_0')$$

Let $c = f(x_0) = f(x_0')$, we have $g(c) \neq g(c)$, which is impossible.

After that, we can apply what we have proved in **Question 4.2-4(f)**, i.e., every continuous injective function on $[0, 1]$ must be strictly monotonic. One can easily see that $g(x)$ is continuous on $[0, 1]$, because $f(x)$ is continuous on $[0, 1]$, and $f(x)$ maps $[0, 1]$ to itself, so $f^n(x)$ is continuous on $[0, 1]$ for any n. (Use the continuous property of composite function and induction). Thus, we can say that $f(x)$ and $g(x)$ are both strictly monotonic. Also, since

$0 = f(0) < f(1) = 1$, $f(x)$ is strictly increasing. Notice that $f^n(0) = 0$ and $f^n(1) = 1$ for all n, so $g(x)$ is strictly increasing too.

Finally, assume there exists $a \in [0,1]$ such that $g(a) \neq a$. Denote $g(a) = b$, we have $f(g(a)) = a \implies f(b) = a$. Suppose $a < b$, since f is strictly increasing, $f(a) < f(b)$; since g is strictly increasing, $g(a) < g(b)$. Combine with what we derived above, we have the relationship as follows

$$f(b) = a < b = g(a) < g(b) \implies f(b) < b < g(b)$$

However, since f is strictly increasing, we have

$$f(b) < b \implies f^2(b) < f(b) < b \implies \cdots \implies f^{n-1}(b) < b \implies g(b) < b$$

which contradicts $b < g(b)$. Similar contradiction will occur if you assume $a > b$.

Therefore, we conclude that $g(x) = x$ for all $x \in [0,1]$. This means that if $f^n(x) = x$, then $f^{n-1}(x) = x$, and n is arbitrary, by induction, we can finally have $f^1(x) = f(x) = x$.

b) If a function $f : [0,1] \mapsto [0,1]$ is continuous and nondecreasing, then for any point $x \in [0,1]$ at least one of the following situations must occur: either x is a fixed point, or $f^n(x)$ tends to a fixed point. (Here $f^n(x) = f \circ \cdots \circ f(x)$ is the nth iteration of f.)

For $x_0 \in [0,1]$, we have three cases $f(x_0) = x_0$, $f(x_0) > x_0$, $f(x_0) < x_0$. If $f(x_0) = x_0$, then x_0 is a fixed point. For the other two cases, we only consider $f(x_0) > x_0$, since the other case is similar, and you can follow suit with the following procedure to prove it so as to check whether you really understand the problem.

Denote $a_1 = f(x_0)$ and $a_n = f^n(x_0)$, since $f(x)$ is nondecreasing, and $f(x_0) > x_0$, we have

$$f(f(x_0)) \geq f(x_0) \implies a_2 \geq a_1$$

By induction, we could obtain for all $n \geq 1$, $a_{n+1} \geq a_n$, which shows a_n is monotonic. Since f is bounded above and below, a_n is bounded above and below. Monotonic sequence is convergent if and only if it is bounded, hence a_n converges. Denote $\lim\limits_{n\to\infty} a_n = c$, c must be in $[0,1]$, and we have

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n \implies \lim_{n\to\infty} f(f^n(x_0)) = c$$

Since f is continuous, we have

$$\lim_{n\to\infty} f(f^n(x_0)) = c \implies f\left(\lim_{n\to\infty} f^n(x_0)\right) = c \implies f(c) = c$$

which shows the limit of a_n, i.e., $f^n(x_0)$ is a fixed point of f. Similar things will happen when $f(x_0) < x_0$. Therefore, for any $x \in [0,1]$, either x is a fixed point, or $f^n(x)$ tends to a fixed point.

Question 4.2-7. Let $f : [0,1] \mapsto \mathbb{R}$ be a continuous function such that $f(0) = f(1)$. Show that

a) for any $n \in \mathbb{N}$ there exists a horizontal closed interval of length $\frac{1}{n}$ with endpoints on the graph of this function;

This is equivalent to say that for any $n \in \mathbb{N}$, there exists $x_0 \in [0,1]$, such that $f(x_0) = f\left(x_0 + \frac{1}{n}\right)$. To prove it, let $F(x) = f(x+1/n) - f(x)$, where $x \in \left[0, 1 - \frac{1}{n}\right]$. We have

$$F(0) = f(1/n) - f(0)$$
$$F(1/n) = f(2/n) - f(1/n)$$
$$\cdots$$
$$F(1-1/n) = f(1) - f(1-1/n)$$

