MAT2006: Elementary Real Analysis
Homework 5

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Question 7.1-1. The *distance* $d(E_1, E_2)$ *between the sets* $E_1, E_2 \subset \mathbb{R}^m$ is the quantity

$$d(E_1, E_2) := \inf_{x_1 \in E_1, x_2 \in E_2} d(x_1, x_2)$$

Give an example of closed sets E_1 and E_2 in $\mathbb{R}^m$ having no points in common for which $d(E_1, E_2) = 0$.

Construct E_1, E_2 as follows,

$$E_1 = \{\vec{x} \in \mathbb{R}^m \mid \vec{x} = (\sqrt{n_1 + \pi}, 2, 2, \ldots, 2), n_1 \in \mathbb{N}\},$$

$$E_2 = \{\vec{x} \in \mathbb{R}^m \mid \vec{x} = (\sqrt{n_2}, 2, 2, \ldots, 2), n_2 \in \mathbb{N}\}$$

Since E_1 and E_2 are countable sets, they have no limit point, hence they are closed. They cannot have common point because for all n_1, n_2, $\sqrt{n_1 + \pi} \neq \sqrt{n_2}$. The distance of them are truly zero, because

$$\inf_{x_1 \in E_1, x_2 \in E_2} d(x_1, x_2) = \inf_{n_1, n_2 \in \mathbb{N}} \left|\sqrt{n_1 + \pi} - \sqrt{n_2}\right| \leq \inf_{n_1, n_2 \in \mathbb{N}} \frac{|n_1 - n_2| + \pi}{\sqrt{n_1 + \pi} + \sqrt{n_2}} = 0$$

Therefore, such E_1, E_2 are the sets that we want to find.

Question 7.1-2. Show that

a) the closure $\overline{E}$ in $\mathbb{R}^m$ of any set $E \subset \mathbb{R}^m$ is a closed set in $\mathbb{R}^m$;

We tend to prove the complement of $\overline{E}$, namely $\overline{E}^c$, is open. Hence, we only need to prove any points of $\overline{E}^c$ are interior points. However, if $x \in \overline{E}^c$, then $x \notin \overline{E}$. Since $\overline{E}$ is a closed set, so x is not the limit point of $\overline{E}$, and must have a neighborhood $B(x)$ which has no intersection with $\overline{E}$. Since $B(x)$ has no intersection with $\overline{E}$, it is contained in $\overline{E}^c$. Notice that x is chosen arbitrarily, which means every $x \in \overline{E}^c$ is an interior point of $\overline{E}^c$. This shows that $\overline{E}^c$ is open, so $\overline{E}$ is closed.

b) the set ∂E of boundary points of any set $E \subset \mathbb{R}^m$ is a closed set;

Boundary point is the point whose arbitrary neighborhood has intersection with both E and E^c. Denote $\mathrm{Int}(E; E^c)$ as the complement of ∂E. For any $x \notin \partial E$, x has at least one neighborhood which lies either entirely in E or E^c. For such neighborhood, let's denote it as $B(x)$. For any $y \in B(x)$, it is an interior point of $B(x)$, hence has a neighborhood with no intersection with either E or E^c. Thus $y \in \text{Int}(E; E^c)$ for all $y \in B(x)$. This shows that $B(x) \subset \text{Int}(E; E^c)$, which implies x is an interior point of $\text{Int}(E; E^c)$. Since x is arbitrarily chosen, $\text{Int}(E; E^c)$ is open, and its complement ∂E is closed.

c) if G is an open set in $\mathbb{R}^m$ and F is closed in $\mathbb{R}^m$, then $G \backslash F$ is open in $\mathbb{R}^m$.

Since $G \backslash F$ means $G \cap F^c$, and F^c is open in $\mathbb{R}^m$, we ought to show that the intersection of two open set in $\mathbb{R}^m$ is still open. This is trivial, denote $K = G \cap F^c$, for any $x \in K$, x must be in both G and F^c. In this way, there exists $\delta_1 > 0$, $\delta_2 > 0$ such that $B_{\delta_1}(x) \subset G$ and $B_{\delta_2}(x) \subset F^c$. Take $\delta = \min\{\delta_1, \delta_2\}$, then $B_\delta(x) \subset B_{\delta_1}(x) \subset G$ and $B_\delta(x) \subset B_{\delta_2}(x) \subset F^c$. Hence, $B_\delta(x) \subset K$, so x is an interior point of K. Since it is chosen arbitrarily in K, K is an open set, and the proof is finished.

