MAT2006: Elementary Real Analysis
Homework 4

W. Lee.

Due date: Tomorrow

Question 6.1-3. About *The Lebesgue criterion.*

a) Verify directly (without using the Lebesgue criterion) that the Riemann function of Example 2 is integrable.

Recall the Riemann function on $[0, 1]$ (any other finite intervals can be proved integrable using the same argument)

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q}\ (q \in \mathbb{N}^+, p \in \mathbb{Z}\backslash\{0\}) \\ 1 & x = 0 \\ 0 & x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

Here p and q are relatively coprime. For any $\epsilon > 0$, if $f(x) \geq \epsilon/2$, we have

$$f(x) \geq \frac{\epsilon}{2} \implies \frac{1}{q} \geq \frac{\epsilon}{2} \implies q \leq \frac{2}{\epsilon}$$

Since $f(x)$ is bounded, the number of $x = p/q$ is finite. Suppose there are k such points in total, denote them as $t_1 < \cdots < t_k$. Take partition $P = \{0 = x_0, x_1, \ldots, x_{2k-1} = 1\}$, such that $t_i \in (x_{2i-2}, x_{2i-1})$, for all $2 \leq i \leq k-1$, and $x_{2i-1} - x_{2i-2} < \epsilon/(2k)$, for all $1 \leq i \leq k$.

Notice that $\epsilon \leq 2$, otherwise q does not exist and $f(x) > 1$ (Impossible). Also, we know that for any x_i, $0 \leq i \leq 2k-1$, $M_i - m_i \leq 1$, and $\sum \Delta x_i = 1$. Also, for any interval (x_{2i-1}, x_{2i}), that does not contain t_i, $f(x) < \epsilon/2$, and hence $M_{2i} - m_{2i} < \epsilon/2$. Consider

$$\begin{aligned} U(P, f) - L(P, f) &= \sum_{i=1}^{2k-1} (M_i - m_i)\Delta x_i \\ &= \sum_{j=1}^{k} (M_{2j-1} - m_{2j-1})(x_{2j-1} - x_{2j-2}) + \sum_{l=1}^{k-1} (M_{2l} - m_{2l})(x_{2l} - x_{2l-1}) \\ &\leq \sum_{j=1}^{k} 1 \cdot (x_{2j-1} - x_{2j-2}) + \sum_{l=1}^{k-1} \frac{\epsilon}{2}(x_{2l} - x_{2l-1}) \\ &\leq \sum_{j=1}^{k} 1 \cdot \frac{\epsilon}{2k} + \frac{\epsilon}{2}\sum_{l=1}^{k-1}(x_{2l} - x_{2l-1}) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Hence, we have $f(x) \in \mathcal{R}[0, 1]$.

b) Show that a bounded function f belongs to $\mathcal{R}[a,b]$ if and only if for any two numbers $\epsilon > 0$ and $\delta > 0$ there is a partition P of $[a,b]$ such that the sum of the lengths of the intervals of the partition on which the oscillation of the function is larger than ϵ is at most δ.

We first prove that for a bounded function f,

$$\left(\forall\, \epsilon > 0, \delta > 0, \exists\, P,\ s.t.\ \sum_{i=1}^{k} |I_i| < \delta,\ I_i \in \left\{ [x_{i-1}, x_i] \in P \,\middle|\, \sup_{s,t\in[x_{i-1},x_i]} |f(s) - f(t)| > \epsilon \right\}\right) \implies (f \in \mathcal{R}[a,b])$$

Suppose $|f(x)| \le M$ for $x \in [a,b]$. For arbitrary $\epsilon' > 0$, take $\epsilon = \epsilon'/2(b-a)$ and $\delta = \epsilon'/4M$. Then there exists a partition $P^* = \{a = x_0, x_1, \ldots, x_n = b\}$, such that

$$\sum_{i=1}^{k} |I_i| < \frac{\epsilon'}{4M}, \quad I_i \in A = \left\{ [x_{i-1}, x_i] \in P^* \,\middle|\, \sup_{s,t\in[x_{i-1},x_i]} |f(s) - f(t)| > \frac{\epsilon'}{2(b-a)} \right\}$$

Consider Darboux upper sum and lower sum, denote $\Delta x_i = x_i - x_{i-1}$, $M_i = \sup\limits_{x\in[x_{i-1},x_i]} f(x)$ and $m_i = \inf\limits_{x\in[x_{i-1},x_i]} f(x)$, we can derive

$$\begin{aligned}
U(P^*, f) - L(P^*, f) &= \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \\
&= \sum_{i\in\{i|I_i\in A\}} (M_i - m_i)\Delta x_i + \sum_{i\in\{i|I_i\notin A\}} (M_i - m_i)\Delta x_i \\
&\le \sum_{i\in\{i|I_i\in A\}} 2M\Delta x_i + \sum_{i\in\{i|I_i\notin A\}} \frac{\epsilon'}{2(b-a)}\Delta x_i \\
&= 2M \sum_{i\in\{i|I_i\in A\}} \Delta x_i + \frac{\epsilon'}{b-a} \sum_{i\in\{i|I_i\notin A\}} \Delta x_i \\
&< 2M\frac{\epsilon'}{4M} + \frac{\epsilon'}{2(b-a)}(b-a) \\
&= \frac{\epsilon'}{2} + \frac{\epsilon'}{2} = \epsilon'
\end{aligned}$$

Hence, $f \in \mathcal{R}[a,b]$.

Conversely, for a bounded function f, we need to prove

$$\left(\exists\, \epsilon > 0, \delta > 0, \forall\, P,\ \sum_{i=1}^{k} |I_i| \ge \delta,\ I_i \in \left\{ [x_{i-1}, x_i] \in P \,\middle|\, \sup_{s,t\in[x_{i-1},x_i]} |f(s) - f(t)| > \epsilon \right\}\right) \implies (f \notin \mathcal{R}[a,b])$$

This is pretty easy, because for any partition P, using the same notations as the former proof except

$$A = \left\{ [x_{i-1}, x_i] \in P \,\middle|\, \sup_{s,t\in[x_{i-1},x_i]} |f(s) - f(t)| > \epsilon \right\}$$

we can derive

$$\begin{aligned}
U(P,f)-L(P,f) &= \sum_{i=1}^{n}(M_i-m_i)\Delta x_i \\
&\geq \sum_{i\in\{i|I_i\in A\}}(M_i-m_i)\Delta x_i \\
&> \sum_{i\in\{i|I_i\in A\}}\epsilon\Delta x_i \\
&= \epsilon\sum_{i\in\{i|I_i\in A\}}\Delta x_i \\
&\geq \epsilon\delta > 0
\end{aligned}$$

Hence there exists some $\epsilon' = \epsilon\delta > 0$, such that for any partition,

$$U(P,f)-L(P,f) > \epsilon'$$

We conclude that $f \notin \mathcal{R}[a,b]$.

c) Show that $f \in \mathcal{R}[a,b]$ if and only if f is bounded on $[a,b]$ and for any $\epsilon > 0$ and $\delta > 0$ the set of points in $[a,b]$ at which f has oscillation larger than ϵ can be covered by a finite set of open intervals the sum of whose lengths is less than δ.