Sum them up, we have

$$\sum_{k=0}^{n-1} F(k/n) = f(1) - f(0) = 0$$

Thus, there are only two cases, the first is that all $F(k/n)$ are zero, the second is that at least two of them have different signs. Denote them as $F(k_1/n)$ and $F(k_2/n)$, W.O.L.G., we assume $F(k_1/n) > 0$, $F(k_2/n) < 0$ and $k_1 < k_2$. Apply Intermediate Value Theorem, we can guarantee there exists $x_0 \in [k_1/n, k_2/n]$, such that $F(x_0) = 0$, i.e.,

$$f\left(x_0 + \frac{1}{n}\right) - f(x_0) = 0$$

Thus, we finish the proof.

b) if the number l is not of the form $\frac{1}{n}$ there exists a function of this form on whose graph one cannot inscribe a horizontal chord of length l.

Denote the length of interval as $\delta > 0$, if $\delta \neq \frac{1}{n}$, then we can construct a function g, such that $g \in \mathcal{C}(\mathbb{R})$, and $g(0) = 0$ but $g(1) = c \neq 0$. We add another condition that g is periodic with period δ. Notice that if $\delta = \frac{1}{n}$, then g does not exist, because 1 is the multiple of $\frac{1}{n}$, so $g(0) = g(1)$. Hence, for any $\delta > 0$, we could find a function $f(x) = g(x) - cx$, and $f(0) = g(0) = 0$, $f(1) = g(1) - c = 0$.

$$f(x+\delta) - f(x) = g(x+\delta) - c(x+\delta) - g(x) + cx = c\delta \neq 0, \text{ for all } x$$

Therefore, if the number l is not of the form $\frac{1}{n}$, such property in part a) fails.

Question 4.2-8. The *modulus of continuity* of a function $f : E \mapsto \mathbb{R}$ is the function $\omega(\delta)$ defined for $\delta > 0$ as follows:

$$\omega(\delta) = \sup_{\substack{|x_1 - x_2| < \delta \\ x_1, x_2 \in E}} |f(x_1) - f(x_2)|$$

Thus, the least upper bound is taken over all pairs of points x_1, x_2 of E whose distance apart is less than δ.

Show that

a) the modulus of continuity is a nondecreasing nonnegative function having the limit $\omega(+0) = \lim\limits_{\delta \to +0} \omega(\delta)$;

First, by definition $\omega(\delta)$ is the supremum of some absolute value, so $\omega(\delta) \geq 0$ for all δ.

Second, if for some $\delta_1 < \delta_2$, we have $\omega(\delta_1) > \omega(\delta_2)$, denote $\omega(\delta_1) = \omega(\delta_2) + L$, where $L > 0$. Also denote $c = \omega(\delta_1)$ and $d = \omega(\delta_2)$. Since c is least upper bound, for any $\epsilon > 0$, there exists $|f(x_1) - f(x_2)| > c - \epsilon$, we take $\epsilon = L$, and then we have some $|f(x_1) - f(x_2)| > c - L = d$. Since $|x_1 - x_2| < \delta_1 < \delta_2$, d is upper bound of $|f(x_1) - f(x_2)|$, which is a contradiction. Hence, $\omega(\delta)$ is nondecreasing.

Finally, we need to prove for a nondecreasing bounded below function on $(0, h)$, its right hand side limit at 0 must exist. Denote $E = \{\omega(\delta) \mid \delta \in (0, h)\}$, since it is bounded below, it must have greatest lower bound, denoted as b. We need to prove $\lim_{\delta \to +0} \omega(\delta) = b$.
By definition of greatest lower bound, $b \leq \omega(\delta)$ for all $\delta \in (0, h)$. Also, for any $\epsilon > 0$, there exists $\delta^* \in (0, h)$ such that $\omega(\delta^*) < b + \epsilon$. Thus, for any $\epsilon > 0$, take $h = \delta^*$, for all $0 < \delta < h$, we have $b - \epsilon < b \leq \omega(\delta) \leq \omega(\delta^*) < b + \epsilon$, which shows that $|\omega(\delta) - b| < \epsilon$, and hence

$$\lim_{\delta \to +0} \omega(\delta) = b$$

Warning! In this question, we can only consider function such that for each fixed δ, the supremum defined above is finite. For function like $1/x$ in $(0, 1)$, the modulus of continuity is always infinite, which is meaningless.

b) for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any points $x_1, x_2 \in E$ the relation $|x_1 - x_2| < \delta$ implies $|f(x_1) - f(x_2)| < \omega(+0) + \varepsilon$;