Question 7.1-3. Show that if $K_1 \supset K_2 \supset \cdots K_n \supset \cdots$ is a sequence of nested nonempty compact sets, then $\bigcap_{i=1}^{\infty} K_i \neq \varnothing$.

First we denote $G_i = {K_i}^c$, where G_i is open because K_i is compact and hence closed. Suppose $\bigcap_{i=1}^{\infty} K_i = \varnothing$, then for all points $x \in K_1$, there exists some i such that $x \notin K_i$, hence $x \in G_i$. This means $\{G_i\}_{i=1}^{\infty}$ is an open cover of K_1. Since K_1 is compact, there exists a finite subcover of it in $\{G_i\}_{i=1}^{\infty}$. Denote it as $\{G_{p_j}\}_{j=1}^{n}$ where p_j is the index of G. Hence,

$$K_1 \subset \bigcup_{j=1}^{n} G_{p_j} \Longrightarrow K_1 \cap K_{p_1} \cap \cdots \cap K_{p_n} = \varnothing$$

But this is impossible, because if we denote $\alpha = \max\{1, p_1, p_2, \ldots, p_n\}$, then

$$K_1 \cap K_{p_1} \cap \cdots \cap K_{p_n} = K_\alpha \neq \varnothing$$

Therefore, contradiction shows that $\bigcap_{i=1}^{\infty} K_i \neq \varnothing$

Question 7.1-4.

a) In the space $\mathbb{R}^k$ a two-dimensional sphere S^2 and a circle S^1 are situated so that the distance from any point of the sphere to any point of the circle is the same. Is this possible?

Yes, this is possible. Consider in the space $\mathbb{R}^5$, a sphere centered at origin $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, 0, 0)$, with radius R, such that $x_1^2 + x_2^2 + x_3^2 = R^2$, $x_4 = x_5 = 0$. Also consider a circle centered at origin, with radius r, such that $x_4^2 + x_5^2 = r^2$, $x_1 = x_2 = x_3 = 0$. Thus, for arbitrary point on sphere S^2, it can be expressed as $(a, b, c, 0, 0)$; for arbitrary point on circle S^1, it can be expressed as $(0, 0, 0, d, e)$. The distance between them is

$$d = \sqrt{(a-0)^2 + (b-0)^2 + (c-0)^2 + (0-d)^2 + (0-e)^2} = \sqrt{R^2 + r^2}$$

which is a constant. Thus, such situation is possible at least in Euclidean space with dimension larger than or equal to 5.

b) Consider problem a) for sphere S^m, S^n of arbitrary dimension in $\mathbb{R}^k$. Under what relation m, n, and k is this situation possible?

Notice that here the sphere is defined in the generalized manner. Therefore, the sphere S^n is defined as

$$S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = r\}$$

From part a), we know that at least when $k \geq m+n+2$, the situation can be possible. From the definition, we know k is at least $1+\max\{m,n\}$. W.L.O.G., let's assume $m > n$ and S^m is centered at origin with $x_{m+2} = \cdots = x_k = 0$, then $k \geq m+1$. Consider the points satisfying the distance between each of them and all points on S^m are all equal, they must satisfy the expression $(\underbrace{0,0,\ldots,0}_{m+1}, x_{m+2}, \ldots, x_k)$ (Check this is really true). All such points constitute a $\mathbb{R}^{k-m-1}$ space. Since we need all points on S^n lies in $\mathbb{R}^{k-m-1}$ space, $k-m-1 \geq n+1$. This shows that $k \geq m+n+2$ is necessary. Hence the necessary and sufficient condition is $k \geq m+n+2$.