It is noteworthy that boundness is a necessary condition for Riemann-integrability (Please accept this, don't rebut). The following proof is too tedious, so it will be divided into three parts.

Part I. Lemma

First, we prove the **lemma** as follows

$$\forall\, \epsilon > 0,\ \text{if}\ \forall\, x\in[a,b],\ \omega(f,x)<\epsilon,\quad \text{then}\ \exists\, \delta>0,\ \forall\, x\in[a,b],\ \omega\big(f,N_\delta(x)\cap[a,b]\big)\leq\epsilon$$

Since $\omega(f,x) = \lim\limits_{\delta\to 0+}\omega\big(f,N_\delta(x)\big)$, for each $x\in[a,b]$, we could find a δ_x (depends on x), such that $\omega\big(f,N_{\delta_x}(x)\cap[a,b]\big)<\epsilon$. Consider that $[a,b]$ is compact, we can choose $x_1,\dots,x_n$ in $[a,b]$ such that

$$[a,b]\subset\bigcup_{j=1}^{n}V_j,\quad V_j = \left(x_j-\frac{\delta_{x_j}}{2},x_j+\frac{\delta_{x_j}}{2}\right)$$

Now, let $\delta = \min\{\delta_{x_1}/2,\dots,\delta_{x_n}/2\}$, if $s,t\in[a,b]$, $|s-t|<\delta$, and $s\in V_j$, then we have $t\in(x_j-\delta_{x_j},x_j+\delta_{x_j})$. This is because

$$|t-x_j|\leq|t-s|+|s-x_j|<\delta+\delta_{x_j}/2<\delta_{x_j}$$

In conclusion, since $\omega\big(f,N_{\delta_{x_j}}(x_j)\big)<\epsilon$ for all $j=1,\dots,n$. This exactly means that for any $|s-t|<\delta$, $|f(s)-f(t)|<\epsilon$, which implies $\omega\big(f,N_\delta(x)\cap[a,b]\big)\leq\epsilon$.

Part II. "If" direction

Next, we prove for bounded function f,

$$\left(\forall\,\epsilon>0,\delta>0,\exists\,\{I_i\}_{i=1}^{n},\ s.t.\ \sum_{i=1}^{n}|I_i|<\delta,\ \bigcup_{i=1}^{n}I_i\supset\{x\in[a,b]\,|\,\omega(f,x)>\epsilon\}\right)\Longrightarrow(f\in\mathcal{R}[a,b])$$

Suppose $|f(x)| \le M$ for $x \in [a,b]$. For arbitrary $\epsilon' > 0$, take $\epsilon = \epsilon'/2(b-a)$ and $\delta_1 = \epsilon'/8M$. Then there exists a collection of finite number of open intervals $\{I_i\}_{i=1}^n$, $I_i = (u_i, v_i)$ (Notice that u_i, v_i may not in interval $[a,b]$) such that

$$\sum_{i=1}^n |I_i| < \delta_1 = \frac{\epsilon'}{8M}, \quad \bigcup_{i=1}^n I_i \supset \left\{ x \in [a,b] \,\middle|\, \omega(f,x) > \frac{\epsilon'}{2(b-a)} \right\}$$

Thus, apply the lemma, we have $\forall\ \epsilon' > 0$, there exists $\delta_2 > 0$, for intervals $[x_{i-1}, x_i]$ with length less than δ_2, which are contained in K, $K = [a,b] \backslash \cup_{i=1}^n I_i$, we have

$$\omega(f, [x_{i-1}, x_i]) \le \frac{\epsilon'}{2(b-a)}$$

Take any partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ whose mesh $\left(\max_i \{x_i - x_{i-1}\}\right)$ is less than or equal to δ, where $\delta = \min\{\delta_1/(2n), \delta_2\}$. We can divide all subintervals of P into two groups, one group comprises subintervals that are contained in K, and the remaining subintervals forms the other group. Denote the first group of subintervals as A, the second group as B.

Thus, we can derive (Pay attention to the formula in **red**, think about why such inequality holds),

$$\begin{aligned}
U(P,f) - L(P,f) &= \sum_{i=1}^n (M_i - m_i)\Delta x_i \quad \text{Here} \quad M_i - m_i = \omega(f, [x_{i-1}, x_i]) \\
&= \sum_{i \in \{i | [x_{i-1}, x_i] \in A\}} \omega(f, [x_{i-1}, x_i])\Delta x_i + \sum_{i \in \{i | [x_{i-1}, x_i] \in B\}} \omega(f, [x_{i-1}, x_i])\Delta x_i \\
&\le \frac{\epsilon'}{2(b-a)} \sum_{i \in \{i | [x_{i-1}, x_i] \in A\}} \Delta x_i + 2M \sum_{i \in \{i | [x_{i-1}, x_i] \in B\}} \Delta x_i \\
&\le \frac{\epsilon'}{2(b-a)}(b-a) + 2M(2n\delta + \delta_1) \\
&\le \frac{\epsilon'}{2} + 2M(\delta_1 + \delta_1) \\
&\le \frac{\epsilon'}{2} + 2M \cdot 2\frac{\epsilon'}{8M} \\
&= \frac{\epsilon'}{2} + \frac{\epsilon'}{2} = \epsilon'
\end{aligned}$$

Hence, we can conclude that $f \in \mathcal{R}[a,b]$.

Part III. "Only if" direction

Then, we prove that if $f \in \mathcal{R}[a,b]$, then f is bounded (easy to prove, refer to textbook, and don't refute...), and

$$\left(\forall\ \epsilon > 0, \delta > 0, \exists\ \{I_i\}_{i=1}^n,\ s.t. \sum_{i=1}^n |I_i| < \delta, \bigcup_{i=1}^n I_i \supset \{x \in [a,b] \,|\, \omega(f,x) > \epsilon\} \right)$$

Denote $D_N = \{x \in [a,b] \,|\omega(f,x) > 1/N\}$, for all $N \in \mathbb{N}$. Since $f \in \mathcal{R}[a,b]$, there exists partition P, such that $U(P,f) - L(P,f) < \delta'$, for all $\delta' > 0$. Hence, we take $\delta' = \delta/(2N)$, then for any given $\delta > 0$, there exists $P = \{a = x_0, x_1, \ldots, x_n = b\}$, such that $U(P,f) - L(P,f) < \delta/(2N)$.