If the statement is wrong, then there exists $\epsilon > 0$, such that $\forall\ \delta > 0$, there exists x_1, x_2, $|x_1 - x_2| < \delta$ implies $|f(x_1) - f(x_2)| \geq \omega(+0) + \epsilon$. Let $b = \omega(+0)$, then we have $|\omega(\delta)| - b| \geq \epsilon$. However, from part a) we know that $\forall\ \epsilon > 0$, there exists $h > 0$, such that for $0 < \delta < h$, we have $|\omega(\delta) - b| < \epsilon/2$. This gives us a contradiction. Hence the statement is correct, meaning that b is the greatest lower bound of the range of function $\omega(\delta)$.

c) if E is a closed interval, an open interval, or a half-open interval, the relation

$$\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$$

As long as E is convex, we can ensure the subadditive property of modulus function $\omega(\delta)$.

$$\begin{aligned}
\omega(\delta_1 + \delta_2) &= \sup_{\substack{|x_1 - x_2| < \delta_1 + \delta_2 \\ x_1, x_2 \in E}} |f(x_1) - f(x_2)| \\
&\leq \sup_{\substack{|x_1 - x^*| < \delta_1 \\ |x^* - x_2| < \delta_2 \\ x_1, x_2, x^* \in E}} \Big(|f(x_1) - f(x^*)| + |f(x^*) - f(x_2)| \Big) \qquad (1) \\
&= \sup_{\substack{|x_1 - x^*| < \delta_1 \\ |x^* - x_2| < \delta_2 \\ x_1, x_2, x^* \in E}} |f(x_1) - f(x^*)| + \sup_{\substack{|x_1 - x^*| < \delta_1 \\ |x^* - x_2| < \delta_2 \\ x_1, x_2, x^* \in E}} |f(x^*) - f(x_2)| \\
&\leq \sup_{\substack{|x_1 - x^*| < \delta_1 \\ x_1, x^* \in E}} |f(x_1) - f(x^*)| + \sup_{\substack{|x^* - x_2| < \delta_2 \\ x^*, x_2 \in E}} |f(x^*) - f(x_2)| \\
&= \omega(\delta_1) + \omega(\delta_2)
\end{aligned}$$

Notice that in (1), the reason why such x^* can be find for all $|x_1 - x_2| < \delta_1 + \delta_2$ is that E is convex, for each pair of $x_1, x_2 \in E$, we take the convex combination of them, i.e.,

$$x^* = \frac{\delta_1}{\delta_1 + \delta_2} x_1 + \frac{\delta_2}{\delta_1 + \delta_2} x_2$$

You can see that $|x^* - x_1| < \delta_1$ and $|x^* - x_2| < \delta_2$, and by the definition of convex set, such convex combination must lie in the set E, hence such x^* is well defined.

d) the moduli of continuity of the functions x and $\sin(x^2)$ on the whole real axis are respectively $\omega(\delta) = \delta$ and the constant $\omega(\delta) = 2$ in the domain $\delta > 0$;

First, consider the moduli of continuity of $f(x) = x$, which is defined as

$$\omega(\delta) = \sup_{|x_1 - x_2| < \delta} |x_1 - x_2|$$

Since $x_1, x_2 \in \mathbb{R}$, $|x_1 - x_2|$ can be any value in $[0, \delta)$, applying Theorem 2.28 in Rudin's book, we have $\sup |x_1 - x_2|$ must be in the closure of $[0, \delta)$, i.e., $[0, \delta]$. For any value $\delta^* \neq \delta$, it cannot be an upper bound of $[0, \delta)$, because the element $(\delta + \delta^*)/2$ is in $[0, \delta)$, but it is larger than δ^*. Hence the only possible value of $\sup |x_1 - x_2|$ is δ, which means $\omega(\delta) = \delta$.

Second, consider the moduli of continuity of $f(x) = \sin(x^2)$, i.e.,

$$\omega(\delta) = \sup_{|x_1 - x_2| < \delta} |\sin(x_1^2) - \sin(x_2^2)|$$

For any $x_1, x_2 \in \mathbb{R}$, we have

$$|\sin(x_1^2) - \sin(x_2^2)| = 2 \left| \cos\left(\frac{x_1^2 + x_2^2}{2}\right) \right| \left| \sin\left(\frac{x_1^2 - x_2^2}{2}\right) \right| \leq 2$$

Hence, for any $\delta > 0$, $\omega(\delta) \leq 2$. However, for any $\delta > 0$, we can find

$$x_1 \geq \frac{2\pi - \delta^2/4}{\delta},\ x_2 = x_1 + \frac{\delta}{2}$$

which yields

$$|x_1 - x_2| < \delta,\ x_2^2 - x_1^2 \geq 2\pi$$

Hence we can find $s^2, t^2 \in [x_1^2, x_2^2]$, such that $\sin(s^2) = 1$, $\sin(t^2) = -1$, and $s, t \in [x_1, x_2]$. Since $|s - t| < \delta$, we conclude that for any $\delta > 0$, $\omega(\delta) \geq 2$. Therefore, $\omega(\delta) = 2$.

e) a function f is uniformly continuous on E if and only if $\omega(+0) = 0$.