Question 7.2-1. Let $f \in \mathcal{C}(\mathbb{R}^m;\mathbb{R})$. Show that

a) the set $E_1 = \{x \in \mathbb{R}^m \mid f(x) < c\}$ is open in $\mathbb{R}^m$;

We tend to prove every point in E_1 is interior point. Take arbitrary $x_0 \in E_1$, we need to prove $B_\delta(x_0) \subset E_1$ for some $\delta > 0$. Since f is continuous, then for any $\epsilon > 0$, there exists $\delta > 0$, such that $|f(x) - f(x_0)| < \epsilon$. Take $\epsilon = [c - f(x_0)]/2$, since $f(x_0) < c$, $\epsilon > 0$. Then there exists $\delta > 0$, such that for $x \in B_\delta(x_0)$,

$$f(x) < f(x_0) + \epsilon = \frac{c + f(x_0)}{2} < \frac{c+c}{2} = c$$

Hence $B_\delta(x_0) \subset E_1$, and we finish our proof.

b) the set $E_2 = \{x \in \mathbb{R}^m \mid f(x) \leq c\}$ is closed in $\mathbb{R}^m$;

We tend to prove ${E_2}^c = \{x \in \mathbb{R}^m \mid f(x) > c\}$ is open, then E_2 must be closed. The method to prove ${E_2}^c$ is open is the same as part a). Take arbitrary $x_0 \in {E_2}^c$, we need to prove $B_\delta(x_0) \subset {E_2}^c$ for some $\delta > 0$. Since f is continuous, then for any $\epsilon > 0$, there exists $\delta > 0$, such that $|f(x) - f(x_0)| < \epsilon$. Take $\epsilon = [f(x_0) - c]/2$, since $f(x_0) > c$, $\epsilon > 0$. Then there exists $\delta > 0$, such that for $x \in B_\delta(x_0)$,

$$f(x) > f(x_0) - \epsilon = \frac{c + f(x_0)}{2} > \frac{c+c}{2} = c$$

Hence $B_\delta(x_0) \subset {E_2}^c$, and we finish our proof.

c) the set $E_3 = \{x \in \mathbb{R}^m \mid f(x) = c\}$ is closed in $\mathbb{R}^m$;

Notice that the complement of E_3, namely ${E_3}^c = \{x \in \mathbb{R}^m \mid f(x) \neq c\}$, is the union of E_1 and ${E_2}^c$. Hence, we only need to prove the union of two open sets is still open, but this is too easy. For arbitrary point in ${E_3}^c$, it is either in E_1 or ${E_2}^c$. Hence it is an interior point of E_1 or ${E_2}^c$. In this way, it has a neighborhood contained in E_1 or ${E_2}^c$, thus in ${E_3}^c$. This shows ${E_3}^c$ is open, and its complement E_3 is closed.

d) if $f(x) \to +\infty$ as $x \to +\infty$, then E_2 and E_3 are compact in $\mathbb{R}^m$;

We have proved that E_2 and E_3 are closed. Thus we only need to prove they are bounded. Since $E_3 \subset E_2$, we only need to prove that E_2 is bounded. Suppose it is not bounded, then by definition, we have

$$\forall\, M > 0,\ M \in \mathbb{N},\ \exists x_M \in E_2,\ s.t.\ \|x_M\| \geq M$$

Since $f(x) \to \infty$ as $x \to \infty$, for any fixed c, there exists N such that $f(x_M) > c$ if $M \geq N$. This means that x_M is not in E_2, which is a contradiction. Thus, E_2 is bounded, and E_3 is automatically bounded. Therefore, E_2, E_3 are both compact.

e) for any $f : \mathbb{R}^m \mapsto \mathbb{R}$ the set $E_4 = \{x \in \mathbb{R}^m \,|\, \omega(f; x) \geq \epsilon\}$ is closed in $\mathbb{R}^m$.

We tend to prove ${E_4}^c = \{x \in \mathbb{R}^m \,|\, \omega(f; x) < \epsilon\}$ is open. For arbitrary point $x \in {E_4}^c$, we need to prove it is an interior point, i.e., for some neighborhood of x, all points y in it satisfy $\omega(f; y) < \epsilon$. It suffices to show at least one of its neighborhood satisfies $\omega(f; B_\delta(x)) < \epsilon$ (then all point in it will satisfy $\omega(f; y) < \epsilon$).

This is trivial, because $\omega(f; x) = \lim_{\delta \to 0+} \omega(f; B_\delta(x))$, and if for any $\delta > 0$, $\omega(f; B_\delta(x)) \geq \epsilon$, then the limit $\omega(f; x)$ must be no less than ϵ, which contradicts the fact that $\omega(f; x) < \epsilon$. Hence, ${E_4}^c$ is really an open set, which means E_4 is closed.