We divide the subintervals of P into two groups in the same manner as preceding proof, i.e.,

$$A = \{[x_{i-1}, x_i] \in [a,b] \,|\, [x_{i-1}, x_i] \cap D_N = \varnothing\}, \quad B = \{[x_{i-1}, x_i] \in [a,b] \,|\, [x_{i-1}, x_i] \cap D_N \neq \varnothing\}$$

Then use $M_i, m_i, \Delta x_i$ with the same meaning as those in former proof, we have

$$U(P,f) - L(P,f) = \sum_{i \in \{i | [x_{i-1}, x_i] \in A\}} (M_i - m_i)\Delta x_i + \sum_{i \in \{i | [x_{i-1}, x_i] \in B\}} (M_i - m_i)\Delta x_i < \frac{\delta}{2N}$$

For $i \in \{i \,|\, [x_{i-1}, x_i] \in B\}$, at least one point $x \in [x_{i-1}, x_i]$ satisfies $\omega(f,x) \geq 1/N$, hence $M_i - m_i \geq 1/N$, and we have

$$\sum_{i \in \{i | [x_{i-1}, x_i] \in B\}} \Delta x_i \leq N \sum_{i \in \{i | [x_{i-1}, x_i] \in B\}} (M_i - m_i)\Delta x_i < N \frac{\delta}{2N} = \frac{\delta}{2}$$

Denote all subintervals (x_{i-1}, x_i) in group B as $\{I_i\}_{i=1}^k$, then all but finitely many points in D_N is covered by $\{I_i\}_{i=1}^k$. To cover the remaining finitely many points (i.e., end points x_i of subintervals in group B), we can use another collection of $\{I_i\}_{i=k+1}^{k+n}$, with $|I_{k+i}| = \delta/(2^{i+1})$, then the total length of open interval that can cover D_N is

$$\sum_{i=1}^{k+n} |I_i| = \sum_{i=1}^{k} |I_i| + \sum_{i=k+1}^{k+n} |I_i| < \frac{\delta}{2} + \frac{\delta}{4} + \frac{\delta}{8} + \ldots + \frac{\delta}{2^{n+1}} < \delta$$

Therefore, such $\{I_i\}_{i=1}^{k+n}$ is indeed the open cover we need to find for D_N. Since for all $\epsilon > 0$, we can find N such that $1/N \leq \epsilon$, $\{x \in [a,b] \,|\, \omega(f,x) > \epsilon\}$ can always be covered by some D_N, thus by $\{I_i\}_{i=1}^{k+n}$. In this case, we complete the proof.

d) Using the preceding problem, prove the Lebesgue criterion for Riemann integrability of a function.

We first prove that if f is Riemann integrable on $[a,b]$, then f is bounded on $[a,b]$ and f is continuous on $[a,b]$ almost everywhere. Since boundness is a necessary condition, so we only need to prove that f is continuous almost everywhere. First we should know that for any point $x \in [a,b]$, $f(x)$ is continuous at x if and only if $\omega(f,x) = 0$. Then, consider the D_N we construct in part c), if we denote $D = \bigcup_{N=1}^{\infty} D_N$, then D is just the set of all discontinuous points of f on $[a,b]$. In this case, from part c), we know that each D_N is of measure zero. Since countable union of sets with zero measure must be still of measure zero, (This is easy to prove, for each set $S_n \subset \bigcup_{k=1}^{p_n} I_{n,k}$ with zero measure, just take $\sum_{k=1}^{p_n} |I_{n,k}| < \epsilon/2^n$) D must be of zero measure, which completes the proof.

Then we prove that if f is bounded on $[a,b]$ and f is continuous almost everywhere on $[a,b]$, then f is Riemann integrable on $[a,b]$. Since f is continuous almost everywhere, D is of measure zero. Again, we write $D = \bigcup_{N=1}^{\infty} D_N$ with the same D_N as part c), then for all $\epsilon > 0$, we can find some N such that $\{x \in [a,b] \,|\, \omega(f,x) > \epsilon\} \subset \{x \in [a,b] \,|\, \omega(f,x) > 1/N\}$. Such D_N is also of measure zero, thus can be covered by a collection of open interval with arbitrary total length, therefore we prove that

$$\left(\forall\, \epsilon > 0, \delta > 0, \exists\, \{I_i\}_{i=1}^n,\ s.t. \sum_{i=1}^n |I_i| < \delta,\ \bigcup_{i=1}^n I_i \supset \{x \in [a,b] \,|\, \omega(f,x) > \epsilon\} \right)$$

By part c), since f is bounded, $f \in \mathcal{R}[a,b]$.

Question 6.1-4. Show that if $f, g \in \mathcal{R}[a,b]$ and f and g are real-valued, then $\max\{f,g\} \in \mathcal{R}[a,b]$ and $\min\{f,g\} \in \mathcal{R}[a,b]$.

First we know that

$$\max\{f,g\} = \frac{f(x)+g(x)+|f(x)-g(x)|}{2} \quad \text{and} \quad \min\{f,g\} = \frac{f(x)+g(x)-|f(x)-g(x)|}{2}$$

Then we need to prove that if $h(x)$ is integrable on $[a,b]$, so is $|h(x)|$. For any $\epsilon > 0$, there exists a partition P, such that $\sum_{i=1}^{n}(M_i - m_i)\Delta x_i < \epsilon$, where M_i and m_i denote the supremum and infimum of $h(x)$ on $[x_{i-1}, x_i]$. Also denote the supremum and infimum of $|h(x)|$ as M_i' and m_i'. If M_i and m_i are both nonnegative, then of course $M_i - m_i = M_i' - m_i'$. The same thing would happen if M_i and m_i are both nonpositive. If $M_i > 0 > m_i$, then $M_i' - m_i' < M_i - m_i$. Hence, for all $\epsilon > 0$, there exists partition P such that

$$\sum_{i=1}^{n}(M_i' - m_i')\Delta x_i \le \sum_{i=1}^{n}(M_i - m_i)\Delta x_i < \epsilon$$

In this way, $|h(x)|$ is integrable. Therefore, $\max\{f,g\}$ and $\min\{f,g\}$ are both integrable on $[a,b]$.

Question 6.1-5. Show that

a) if $f, g \in \mathcal{R}[a,b]$ and $f(x) = g(x)$ almost everywhere on $[a,b]$, then $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$;

Since $f, g \in \mathcal{R}[a,b]$, we have $f - g \in \mathcal{R}[a,b]$. Thus we only need to prove $\int_a^b (f-g)\,dx = 0$. Consider the partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$, denote $M_P = \max_i\{|x_i - x_{i-1}|\}$, and $m_P = \min_i\{|x_i - x_{i-1}|\}$.

Since $f(x) = g(x)$ almost everywhere, for any $\epsilon > 0$, there exists a covering of set $\{x \,|\, f(x) \neq g(x)\}$ by a system of intervals $\{I_k\}$ with $\sum_{k=1}^{\infty}|I_k| \le \epsilon$. Take $\epsilon = m_P$, then we have for each interval $[x_{i-1}, x_i]$,

$$|x_i - x_{i-1}| \ge m_P > \sum_{k=1}^{\infty}|I_k|$$

This means that in each interval $[x_{i-1}, x_i]$, there exists ξ_i, such that $f(\xi_i) = g(\xi_i)$. Otherwise,

$$[x_{i-1}, x_i] \subset \bigcup_{k=1}^{\infty} I_k \Longrightarrow \sum_{k=1}^{\infty}|I_k| \ge |x_i - x_{i-1}| \ge m_P$$

This contradicts the fact that $|x_i - x_{i-1}| > \sum_{k=1}^{\infty}|I_k|$.

Therefore, we can construct the Riemann sum with partition P and function value $f(\xi_i) - g(\xi_i)$ in each interval $[x_{i-1}, x_i]$,

$$\int_a^b f(x) - g(x)\,dx = \lim_{M_P \to 0}\left[\sum_{i=1}^{n}(f(\xi_i) - g(\xi_i))\Delta x_i\right] = \lim_{M_P \to 0} 0 = 0$$

Finally, we obtain

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$$

b) if $f \in \mathcal{R}[a,b]$ and $f(x) = g(x)$ almost everywhere on $[a,b]$, then g can fail to be Riemann-integrable on $[a,b]$, even if g is defined and bounded on $[a,b]$.