First prove if $\omega(+0) = 0$, f is uniformly continuous. By part b), let $\omega(+0) = 0$, we have the following statement,

$$\forall\ \varepsilon > 0, \exists\ \delta > 0 \text{ s.t. } \forall\ x_1, x_2 \in E, |x_1 - x_2| < \delta, \text{ we have } |f(x_1) - f(x_2)| < \varepsilon$$

This is exactly the definition of uniform continuous of $f(x)$, so the proof is finished.

Then we prove if f is uniformly continuous, $\omega(+0) = 0$. If $\omega(+0) \neq 0$, then there exists $\epsilon > 0$ such that $\omega(\delta) \geq \epsilon$ for any $\delta > 0$ because it is monotonic by part a). However, the uniform continuous ensure that there exists $h > 0$ such that for all $|x_1 - x_2| < h$, $|f(x_1) - f(x_2)| < \epsilon/2$. Hence, their supremum, i.e., $\omega(\delta) \le \epsilon/2$, which gives a contradiction.

(Hint: For the existence of ϵ such that $\omega(\delta) \ge \epsilon$ for all δ when $\omega(+0) \ne 0$, first you could show $\omega(+0)$ is the lower bound of all $\omega(\delta)$ by using nondecreasing property, and then you could easily show that $\omega(+0)$ is also nonnegative.)

Question 4.2-9. Let f and g be bounded functions defined on the same set X. The quantity $\Delta = \sup_{x\in X} |f(x) - g(x)|$ is called the *distance* between f and g. It shows how well one function approximates the other on the given set X. Let X be a closed interval $[a, b]$. Show that if $f, g \in \mathcal{C}[a, b]$, then $\exists\ x_0 \in [a, b]$, where $\Delta = |f(x_0) - g(x_0)|$, and that such is not the case in general for arbitrary bounded functions.

This may be the easiest one among all questions. Let $h(x) = f(x) - g(x)$ defined on $[a, b]$. Since f, g is continuous, h is also continuous on $[a, b]$. Also, one can easily prove that if h is continuous, then $|h(x)|$ is continuous. We can see $\Delta = \sup |h(x)|$, and since $[a, b]$ is compact on $\mathbb{R}$, by Theorem 4.15 in Rudin's book, we conclude that the range of $|h(x)|$ is closed and bounded. Since the supremum and infimum of a closed and bounded set must lie in this set (Theorem 2.28 in Rudin's book), Δ is in the range of $|h(x)|$. This exactly means that there exists a $x_0 \in [a, b]$ such that $\Delta = |h(x_0)| = |f(x_0) - g(x_0)|$.

Such statement is obvious incorrect in general, because we can take $f(x) = x$ on $[0, 1)$ and $f(x) = 0$ at $x = 1$. Also take $g(x) = 0$ on $[0, 1]$. Then X in the above statement is closed interval, and f, g are bounded. However, it is easy to see $\Delta = 1$, but the range of $|f(x) - g(x)|$ is $[0, 1)$, so there does not exist $x_0 \in [0, 1]$ such that $f(x_0) - g(x_0) = f(x_0) = 1$.

Question 4.2-10. Let $P_n(x)$ be a polynomial of degree n. We are going to approximate a bounded function $f : [a, b] \mapsto \mathbb{R}$ by polynomials. Let

$$\Delta(P_n) = \sup_{x\in[a,b]} |f(x) - P_n(x)| \quad \text{and} \quad E_n(f) = \inf_{P_n} \Delta(P_n)$$

where the infimum is taken over all polynomials of degree n. A polynomial P_n is called a *polynomial of best approximation* of f if $\Delta(P_n) = E_n(f)$.