Question 7.2-2. Show that the mapping $f : \mathbb{R}^m \mapsto \mathbb{R}^n$ is continuous if and only if the preimage of every open set in $\mathbb{R}^n$ is an open set in $\mathbb{R}^m$.

Suppose f is continuous on $\mathbb{R}^m$ and V is an open set in $\mathbb{R}^n$. Denote the preimage of V as W. We need to show every point of W is an interior point of W. Suppose $p \in \mathbb{R}^m$ and $f(p) \in \mathbb{R}^n$. Since V is open, there exists $\epsilon > 0$ such that $B_\epsilon(f(p)) \subset V$. Since f is continuous at p, there exists $\delta > 0$ such that $d(f(x), f(p)) < \epsilon$ for all $x \in B_\delta(p)$. Then all points in $B_\delta(p)$ are mapped into $B_\epsilon(f(p))$, thus in V. Hence $B_\delta(p) \subset W$. Again, p is arbitrarily chosen in $\mathbb{R}^m$, every point in W is an interior point and the proof is completed.

Conversely, suppose W is open in $\mathbb{R}^m$ for every open set V in $\mathbb{R}^n$. Fix $p \in \mathbb{R}^m$ and $\epsilon > 0$, let V be $B_\epsilon(f(p))$. Since V is open, W is also open. Thus, there exists $\delta > 0$ such that $B_\delta(p) \subset W$. For all $x \in B_\delta(p)$ (actually for all $x \in W$), $f(x) \in V = B_\epsilon(f(p))$. This is just saying for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $x \in B_\delta(p)$, we have $d(f(x), f(p)) < \epsilon$, thus $f(x)$ is continuous at p. Since p is taken arbitrarily, f is continuous on $\mathbb{R}^m$.

The following question is not required, since connectedness wasn't taught in MAT2006 but MAT1003.

Question 7.2-3. Show that

a) the image $f(E)$ of a connected set $E \subset \mathbb{R}^m$ under a continuous mapping $f : E \mapsto \mathbb{R}^n$ is a connected set;

If we regard all "connected" here as generally defined conception, then the proof is as follows.

Suppose $f(E)$ is not connected, then $f(E) = A \cup B$, where A and B are nonempty separated subsets of E. Recall that A and B is separated if and only if $A \cap \overline{B} = B \cap \overline{A} = \varnothing$. Denote $G = f^{-1}(A)$ and $H = f^{-1}(B)$. Then $E = G \cup H$, and neither G nor H is empty (A and B are nonempty).

Since $A \subset \overline{A}$, $G = f^{-1}(A) \subset f^{-1}(\overline{A})$. Don't forget that the preimage of any closed set under continuous function is still closed, so $f^{-1}(\overline{A})$ is closed. Therefore, $\overline{G} \subset f^{-1}(\overline{A})$. It follows that $f(\overline{G}) \subset \overline{A}$. Remember we defined $f(H) = B$, and $\overline{A} \cap B$ is empty, so $\overline{G} \cap H$ must be empty (otherwise their image must have common point). Similarly, you can show that $\overline{H} \cap G$ by the exactly same procedure (since G and H are equivalent to each other). This shows that G and H are nonempty separated sets, meaning that E is not connected and yielding a contradiction. Therefore, $f(E)$ is connected.

If we regard all "connected" here as "pathwise-connected" as defined in Zorich's textbook, then the proof is as follows.

Since E is pathwise-connected, for arbitrary two points x, y in E, there exists a path (continuous function) $\Gamma : [a, b] \mapsto E$ such that $\Gamma(a) = x, \Gamma(b) = y$. Since f is continuous on E, the restriction of it (denoted as g) on $\Gamma([a, b])$ is still continuous. Then we can construct a function $h : [a, b] \mapsto f(E)$, defined by $h = g \circ \Gamma$, which is still continuous on $[a, b]$. For any two points c, d in $f(E)$, we can find their preimage in E, denoted as c_0, d_0, and $g(c_0) = c, g(d_0) = d$. Then we can find such h that $h(a) = g(c_0) = c$ and $h(b) = g(d_0) = d$. This means h is a path from c to d, which lies in $f(E)$. Hence, $f(E)$ is pathwise-connected.

b) the union of connected sets having a point in common is a connected set;

If we regard all "connected" here as generally defined conception, then the proof is as follows.