The counterexample is easy to find. Consider

$$f(x) = 0 \quad x \in [a,b]$$

$$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [a,b] \\ 0 & x \in (\mathbb{R} \backslash \mathbb{Q}) \cap [a,b] \end{cases}$$

We can see $f(x)$ is obviously integrable, and $f(x) = g(x)$ almost everywhere on $[a,b]$, but $g(x)$ is not integrable. To prove $f(x) = g(x)$ on $[a,b]$ almost everywhere, since take open cover such that $|I_i| = \epsilon/2^i$, and each I_i covers one rational point in $[a,b]$.

Question 6.2-1. Show that if $f \in \mathcal{R}[a,b]$ and $f(x) \geq 0$ on $[a,b]$, then the following statements are true.

a) If the function $f(x)$ assumes a positive value $f(x_0) > 0$ at a point of continuity $x_0 \in [a,b]$, then the strict inequality

$$\int_a^b f(x)\,dx > 0$$

holds.

If $f(x_0) > 0$, then there exists a neighborhood within which all values of f are larger than $\frac{1}{2}f(x_0)$. Take the intersection of $[a,b]$ and that neighborhood, it ensures us an interval I with length $\delta > 0$. Since all $f(x)$ are positive, we have

$$\int_a^b f(x)\,dx \geq \int_I f(x)\,dx > \int_I \frac{1}{2}f(x_0)\,dx = \frac{1}{2}f(x_0)\delta > 0$$

The details of the above argument need to be completed. (Use the definition of continuity, you know..., those standard procedures.)

b) The condition $\int_a^b f(x)\,dx = 0$ implies that $f(x) = 0$ at almost all points of $[a,b]$.

Suppose $f(x_0) > 0$ for $x_0 \in [a,b]$, and the set of x_0 is not with measure zero. According to Lebesgue criterion (see **Question 6.1-3**), since $f \in \mathbb{R}[a,b]$, the set of all discontinuous points must be of measure zero. This implies that there exists continuous point $\xi \in [a,b]$ such that $f(\xi) > 0$. Otherwise, if all points ξ satisfying $f(\xi) > 0$ is discontinuous, the set of discontinuous points will have positive measure, which is a contradiction.

By part a), we know that if $f(x)$ assumes a positive value $f(\xi) > 0$ at a point of continuity $\xi \in [a,b]$, then

$$\int_a^b f(x)\,dx > 0$$

which contradicts the assumption

$$\int_a^b f(x)\,dx = 0$$

Hence, the set of all points x_0 satisfying $f(x_0) > 0$ must have zero measure, i.e., $f(x) = 0$ almost everywhere.

Question 6.2-2. Show that if $f \in \mathcal{R}[a,b]$, $m = \inf_{(a,b)} f(x)$, and $M = \sup_{(a,b)} f(x)$, then

a) $\int_a^b f(x)\,dx = \mu(b-a)$, where $\mu \in [m, M]$;

Since we have

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx \tag{1}$$

We have

$$\frac{1}{b-a}\int_a^b f(x)\,dx \in [m, M]$$

Thus, there exists $\mu \in [m, M]$, such that

$$\mu = \frac{1}{b-a}\int_a^b f(x)\,dx \iff \int_a^b f(x)\,dx = \mu(b-a)$$

Warning, the above argument seems smart, but it is not rigorous. The correct method goes as below. Construct $g(x)$ such that $g(a) = g(b) = M$, and $g(x) = f(x)$ for all points in (a,b). Since $g(x)$ only have at most two more discontinuous points than $f(x)$, that $f(x)$ is integrable will imply $g(x)$ is also integrable. By **Question 6.1-5(a)**, the two integral are equal. Hence we can evaluate the integral of $g(x)$ instead of $f(x)$, because $g(x)$ is bounded by $[m, M]$ on $[a,b]$, while $f(x)$ may not (Thus equation (1) here is not rigorous). The following procedure is exactly the same.

b) if f is continuous on $[a,b]$, there exists a point $\xi \in (a,b)$ such that

$$\int_a^b f(x)\,dx = f(\xi)(b-a)$$

If f is continuous on $[a,b]$, f will be bounded between $[m, M]$. There exist $c, d \in [a,b]$ such that $f(c) = m$, $f(d) = M$. For each value of (m, M), by Intermediate Value Theorem, $f(\xi)$ with $\xi \in (a,b)$ can assume it. However, if the integral is equal to $m(b-a)$ or $M(b-a)$, then $f(x)$ must be constant function m or M (Check this is really true). In this case, m, M can still be obtained by $f(\xi)$ with $\xi \in (a,b)$. Hence, let $\mu = f(\xi)$, we have

$$\int_a^b f(x)\,dx = f(\xi)(b-a)$$

Question 6.2-3. Show that if $f \in \mathcal{C}[a,b]$, $f(x) \ge 0$ on $[a,b]$, and $M = \max_{[a,b]} f(x)$, then

$$\lim_{n\to\infty}\left(\int_a^b f^n(x)\,dx\right)^{1/n} = M$$

We first prove that

$$\overline{\lim_{n\to\infty}} \left(\int_a^b f^n(x)\, dx \right)^{1/n} \leq M$$

This is obvious, since

$$\left(\int_a^b f^n(x)\, dx \right)^{1/n} \leq \left(\int_a^b M^n\, dx \right)^{1/n} = M(b-a)^{\frac{1}{n}}$$

For any $b - a > 0$, we have

$$\overline{\lim_{n\to\infty}}\, M(b-a)^{\frac{1}{n}} = M$$

Then we prove the other direction

$$\underline{\lim_{n\to\infty}} \left(\int_a^b f^n(x)\, dx \right)^{1/n} \geq M$$

We claim that $\forall\, \epsilon > 0$,

$$\underline{\lim_{n\to\infty}} \left(\int_a^b f^n(x)\, dx \right)^{1/n} \geq M - \epsilon$$

Since f is continuous on $[a, b]$, and $M = \max_{[a,b]} f(x)$, there exists $c \in [a, b]$, such that $f(c) = M$. Also, $\forall\, \epsilon > 0$, there exists $\delta > 0$, such that $|f(x) - f(c)| < \epsilon$ because of the continuity of $f(x)$ at c. Thus, $f(x) \geq M - \epsilon$ for all $x \in (c - \delta, c + \delta) \cap [a, b] = I$. Denote the length of I as $l > 0$, then we have

$$\left(\int_a^b f^n(x)\, dx \right)^{1/n} \geq \left(\int_I f^n(x)\, dx \right)^{1/n} \geq \left(\int_I (M-\epsilon)\, dx \right)^{1/n} \geq l(M-\epsilon)^n$$

Therefore, as $n \to \infty$,

$$\left(\int_a^b f^n(x)\, dx \right)^{1/n} \geq (M-\epsilon)\delta^{\frac{1}{n}} \to M - \epsilon$$