Show that

a) there exists a polynomial $P_0(x) \equiv a_0$ of best approximation of degree zero;

Denote $U = \sup f([a, b])$ and $L = \inf f([a, b])$, since f is bounded function. We can see that when $a_0 = \frac{1}{2}(U + L)$, i.e., $P_0(x) = \frac{1}{2}(U + L)$, $P_0(x)$ is a polynomial of best approximation.

$$\inf_{P_n} \sup_{x\in[a,b]} |f(x) - P_n(x)| = \inf_{a_0} \max\{|U - a_0|, |L - a_0|\}$$

Since we have

$$\max\{|U - a_0|, |L - a_0|\} = \frac{|U - a_0| + |L - a_0| + ||U - a_0| - |L - a_0||}{2}$$

Notice that the minimum value of $|U - a_0| + |L - a_0|$ is $|U - L|$, and the minimum value of $||U - a_0| - |L - a_0||$ is zero. They can be obtained by the same a_0, and such a_0 is unique because only if $a_0 = \frac{1}{2}(U + L)$ will the second term vanish. Therefore,

$$\inf_{a_0} \max\{|U - a_0|, |L - a_0|\} = \frac{|U - L|}{2}$$

and such infimum is obtained only by the polynomial of order zero,

$$P_0(x) = a_0 = \frac{1}{2}(U + L)$$

b) among the polynomials $Q_\lambda(x)$ of the form $\lambda P_n(x)$, where P_n is a fixed polynomial, there is a polynomial Q_{λ_0} such that

$$\Delta(Q_{\lambda_0}) = \min_{\lambda \in \mathbb{R}} \Delta(Q_\lambda)$$

For simplicity, we can let $k = \inf_{\lambda \in \mathbb{R}} \Delta(Q_\lambda)$, where k is a constant number. Since k is the greatest lower bound, for any $\epsilon > 0$, there exist λ, such that

$$k + \epsilon > \Delta(Q_\lambda)$$

Therefore we can find a sequence of $\{\lambda_n\}$, such that

$$\Delta(Q_{\lambda_n}) \to k$$

Since here P_n is fixed, $\Delta(Q_{\lambda_n}) = \Delta(\lambda_n)$ is a convergent sequence with respect to λ_n. Thus, $\Delta(\lambda_n)$ must be bounded, and since $f(x)$ is bounded, $P_n(x)$ is bounded, we only need λ_n is bounded. (You can easily prove by contradiction that if not, the whole $\Delta(\lambda_n)$ will be unbounded.)

By Bolzano-Weierstrass Theorem, a bounded sequence must have a convergent subsequence, so we choose λ_{n_k} which converges to λ^*. Now here comes the crux of the problem, we need to prove $\Delta(\lambda)$ is a continuous function even if $f(x)$ is not continuous. Since polynomial $P_n(x)$ must be bounded on $[a, b]$, we can suppose $|P_n(x)| \leq M$. For any $\epsilon > 0$, take $\delta = \epsilon/(2M)$, for any $|\lambda_1 - \lambda_2| < \delta$,

$$\begin{aligned}
|\Delta(\lambda_1) - \Delta(\lambda_2)| &= \sup_{x \in [a,b]} |f(x) - \lambda_1 P_n(x)| - \sup_{x \in [a,b]} |f(x) - \lambda_2 P_n(x)| \\
&= \sup_{x \in [a,b]} \Big(|f(x) - \lambda_1 P_n(x)| - |f(x) - \lambda_2 P_n(x)|\Big) \\
&\leq \sup_{x \in [a,b]} \Big(|f(x) - \lambda_1 P_n(x) - f(x) + \lambda_2 P_n(x)|\Big) \\
&= \sup_{x \in [a,b]} |\lambda_1 P_n(x) - \lambda_2 P_n(x)| \\
&\leq \sup_{x \in [a,b]} |\lambda_1 - \lambda_2||P_n(x)| \\
&\leq \sup_{x \in [a,b]} |\lambda_1 - \lambda_2| M \\
&\leq \sup_{x \in [a,b]} \frac{\epsilon}{2M} \cdot M < \epsilon
\end{aligned}$$

Hence, $\Delta(\lambda)$ is a continuous function on $[a, b]$. Therefore, we have

$$k = \lim_{k\to\infty} \Delta(\lambda_{n_k}) = \Delta\left(\lim_{k\to\infty} \lambda_{n_k}\right) = \Delta(\lambda^*)$$

Therefore, we find such $\lambda_0 = \lambda^*$ that satisfies

$$\Delta(Q_{\lambda_0}) = k = \inf_{\lambda\in\mathbb{R}} \Delta(Q_\lambda) = \min_{\lambda\in\mathbb{R}} \Delta(Q_\lambda)$$

c) if there exists a polynomial of best approximation of degree n, there also exists a polynomial of best approximation of degree $n+1$;

(Sketch only)