Denote a collection of connected sets as $\{A_\alpha\}_I$, where I is the index set. Denote the common point $p \in \bigcap_{\alpha\in I} A_\alpha$, and $Y = \bigcup_{\alpha\in I} A_\alpha$. Suppose Y is not connected, then $Y = C \cup D$, where C, D is nonempty separated sets. Since p must be either in C or D, W.O.L.G., suppose $p \in C$. Since A_α is connected, it must lie in either C or D. However, $p \in A_\alpha$ and $p \in C$, so A_α lie in C for all $\alpha \in I$. This yields that $\bigcup_{\alpha\in I} A_\alpha = C$, showing that D is empty, which is a contradiction. So Y must be connected.

If we regard all "connected" here as "pathwise-connected" as defined in Zorich's textbook, then the proof is as follows.

First we denote the common point as p, and the collection of connected sets as $Y = \{A_\alpha\}_I$. Choose arbitrary two points in Y, then suppose these two points x_1, x_2 are in A_{α_1} and A_{α_2}. If $\alpha_1 = \alpha_2$, it's trivial that there exists a path connecting them. If $\alpha_1 \neq \alpha_2$, then there exists a path $\Gamma_1 : [a, b] \mapsto Y$, such that $\Gamma(a) = x_1, \Gamma(b) = p$. There also exists another path $\Gamma_2 : [b, c] \mapsto Y$ such that $\Gamma(b) = p, \Gamma(c) = x_2$. By pasting lemma (or gluing lemma), since $[a, b]$ and $[b, c]$ are closed, Γ_1, Γ_2 are continuous, and $\Gamma_1(x) = \Gamma_2(x)$ for every $x \in [a, b] \cap [b, c]$, we obtain the combination of Γ_1, Γ_2, denoted as $\Gamma : [a, c] \mapsto Y$, which is continuous and defined by setting $\Gamma(x) = \Gamma_1(x)$ if $x \in [a, b]$, and $\Gamma(x) = \Gamma_2(x)$ if $x \in [b, c]$. Hence Γ is a path from x_1 to x_2 in Y, meaning that Y is also pathwise-connected.

c) the hemisphere $(x^1)^2 + \cdots + (x^m)^2 = 1$, $x^m \geq 0$, is a connected set;

Here it suffices to show that this set is pathwise-connected set, because pathwise-connected set must be connected (you can prove it easily). We first prove $A \backslash \{0\}$, where $A = \{(x^1, \ldots, x^m) \,|\, x^m \geq 0\}$, is pathwise connected. We take arbitrary two points in $A \backslash \{0\}$, and connect them with a line, then there are two cases, one is this line going through 0, the other is this line not going through 0. For the second case, this line is a path in $A \backslash \{0\}$; for the first case, we pick another point in $A \backslash \{0\}$ that is not on the line, link the two points with this new point, then the broken line is a path in $A \backslash \{0\}$. Hence, $A \backslash \{0\}$ is a pathwise-connected set.

Construct a mapping $f : A \backslash \{0\} \mapsto B$, where B is the hemisphere in the question, and $f(\vec{x}) = \vec{x}/\|\vec{x}\|$. One can check this function f is surjective and continuous, then by part a), since the preimage is pathwise-connected, the image B is also pathwise-connected. Since the surjectivity is too trivial, we only prove the continuity of f. For arbitrary $\epsilon > 0$, for any fixed x_0 in the domain, take $\delta = \|\vec{x_0}\|\epsilon/2$, for any $\|\vec{x} - \vec{x_0}\| < \delta$, we have