Thus,

$$\underline{\lim_{n\to\infty}} \left(\int_a^b f^n(x)\, dx \right)^{1/n} \geq M$$

Question 6.3-1. Using the integral, find

a) $\displaystyle\lim_{n\to\infty} \left[\frac{n}{(n+1)^2} + \cdots + \frac{n}{(2n)^2} \right];$

The original equation can be regarded as

$$\begin{aligned}
\lim_{n\to\infty} \left[\frac{n}{(n+1)^2} + \cdots + \frac{n}{(2n)^2} \right] &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{(1+i/n)^2} \frac{1}{n} \\
&= \int_0^1 \frac{1}{(1+x)^2}\, dx \\
&= -\frac{1}{1+x}\bigg|_0^1 = \frac{1}{2}
\end{aligned}$$

Here we take step size as $\frac{1}{n}$ on $[0, 1]$, and approximate the area by the minimum value of each small interval.

b) $\lim_{n\to\infty} \frac{1^\alpha + 2^\alpha + \cdots + n^\alpha}{n^{\alpha+1}}$, if $\alpha \geq 0$.

The original equation can be regarded as

$$\begin{aligned}\lim_{n\to\infty} \frac{1^\alpha + 2^\alpha + \cdots + n^\alpha}{n^{\alpha+1}} &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^\alpha \frac{1}{n} \\ &= \int_0^1 x^\alpha \, dx \quad (\alpha \geq 0) \\ &= \frac{1}{\alpha+1}\end{aligned}$$

Here we take step size as $\frac{1}{n}$ on $[0,1]$, and approximate the area by the minimum value of each small interval.

Question 6.3-2.

a) Show that any continuous function on an open interval has a primitive on that interval.

Notice that continuous function on open interval may be not integrable!

Hence, for function $f(x)$ on open interval (a,b), we cannot directly use the integral $\int_a^x f(t)\,dt$ to be its primitive. However, $f(x)$ is indeed integrable on any closed interval contained in (a,b). Thus, we have

$$F(x) = \int_{\frac{a+b}{2}}^{x} f(t)\,dt \quad \text{is defined on } (a,b)$$

Now we prove that $F(x)$ is a primitive of $f(x)$ on open interval (a,b).

Since $f(x)$ is continuous at $x \in (a,b)$, it is uniform continuous on any closed interval $[x, x+h]$ for $x, x+h \in (a,b)$. For any $\epsilon > 0$, there exists $\delta > 0$, such that for all $|x-y| < 2\delta$, $|f(x) - f(y)| < \epsilon$. Consider small enough $h > 0$ such that $x + h \in (a,b)$,

$$\begin{aligned}\frac{F(x+h) - F(x)}{h} - f(x) &= \frac{1}{h}\left(\int_{\frac{a+b}{2}}^{x+h} f(t)\,dt - \int_{\frac{a+b}{2}}^{x} f(t)\,dt - \int_x^{x+h} f(x)\,dt\right) \\ &= \frac{1}{h}\int_x^{x+h} f(t) - f(x)\,dt \\ &\leq \frac{1}{h}\int_x^{x+h} |f(t) - f(x)|\,dt \\ &< \frac{1}{h}\int_x^{x+h} \epsilon\,dt \\ &= \epsilon\end{aligned}$$

Hence, as $h \to 0$, we have

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

for all $x \in (a,b)$. Therefore, $F(x)$ is a primitive of $f(x)$ on (a,b). It shows that a function $f(x)$ with a primitive on (a,b) may not be integrable on (a,b).

b) Show that if $f \in \mathcal{C}^{(1)}[a,b]$, then f can be represented as the difference of two nondecreasing functions on $[a,b]$.

Since $f \in \mathcal{C}^{(1)}[a,b]$, $f'(x)$ is continuous on $[a,b]$. Denote $f'(x)^+$ as the positive part of the function f', and $f'(x)^-$ as the negative part of the function f'. We have

$$f'(x)^+ = \max\{f(x), 0\} \geq 0 \qquad f'(x)^- = -\min\{f(x), 0\} \geq 0$$

It is clear that $f'(x) = f'(x)^+ - f'(x)^-$. From **Question 6.1-4** we can see that $f'(x)^+$ and $f'(x)^-$ are both integrable on $[a,b]$, because $f'(x)$ is integrable on $[a,b]$. Hence, we have

$$f(x) = \int_a^x f'(t)\, dt = \int_a^x f'(t)^+ - f'(t)^-\, dt = \int_a^x f'(t)^+\, dt - \int_a^x f'(t)^-\, dt$$

Since both $f'(x)^+$ and $f'(x)^-$ are nonnegative, the integral above are both nondecreasing, hence $f(x)$ can be represented as the difference of two nondecreasing functions $\int_a^x f'(t)^+\, dt$ and $\int_a^x f'(t)^-\, dt$.

Question 6.3-4. Show that if $f \in \mathcal{C}(\mathbb{R})$, then for any fixed closed interval $[a,b]$, given $\epsilon > 0$ one can choose $\delta > 0$ so that the inequality $|F_\delta(x) — f(x)| < \epsilon$ holds on $[a,b]$, where F_δ is the average of the function defined as

$$\frac{1}{2\delta}\int_{x-\delta}^{x+\delta} f(t)\, dt$$

Consider

$$\begin{aligned}
|F_\delta(x) — f(x)| &= \left| \frac{1}{2\delta}\int_{x-\delta}^{x+\delta} f(t)\, dt - f(x) \right| \\
&= \left| \frac{1}{2\delta}\int_{x-\delta}^{x+\delta} f(t)\, dt - \frac{1}{2\delta}\int_{x-\delta}^{x+\delta} f(x)\, dt \right| \\
&= \left| \frac{1}{2\delta}\int_{x-\delta}^{x+\delta} f(t) - f(x)\, dt \right| \\
&\leq \frac{1}{2\delta}\int_{x-\delta}^{x+\delta} |f(t) - f(x)|\, dt
\end{aligned}$$

Since f is continuous on any closed interval on $\mathbb{R}$, $\forall\, \epsilon > 0$, there exists $\delta > 0$, such that $|f(x) - f(y)| < \epsilon$, for all $|x - y| < 2\delta$. Thus, we have

$$|F_\delta(x) — f(x)| < \frac{1}{2\delta}\int_{x-\delta}^{x+\delta} \epsilon\, dt = \epsilon$$

Since the continuous on $[a,b]$ is uniform, the δ is independent on x and only dependent on ϵ.