Since $\Delta(P_n) = \sup_{x\in[a,b]} |f(x) - P_n(x)|$, we could write

$$P_n(x) = Q_{\vec{\lambda}}(x) = \lambda_n x^n + \lambda_{n-1}x^{n-1} + \ldots + \lambda_1 x + \lambda_0$$

Then denote $k = \inf_{\vec{\lambda}\in\mathbb{R}^{n+1}} \Delta(Q_{\vec{\lambda}})$, where k is a constant number. Since k is the greatest lower bound, for any $\epsilon > 0$, there exist $\vec{\lambda} = [\lambda_n\ \lambda_{n-1}\ \cdots\ \lambda_1\ \lambda_0]$, such that

$$k + \epsilon > \Delta(Q_{\vec{\lambda}})$$

Therefore we can find a sequence of vectors $\left\{\vec{\lambda}^{(n)}\right\}$, such that

$$\Delta\left(Q_{\vec{\lambda}^{(n)}}\right) \to k$$

Since $f(x)$ is fixed, $\Delta(Q_{\vec{\lambda}})$ only depends on $\vec{\lambda}$, we can denote $\Delta(Q_{\vec{\lambda}})$ as $\Delta\left(\vec{\lambda}\right)$, which is a multi-variable real-valued function. Since $\Delta\left(Q_{\vec{\lambda}^{(n)}}\right)$ converges, it must be bounded. Now the key point is to prove $\vec{\lambda}^{(n)}$ is bounded, i.e., all of its entries are bounded.

Since $\Delta\left(Q_{\vec{\lambda}^{(n)}}\right)$ is bounded, denote $\vec{x} = [1\ x\ x^2\ \cdots\ x^n]^{\mathrm{T}}$, we have

$$\sup_{\vec{x}\in[a,b]} |f(x) - \vec{\lambda}\vec{x}| \geq \sup_{\vec{x}\in[a,b]} \left||f(x)| - |\vec{\lambda}\vec{x}|\right|$$

Since $f(x)$ is bounded, $|\vec{\lambda}\vec{x}|$ must be bounded, i.e., $|\vec{\lambda}\vec{x}| \leq M$, for all $x \in [a, b]$. Choose n distinct value in $[a, b]$, denote them as $x_1, x_2, \ldots, x_n$. Then we have

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \cdot \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \leq \begin{bmatrix} M \\ M \\ \vdots \\ M \end{bmatrix}$$

Note that the matrix one the left hand side is just the transpose of Vandermonde matrix of order n. The determinant of Vandermonde matrix W_n of order n is given by

$$\det(W_n) = \prod_{1\leq i<j\leq n} (x_i - x_j) \neq 0$$

This is because all x_i are distinct. Since the determinant is nonzero, W_n is invertible, thus we have

$$\begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \leq \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}^{-1} \cdot \begin{bmatrix} M \\ M \\ \vdots \\ M \end{bmatrix}$$

Since all x_i and M are fixed value, we can see all λ_i is bounded above. Similarly, they are all bounded below. Therefore, we proved that sequence $\overrightarrow{\lambda}^{(n)}$ is bounded.

Next you could apply the same procedure as part b). Notice that Bolzano-Weierstrass Theorem can be applied in $\mathbb{R}^n$, and you can also show that $\Delta\left(\overrightarrow{\lambda}\right)$ is continuous with respect to $\overrightarrow{\lambda}$. Finally you can take the limit and move the limit inside the function, so that you can find the $\overrightarrow{\lambda}$ that gives the best approximation.

This prove is valid for any n without the condition that $n-1$ is valid.

d) for any bounded function on a closed interval and any $n = 0, 1, 2, \ldots$ there exists a polynomial of best approximation of degree n.

Since we have proved that if n degree polynomial of best approximation exist, so does $n+1$ degree, and we also proved that when $n = 0$, there exist a polynomial of best approximation in part a), by induction, we could easily see that for any bounded function on closed interval, the polynomial of best approximation of any order shall exist.

Question 4.2-12.

a) Show that for any $n \in \mathbb{N}$ the function $T_n(x) = \cos(n \arccos x)$ defined on the closed interval $[-1, 1]$ is an algebraic polynomial of degree n. (These are the *Chebyshev polynomials.*)

This question is essential for Numerical Analysis. Note that $\theta = \arccos x$ is a one-to-one function mapping from $[-1, 1]$ to $[0, \pi]$. Therefore, we can write (for $\theta \in [0, \pi]$),

$$T_n(\theta) = \cos(n\theta)$$

We can easily see that $T_0(x) = \cos 0 = 1$. For $n \geq 1$,

$$T_{n+1}(\theta) = \cos(n+1)\theta = \cos\theta\cos n\theta - \sin\theta\sin n\theta$$

$$T_{n-1}(\theta) = \cos(n-1)\theta = \cos\theta\cos n\theta + \sin\theta\sin n\theta$$