$$\begin{aligned}
f(\vec{x}) - f(\vec{x_0}) &= \frac{\|\vec{x_0}\|\vec{x} - \vec{x_0}\|\vec{x}\|}{\|\vec{x}\|\|\vec{x_0}\|} \\
&= \frac{\|\vec{x_0}\|\vec{x} - \|\vec{x}\|\vec{x} + \|\vec{x}\|\vec{x} - \vec{x_0}\|\vec{x}\|}{\|\vec{x}\|\|\vec{x_0}\|} \\
&= \frac{(\|\vec{x_0}\| - \|\vec{x}\|)\vec{x} + \|\vec{x}\|(\vec{x} - \vec{x_0})}{\|\vec{x}\|\|\vec{x_0}\|}
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|f(\vec{x}) - f(\vec{x_0})\| &\leq \frac{\Big\|\|\vec{x_0}\| - \|\vec{x}\|\Big\|\|\vec{x}\| + \|\vec{x}\|\Big\|\vec{x} - \vec{x_0}\Big\|}{\|\vec{x}\|\|\vec{x_0}\|} \\
&\leq \frac{\|\vec{x} - \vec{x_0}\|\|\vec{x}\| + \|\vec{x}\|\|\vec{x} - \vec{x_0}\|}{\|\vec{x}\|\|\vec{x_0}\|} \\
&= \frac{2\|\vec{x} - \vec{x_0}\|}{\|\vec{x_0}\|} < \epsilon
\end{aligned}$$

Hence, f is continuous (not uniformly) at every point in domain $A \backslash \{0\}$.

d) the sphere $(x^1)^2 + \cdots + (x^m)^2 = 1$, is a connected set;

Denote the sphere $(x^1)^2+\cdots+(x^m)^2=1$ as A, the hemisphere $(x^1)^2+\cdots+(x^m)^2=1$, $x^m\geq 0$, as B, and the hemisphere $(x^1)^2+\cdots+(x^m)^2=1$, $x^m\leq 0$ as C. Since we have proved B is pathwise-connected in part c), a similar proof will show that C is also pathwise-connected. B and C obviously have a common point (actually infinitely many common points), and by part b), the union of B and C is pathwise-connected. Since $A=B\cup C$, A is pathwise-connected (thus also connected).

e) if $E\subset\mathbb{R}$ and E is connected, then E is an interval in $\mathbb{R}$ (that is, a closed interval, a half-open interval, an open interval, or the entire real line);

Here it suffices to show above statement is true under the assumption that E is only connected (not necessarily pathwise-connected). We only need to prove that if E is connected set, then it satisfies that if $x,y\in E$, then any z between x,y is in E. If $x=y$, this is trivial. If $x\neq y$, W.O.L.G., we assume $x<y$, if there exists some $z\in(x,y)$, but $z\in E$, we can write $E=A_z\cup B_z$, where

$$A_z=E\cap(-\infty,z),\qquad B_z=E\cap(z,\infty)$$

Since $x\in A_z$, $y\in B_z$, A,B are nonempty. Since $A_z\subset(-\infty,z)$, $B_z\subset(z,+\infty)$, they are separated. Hence E is not connected, which is a contradiction. Thus such z does not exist.

Now we can see if E is bounded above and below, then we have three cases. If its supremum and infimum lie in E, then $E=[\inf E,\sup E]$; if one of them lies out of E, then $E=(\inf E,\sup E]$ or $E=[\inf E,\sup E)$; if both of them lie out of E, then $E=(\inf E,\sup E)$.

If E is either unbounded above or below, then we have four cases. If it is unbounded above and cannot assume its infimum, then $E=(\inf E,+\infty)$; if it is unbounded above and can assume its infimum, then $E=[\inf E,+\infty)$. If it is unbounded below and cannot assume its supremum, then $E=(-\infty,\sup E)$; otherwise $E=(-\infty,\sup E]$.

If E is unbounded above and below, then $E=(-\infty,+\infty)=\mathbb{R}$. Hence we finish the proof.

f) if x_0 is an interior point and x_1 an exterior point in relation to the set $M\subset\mathbb{R}^m$, then the support of any path with endpoints x_0,x_1 intersects the boundary of the set M.

Notice that the support of any path with endpoints must be pathwise-connected set since it is the image of closed interval under continuous mapping (see the definition of path). Suppose any path with endpoints x_0,x_1 does not intersect the boundary of the set M, then any point on the path is interior to M or M^c. Thus, the support $\Gamma(I)$ can be divided into two parts, i.e., $\Gamma(I)=A\cup B$, where A and B are disjoint because $A\subset M\backslash\partial M$, $B\subset M^c\backslash\partial M^c$. Thus, A and B are separated sets, because they lie in two nonempty separated sets respectively. This shows that $\Gamma(I)$ is not connected. Therefore, $\Gamma(I)$ is not pathwise-connected, which gives a contradiction indicating that any path with endpoints x_0,x_1 must intersect the boundary of the set M.