Question 6.3-5. Show that

$$\int_1^{x^2} \frac{e^t}{t}\, dt \sim \frac{1}{x^2} e^{x^2} \qquad \text{as } x \to \infty$$

Consider the limit

$$\lim_{n\to\infty} \frac{\displaystyle\int_1^{x^2} \frac{e^t}{t}\, dt}{\dfrac{1}{x^2}e^{x^2}}$$

We can easily verify that both the numerator and denominator tends to infinity as $x \to \infty$. (For the numerator, compare e^t/t with $1/x^p$; for the denominator, use L'Hôpital's rule.) Hence, apply L'Hôpital's rule, we have

$$\begin{aligned}
\lim_{x\to\infty} \frac{\displaystyle\int_1^{x^2} \frac{e^t}{t}\, dt}{\dfrac{1}{x^2}e^{x^2}} &= \lim_{x\to\infty} \frac{\dfrac{e^{x^2}}{x^2}\cdot 2x}{\dfrac{e^{x^2}\cdot 2x^3 - e^{x^2}\cdot 2x}{x^4}} \\
&= \lim_{x\to\infty} \frac{\dfrac{2e^{x^2}}{x}}{\dfrac{2e^{x^2}x^2 - 2e^{x^2}}{x^3}} \\
&= \lim_{x\to\infty} \frac{\dfrac{1}{x}}{\dfrac{x^2-1}{x^3}} \\
&= \lim_{x\to\infty} \frac{x^2}{x^2-1} = 1
\end{aligned}$$

Question 6.3-7. Show that if $f : \mathbb{R} \mapsto \mathbb{R}$ is a periodic function that is integrable on every closed interval $[a, b] \subset \mathbb{R}$, then the function

$$F(x) = \int_a^x f(t)\, dt$$

can be represented as the sum of a linear function and periodic function.

Let $C = \int_0^T f(t)\, dt$, where T is any period of f (not necessarily minimum period) and $g(t) = f(t) - C/T$, then we have

$$F(x) = \int_a^x f(t)\, dt = h(x) + \phi(x) = \int_a^x g(t)\, dt + \int_a^x \frac{C}{T}\, dt$$

We can easily find out $\phi(x)$ is a linear function because

$$\phi(x) = \int_a^x \frac{C}{T}\, dt = \frac{C}{T}(x-a)$$

Next, consider $h(x+T)$, we have

$$h(x+T) = \int_a^{x+T} g(t)\, dt = \int_a^x g(t)\, dt + \int_x^{x+T} g(t)\, dt = h(x) + \int_x^{x+T} g(t)\, dt$$

We need to show the second term above is zero. We can observe that

$$\int_x^{x+T} g(t)\, dt = \int_x^0 g(t)\, dt + \int_0^T g(t)\, dt + \int_T^{x+T} g(t)\, dt$$

The second term is automatically zero, since

$$\int_0^T g(t)\,dt = \int_0^T f(t)\,dt - \int_0^T \frac{C}{T}\,dt = 0$$

Consider the third term, apply change of variable with $y = t - T$, we have

$$\int_T^{x+T} g(t)\,dt = \int_0^x g(y+T)\,dy$$

However, since $f(t)$ is periodic, so is $g(t)$, and

$$\int_0^x g(y+T)\,dy = \int_0^x g(y)\,dy = -\int_x^0 g(y)\,dy = -\int_x^0 g(t)\,dt$$

Therefore,

$$\int_x^{x+T} g(t)\,dt = 0$$

meaning that $h(x+T) = h(x)$, i.e., $h(x)$ is periodic, and $F(x)$ can be represented as the sum of a linear function $\phi(x)$ and periodic function $h(x)$.

Question 6.5-1. Show that the following functions have the stated properties.

a) $\mathrm{Si}\,(x) = \int_0^x \frac{\sin t}{t}\,dt$ (the *sine integral*) is defined on all of $\mathbb{R}$, is an odd function, and has a limit as $x \to \infty$.

Since $\sin t/t$ converges to 1 as $t \to 0$, and $x = 0$ is the only discontinuous point of it, this function must be integrable on the whole real line $\mathbb{R}$. Hence $\mathrm{Si}\,(x)$ is defined for any $x \in \mathbb{R}$.

To check it is odd function,

$$\mathrm{Si}\,(-x) = \int_0^{-x} \frac{\sin t}{t}\,dt = -\int_{-x}^0 \frac{\sin t}{t}\,dt = -\int_0^x \frac{\sin t}{t}\,dt = -\mathrm{Si}\,(x)$$

The third equality is because $\sin t/t$ is an even function, and the integral on symmetric intervals about y-axis must be the same value.

To check the limit of it as $x \to \infty$, (C is a constant)

$$\int_0^x \frac{\sin t}{t}\,dt = \int_0^1 \frac{\sin t}{t}\,dt + \int_1^x \frac{\sin t}{t}\,dt = C + \int_1^x \frac{-1}{t}\,d(\cos x) = C - \frac{\cos t}{t}\bigg|_1^x - \int_1^x \frac{\cos t}{t^2}\,dt$$

As $x \to \infty$, the second term tends to a constant value, the integral term will converges since

$$\frac{|\cos t|}{t^2} \le \frac{1}{t^2} \quad \text{and} \quad \int_1^\infty \frac{1}{t^2}\,dt \quad \text{converges}$$

Hence, $\mathrm{Si}\,(x)$ has a limit as $x \to \infty$.

b) $\mathrm{si}\,(x) = -\int_x^\infty \frac{\sin t}{t}\,dt$ is defined on all of $\mathbb{R}$ and differs from $\mathrm{Si}\,(x)$ only by a constant;

Suppose the limit in part a) is constant k (and actually $k = \frac{\pi}{2}$), then

$$\mathrm{si}\,(x) = -\int_x^\infty \frac{\sin t}{t}\,dt = -\left(\int_0^\infty \frac{\sin t}{t}\,dt - \int_0^x \frac{\sin t}{t}\,dt\right) = \mathrm{Si}\,(x) - k$$

Hence,

$$\mathrm{Si}\,(x) - \mathrm{si}\,(x) = k$$

This also indicates that $\mathrm{si}\,(x)$ is defined on the whole real line, because $\mathrm{Si}\,(x)$ is defined on whole real line.

c) $\mathrm{Ci}\,(x) = -\int_x^\infty \frac{\cos t}{t}\,dt$ (the *cosine integral*) can be computed for sufficiently large values of x by the approximate formula $\mathrm{Ci}\,(x) \approx \frac{\sin x}{x}$; estimate the region of values where the absolute error of this approximation is less than 10^{-4}.

For large x, $\mathrm{Ci}\,(x)$ definitely exists. Apply integration by part, we have

$$\begin{aligned}\mathrm{Ci}\,(x) = -\int_x^\infty \frac{\cos t}{t}\,dt &= -\int_x^\infty \frac{1}{t}\,d(\sin t)\\ &= -\left[\frac{\sin t}{t}\bigg|_x^\infty + \int_x^\infty \frac{\sin t}{t^2}\,dt\right]\\ &= -\left[-\frac{\sin x}{x} + \int_x^\infty \frac{\sin t}{t^2}\,dt\right]\\ &= \frac{\sin x}{x} - \int_x^\infty \frac{\sin t}{t^2}\,dt\end{aligned}$$

It's easy to see that $\int_x^\infty \frac{\sin t}{t^2}\,dt$ will converge to zero as $x \to \infty$. Thus we need to estimate the absolute value of this integral. If we consider

$$\left|\int_x^\infty \frac{\sin t}{t^2}\,dt\right| \le \int_x^\infty \frac{1}{t^2}\,dt = \frac{1}{x}$$

In this way we need

$$\frac{1}{x} \le 10^{-4} \Longrightarrow x \ge 10^4$$

which is not a reasonable estimate. If we apply integrable by part to estimate the error, we have

$$\begin{aligned}\left|\int_x^\infty \frac{\sin t}{t^2}\,dt\right| &= \left|\frac{\cos x}{x^2} - \int_x^\infty \frac{2\cos t}{t^3}\,dt\right|\\ &\le \frac{1}{x^2} - \frac{1}{t^2}\bigg|_x^\infty\\ &= \frac{2}{x^2}\end{aligned}$$

In this case, we have

$$\frac{2}{x^2} \le 10^{-4} \Longrightarrow x \ge 100\sqrt{2} \approx 142$$

which is much better than the preceding one.