Sum the above two equations, we have

$$T_{n+1}(\theta) + T_{n-1}(\theta) = 2\cos\theta\cos n\theta$$

Since $\theta = \arccos x$, we have $\cos\theta = x$, thus,

$$T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x)$$

From the original definition, we also know that $T_1(x) = \cos(\arccos x) = x$. Therefore we can conclude that $T_0(x)$ and $T_1(x)$ are algebraic polynomials of degree 0 and 1 respectively. We assume that $T_n(x)$ is algebraic polynomial of degree n, for all $n \leq k$, by strong induction, we need to prove $T_{n+1}(x)$ is algebraic polynomial of degree $n+1$. This is obviously true because if

$$T_{n-1}(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0$$

$$T_n(x) = b_nx^n + b_{n-1}x^{n-1} + \ldots + b_1x + b_0$$

then we have

$$\begin{aligned} T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x) \\ &= 2x(b_nx^n + b_{n-1}x^{n-1} + \ldots + b_1x + b_0) - (a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0) \end{aligned}$$

which gives us

$$T_{n+1}(x) = 2b_nx^{n+1} + 2b_{n-1}x^n + (2b_{n-2} - a_{n-1})x^{n-1} + \ldots + (2b_0 - a_1)x - a_0 \tag{1}$$

Thus, by induction we could see that every T_n can be defined by its two consecutive precursors, and since the first two are algebraic polynomials, all of their descendants are algebraic polynomials.

b) Find an explicit algebraic expression for the polynomials T_1, T_2, T_3 and T_4 and draw their graphs.

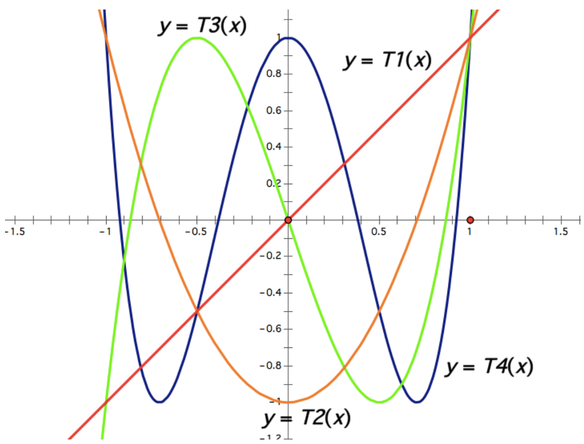
We have find out that $T_0(x) = 1$ and $T_1(x) = x$, next we can use the formula above to find $T_2(x), T_3(x), T_4(x)$.

$$T_2(x) = 2xT_1(x) - T_0(x) = 2x^2 - 1$$

$$T_3(x) = 2xT_2(x) - T_1(x) = 4x^3 - 2x - x = 4x^3 - 3x$$

$$T_4(x) = 2xT_3(x) - T_2(x) = 8x^4 - 6x^2 - 2x^2 + 1 = 8x^4 - 8x^2 + 1$$

The graphs of them are as follows (only consider interval $[-1, 1]$)



c) Find the roots of the polynomial $T_n(x)$ on the closed interval $[-1, 1]$ and the points of the interval where $|T_n(x)|$ assumes its maximum value.

To find the roots of the polynomial $T_n(x)$, let $T_n(\theta) = 0$, and we will have

$$n\theta = \frac{\pi}{2} + k\pi$$

Since $n\theta \in [0, n\pi]$, we know that $k = 0, 1, \ldots, n-1$. Therefore, on interval $[0, \pi]$, there are n values

$$\theta_k = \frac{(2k+1)\pi}{2n}$$

such that $\cos n\theta_k = 0$. Also, since $\theta = \arccos x$ is one to one from $[-1, 1]$ to $[0, \pi]$. Therefore, we can say $x = \cos\theta$, and

$$x_k = \cos\frac{(2k+1)\pi}{2n}$$

and all x_k are distinct. Finally, we have find n distinct roots of $T_n(x)$, and $T_n(x)$ is an algebraic polynomial of degree n, which means it can have at most n roots. This shows that all n roots are of multiplicity one.

To find all points of the extreme value of $T_n(x)$, let $T_n(\theta) = \pm 1$, and we will have

$$\theta_k = \frac{k\pi}{n},\ k = 0, 1, \ldots, n$$

Hence there are $n+1$ points of $T_n(x)$,

$$x_k = \cos\frac{k\pi}{n},\ k = 0, 1, \ldots, n$$

such that

$$T_n(x_k) = \cos n\left[\arccos\left(\cos\frac{k\pi}{n}\right)\right] = \cos n\frac{k\pi}{n} = \cos k\pi = (-1)^k$$

Since the maximum value of a cos function is 1, $|T_n(x_k)| = 1$ ensures that $|T_n(x)|$ reaches its maximum value.

d) Show that among all polynomials $P_n(x)$ of degree n whose leading coefficient is 1 the polynomial $T_n(x)$ is the unique polynomial closest to zero, that is, $E_n(0) = \max\limits_{|x|\le 1}|T_n(x)|$. (For the definition of $E_n(f)$ see Question 4.2-10.)

Since the original $T_n(x)$ is not of unit leading coefficient, we need to convert it into polynomial with leading coefficient 1. To achieve this, we only need to calculate the original leading coefficient of $T_n(x)$. Denote the leading coefficient of $T_n(x)$ as $\hat{a}_n$, we have known that $\hat{a}_0 = 1$, $\hat{a}_1 = 1$, $\hat{a}_2 = 2$. We assume $\hat{a}_n = 2^{n-1}$, since we have

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

Note that the leading coefficient of $T_{n+1}(x)$ is only related to the leading coefficient $T_n(x)$, and from the equation (1) in part a), we know that $\hat{a}_{n+1} = 2\hat{a}_n = 2^n$, meaning that by induction we could say that the leading coefficient of $T_n(x)$ is 2^{n-1} when $n \ge 1$.

Therefore, we can convert $T_n(x)$ to $\widetilde{T}_n(x)$ (where $\widetilde{T}_n(x)$ is the monic polynomial of $T_n(x)$) by

$$\widetilde{T}_n(x) = \frac{1}{2^{n-1}}T_n(x)$$

Also denote all polynomials with coefficient one (monic polynomial) as $\widetilde{P}_n(x)$, and we need to prove that

$$E_n(0) = \inf_{\widetilde{P}_n} \sup_{|x|\le 1} |\widetilde{P}_n(x)| = \max_{|x|\le 1} |\widetilde{T}_n(x)|$$

Since we have proved that the maximum value of $|T_n(x)|$ is 1, the maximum value of $|\widetilde{T}_n(x)|$ is $2^{-(n-1)}$. We only need to prove that for any $\widetilde{P}_n$, on $[-1, 1]$,

$$\max_{|x|\le 1} |\widetilde{T}_n(x)| = \frac{1}{2^{n-1}} \le \max_{|x|\le 1} |\widetilde{P}_n(x)|$$

and the equality can only be obtained when $\widetilde{P}_n = \widetilde{T}_n$.

We prove it by contradiction. Suppose there exist $\widetilde{P}_n \ne \widetilde{T}_n$, such that

$$\frac{1}{2^{n-1}} \ge \max_{|x|\le 1} |\widetilde{P}_n(x)|$$

Denote $Q = \widetilde{T}_n - \widetilde{P}_n$, since both $\widetilde{T}_n$ and $\widetilde{P}_n$ are monic polynomial of degree n, Q is a polynomial of at most $(n-1)$. Consider the value of Q at each extreme point x_k of $T_n(x)$.

$$Q(x_k) = \widetilde{T}_n(x_k) - \widetilde{P}_n(x_k) = \frac{(-1)^k}{2^{n-1}} - \widetilde{P}_n(x_k)$$

Since we have assumed $|\widetilde{P}_n(x)| \le 2^{-(n-1)}$, the conclusion is

$$Q(x_k) \le 0, \quad \text{when } k \text{ is odd} \quad \text{and} \quad Q(x_k) \ge 0, \quad \text{when } k \text{ is even}$$

Since Q is continuous, by IVT, for $k = 0, 1, \dots, n-1$, each of the n intervals $[x_k, x_{k+1}]$ ensures one zero of $Q(x)$.

The only trouble-maker exists when two consecutive closed intervals have the same zero (and each interval has only one zero), e.g. $x_k < 0$, $x_{k+1} = 0$, $x_{k+2} > 0$. It seems that on these two intervals only exists one zero. However, remember that this is a zero of multiplicity two, meaning that it is actually produced by $(x - x_k)(x - x_k) = 0$. Hence, it should be regarded as two repeated roots. In this way, on n intervals we have n roots in total (including repeated roots), but Q is a polynomial of degree $n-1$, meaning that $Q = 0$. Thus, $\widetilde{P}_n = \widetilde{T}_n$, which contradicts our assumption.