Actually, by using MATLAB, one can check that x should be at least 97.7, because $|\mathrm{Ci}\,(97.7) - \sin 97.7/97.7| > 10^{-4}$. This means our estimation is pretty reasonable, but if you want it to be more accurate, you can continue to apply integration by part.

Question 6.5-3. Show that

a) the elliptic integral of first kind

$$F(k,\varphi) = \int_0^{\sin\varphi} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

is defined for $0 \le k < 1$, $0 \le \varphi \le \frac{\pi}{2}$ and can be brought into the form

$$F(k,\varphi) = \int_0^{\varphi} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}$$

The first part has been discussed in lecture. Since the fixed number k is strictly less than 1, we have

$$\begin{aligned}\int_0^{\sin\varphi} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} &\le \frac{1}{\sqrt{1-k^2}} \int_0^{\sin\varphi} \frac{dt}{\sqrt{(1-t^2)}} \\ &= \frac{1}{\sqrt{1-k^2}} \int_0^{\sin\varphi} \frac{dt}{\sqrt{1+t}\sqrt{1-t}} \\ &\le \frac{1}{\sqrt{1-k^2}} \int_0^{\sin\varphi} \frac{dt}{\sqrt{1-t}}\end{aligned}$$

Thus, by comparison test

$$\frac{1}{\sqrt{1-k^2}} \int_0^{\sin\varphi} \frac{dt}{\sqrt{1-t}} \sim \int_0^1 \frac{dt}{\sqrt{1-t}} \sim \int_0^1 \frac{1}{t^{1/2}}\,dt$$

It's easy to see the right hand side improper integral converges, thus the original integral also converges. Therefore, for any fixed $0 \le k < 1$, $F(k,\varphi)$ is defined for all $0 \le \varphi \le \pi/2$.

Take $t = \sin\psi$, where $\psi \in [0, \pi/2]$, apply change of variable, we have

$$\int_0^{\sin\varphi} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\varphi} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}$$

b) the complete elliptic integral of first kind

$$K(k) = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}$$

increases without bound as $k \to 1-$.

First, we need to prove $K(k)$ is increasing function. For any $0 \le k_1 < k_2 < 1$, consider the difference

$$\begin{aligned}K(k_2) - K(k_1) &= \int_0^{\pi/2} \left[\frac{1}{\sqrt{1-k_2^2\sin^2\psi}} - \frac{1}{\sqrt{1-k_1^2\sin^2\psi}}\right] d\psi \\ &= \int_0^{\pi/2} \frac{\sqrt{1-k_1^2\sin^2\psi} - \sqrt{1-k_2^2\sin^2\psi}}{\sqrt{1-k_2^2\sin^2\psi}\sqrt{1-k_1^2\sin^2\psi}}\, d\psi \\ &= \int_0^{\pi/2} \frac{(k_2^2-k_1^2)\sin^2\psi}{\sqrt{1-k_2^2\sin^2\psi}\sqrt{1-k_1^2\sin^2\psi}\left(\sqrt{1-k_1^2\sin^2\psi}+\sqrt{1-k_2^2\sin^2\psi}\right)}\, d\psi \\ &> 0\end{aligned}$$

Hence, $K(k)$ is strictly increasing function on $[0,1)$.

If you are clever enough, maybe you will have an intuition that as $k \to 1-$,

$$K(k) = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}} \sim -\ln\left(\sqrt{1-k^2}\right)$$

Then it is trivial that $K(k)$ will increase without a bound. What we need to do next is to prove the conjecture above is correct.

First we use change of variable, let $\theta = \pi/2 - \psi$, then

$$\int_0^{\pi/2} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\cos^2\theta}} = \int_0^{\pi/4} \frac{d\theta}{\sqrt{1-k^2\cos^2\theta}} + \int_{\pi/4}^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\cos^2\theta}}$$

Then second part of the above integral is proper integral (so it must be bounded) since the denominator of it is larger than $1/\sqrt{2}$. Hence, we only focus on the first part. Let $\epsilon = \sqrt{1-k^2}$, as $k \to 1-$, we have $\epsilon \to 0+$, and we can derive

$$\begin{aligned}
\int_0^{\pi/4} \frac{d\theta}{\sqrt{1-k^2\cos^2\theta}} &= \int_0^{\pi/4} \frac{d\theta}{\sqrt{1-(1-\epsilon^2)\cos^2\theta}} \\
&= \int_0^{\pi/4} \frac{\cos^2\theta+\sin^2\theta}{\sqrt{\cos^2\theta+\sin^2\theta}\sqrt{\cos^2\theta+\sin^2\theta-(1-\epsilon^2)\cos^2\theta}}\, d\theta \\
&= \int_0^{\pi/4} \frac{\cos^2\theta+\sin^2\theta}{\sqrt{\cos^2\theta+\sin^2\theta}\sqrt{\sin^2\theta+\epsilon^2\cos^2\theta}}\, d\theta \\
&= \int_0^{\pi/4} \frac{1+\tan^2\theta}{\sqrt{1+\tan^2\theta}\sqrt{\tan^2\theta+\epsilon^2}}\, d\theta
\end{aligned}$$

Let $s = \tan\theta \in [0,1]$, since $ds = \sec^2\theta d\theta = (1+\tan^2\theta)d\theta$, we have

$$\begin{aligned}
\int_0^{\pi/4} \frac{1+\tan^2\theta}{\sqrt{1+\tan^2\theta}\sqrt{\tan^2\theta+\epsilon^2}}\, d\theta &= \int_0^1 \frac{1}{\sqrt{1+s^2}\sqrt{s^2+\epsilon^2}}\, ds \\
&= \int_0^1 \frac{1}{\sqrt{s^2+\epsilon^2}}\, ds - \int_0^1 \left(1-\frac{1}{\sqrt{1+s^2}}\right)\frac{1}{\sqrt{s^2+\epsilon^2}}\, ds
\end{aligned}$$

Consider the second part above

$$\left(1-\frac{1}{\sqrt{1+s^2}}\right)\frac{1}{\sqrt{s^2+\epsilon^2}} \le \left(1-\frac{1}{\sqrt{1+s^2}}\right)\frac{1}{\sqrt{s^2}} = \left(\frac{\sqrt{1+s^2}-1}{\sqrt{1+s^2}}\right)\frac{1}{s} = \frac{s}{\sqrt{1+s^2}(1+\sqrt{1+s^2})}$$

Since the denominator is larger than or equal to 2, the second part above is also proper integral, hence it is bounded. We only consider the first part, take $u = s/\epsilon \in [0, 1/\epsilon]$, and we have

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{s^2+\epsilon^2}}\, ds &= \int_0^{1/\epsilon} \frac{1}{\sqrt{1+u^2}}\, du \\
&= \ln\left(u+\sqrt{1+u^2}\right)\Big|_0^{1/\epsilon} \\
&= \ln\left(\frac{1}{\epsilon}+\sqrt{1+\frac{1}{\epsilon^2}}\right) \\
&= \ln\frac{1}{\epsilon} + \ln\left(1+\sqrt{1+\epsilon^2}\right) \\
&= -\ln\left(\sqrt{1-k^2}\right) + \ln\left(1+\sqrt{2-k^2}\right)
\end{aligned}$$

Therefore, as $k \to 1-$, we have

$$\int_0^1 \frac{1}{\sqrt{s^2+\epsilon^2}}\, ds \sim -\ln\left(\sqrt{1-k^2}\right)$$

which further implies that

$$K(k) = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}} \sim -\ln\left(\sqrt{1-k^2}\right)$$

However, $-\ln\left(\sqrt{1-k^2}\right)$ tends to positive infinity, so $K(k)$ is unbounded.

Question 6.5-5. Show that

a) the function $\Phi(x) = \frac{1}{\sqrt{\pi}}\int_{-x}^{x} e^{-t^2}\, dt$, called the *error* function and often denoted $\operatorname{erf}(x)$, is defined, **odd**, and infinitely differetiable on $\mathbb{R}$ and has a limit as $x \to \infty$;

The function e^{-t^2} is continuous for all $t \in \mathbb{R}$. Thus, the integral of it on any closed interval $[-x, x]$ is well-defined. Hence the function $\Phi(x)$ is defined on $\mathbb{R}$.

To check $\Phi(x)$ is odd function,

$$\Phi(-x) = \frac{1}{\sqrt{\pi}}\int_{x}^{-x} e^{-t^2}\, dt = -\frac{1}{\sqrt{\pi}}\int_{-x}^{x} e^{-t^2}\, dt = -\Phi(x)$$

To explore the differetiability of $\Phi(x)$, we first observe

$$\Phi(x) = \frac{1}{\sqrt{\pi}}\int_{-x}^{x} e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}}\int_{0}^{x} e^{-t^2}\, dt$$

Then we compute the first order derivative

$$\Phi'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$$

It is easy to prove $\Phi'(x)$ is infinitely differetiable by induction. Hence $\Phi(x)$ is also infinitely differetiable.

To check whether $\Phi(x)$ converges as $x \to \infty$, we only need to compare e^{-t^2} with t^{-2}. Since

$$\lim_{t\to\infty} \frac{e^{-t^2}}{t^{-2}} = 0$$

we know that e^{-t^2} decreases much faster than t^{-2}. We know that the integral of t^{-p} converges as $x \to \infty$ when $p > 1$, hence $\int_1^\infty t^{-2}$ converges. Therefore, $\Phi(x)$ converges as $x \to \infty$.

b) if the limit in a) is equal to 1 (and it is), then

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_{-x}^{x} e^{-t^2}\, dt = 1 - \frac{2}{\sqrt{\pi}} e^{-x^2}\left(\frac{1}{2x} - \frac{1}{2^2x^3} + \frac{1\cdot 3}{2^3x^5} - \frac{1\cdot 3\cdot 5}{2^4x^7} + o\left(\frac{1}{x^7}\right)\right)$$

as $x \to \infty$.

Consider the complementary error function $\operatorname{erfc}(x)$ defined as $\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\, dt$, we have

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\, dt - \operatorname{erfc}(x) = 1 - \operatorname{erfc}(x)$$

Thus, we have (apply integration by part)

$$\begin{aligned}
\operatorname{erf}(x) &= 1 - \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt \\
&= 1 - \frac{2}{\sqrt{\pi}} \int_x^\infty -\frac{1}{2t}\, d(e^{-t^2}) \\
&= 1 - \frac{2}{\sqrt{\pi}} \left[-\frac{1}{2t} e^{-t^2} \Big|_x^\infty - \int_x^\infty \frac{1}{2t^2} e^{-t^2}\, dt \right] \\
&= 1 - \frac{2}{\sqrt{\pi}} \left[\frac{1}{2x} e^{-x^2} - \int_x^\infty \frac{1}{2t^2} e^{-t^2}\, dt \right] \\
&= 1 - \frac{2}{\sqrt{\pi}} \left[\frac{1}{2x} e^{-x^2} - \int_x^\infty -\frac{1}{2^2 t^3}\, d(e^{-t^2}) \right] \\
&= 1 - \frac{2}{\sqrt{\pi}} \left[\frac{1}{2x} e^{-x^2} + \frac{1}{2^2 t^3} e^{-t^2} \Big|_x^\infty + \int_x^\infty \frac{3}{2^2 t^4} e^{-t^2}\, dt \right] \\
&= 1 - \frac{2}{\sqrt{\pi}} \left[\frac{1}{2x} e^{-x^2} - \frac{1}{2^2 x^3} e^{-x^2} + \int_x^\infty \frac{3}{2^2 t^4} e^{-t^2}\, dt \right] \\
&= \cdots
\end{aligned}$$

Continue the procedure for two more times, we can finally get

$$\operatorname{erf}(x) = 1 - \frac{2}{\sqrt{\pi}} e^{-x^2} \left(\frac{1}{2x} - \frac{1}{2^2 x^3} + \frac{1 \cdot 3}{2^3 x^5} - \frac{1 \cdot 3 \cdot 5}{2^4 x^7} + e^{x^2} \int_x^\infty \frac{7 \cdot 5 \cdot 3 \cdot 1}{2^4 t^8} e^{-t^2}\, dt \right)$$

We finally show that the remainder here is $o(x^{-7})$.

$$\begin{aligned}
\lim_{x\to\infty} \frac{e^{x^2} \int_x^\infty \frac{7 \cdot 5 \cdot 3 \cdot 1}{2^4 t^8} e^{-t^2}\, dt}{\frac{1}{x^7}} &= \lim_{x\to\infty} \frac{\int_x^\infty \frac{7 \cdot 5 \cdot 3 \cdot 1}{2^4 t^8} e^{-t^2}\, dt}{\frac{1}{x^7} e^{-x^2}} \\
&= \lim_{x\to\infty} \frac{-\frac{7 \cdot 5 \cdot 3 \cdot 1}{2^4 x^8} e^{-x^2}}{\frac{-7}{x^8} e^{-x^2} + \frac{-2}{x^6} e^{-x^2}} \quad \text{(L'Hôpital's rule)} \\
&= \lim_{x\to\infty} \frac{-\frac{7 \cdot 5 \cdot 3 \cdot 1}{2^4}}{-7 - 2x^2} = 0
\end{aligned}$$

Hence, the remainder here is indeed $o(x^{-7})$, which implies as $x \to \infty$,

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{-x}^x e^{-t^2}\, dt = 1 - \frac{2}{\sqrt{\pi}} e^{-x^2} \left(\frac{1}{2x} - \frac{1}{2^2 x^3} + \frac{1 \cdot 3}{2^3 x^5} - \frac{1 \cdot 3 \cdot 5}{2^4 x^7} + o\left(\frac{1}{x^7}\right) \right)$$