MAT2006: Elementary Real Analysis
Homework 6

W. Lee.

Due date: Today

Question 8.3-2.

a) Draw the graph of the function $z = x^2 + 4y^2$, where (x, y, z) are Cartesian coordinates in $\mathbb{R}^3$.

b) Let $f : C \mapsto \mathbb{R}$ be a numerically valued function defined on a domain $G \subset \mathbb{R}^m$. A *level set* (c-level) of the function is a set $E \subset G$ on which the function assumes only one value ($f(E) = c$). More precisely, $E = f^{-1}(c)$. Draw the level sets in $\mathbb{R}^2$ for the function given in part a).

c) Find the gradient of the function $f(x, y) = x^2 + 4y^2$, and verify that at any point (x, y) the vector grad f is orthogonal to the level curve of the function f passing through the point.

d) Using the results of a), b), and c), lay out what appears to be the shortest path on the surface $z = x^2 + 4y^2$ descending from the point $(2, 1, 8)$ to the lowest point on the surface $(0, 0, 0)$.

e) What algorithm, suitable for implementation on a computer, would you propose for finding the minimum of the function $f(x, y) = x^2 + 4y^2$?

Question 8.3-3. We say that a *vector field* is defined in a domain G of $\mathbb{R}^m$ if a vector $\vec{v}(x) \in T\mathbb{R}^m_x$ is assigned to each point $x \in G$. A vector field $\vec{v}(x)$ in G is called a *potential field* if there is a numerical-valued function $U : G \mapsto \mathbb{R}$ such that $\vec{v}(x) = \text{grad } U(x)$. The function $U(x)$ is called the *potential* of the field $\vec{v}(x)$. (In physics it is the function $-U(x)$ that is usually called the potential, and the function $U(x)$ is called the *force function* when a field of force is being discussed.)

a) On a plane with Cartesian coordinates (x, y) draw the field grad $f(x.y)$ for each of the following functions: $f_1(x, y) = x^2 + y^2$; $f_2(x, y) = -(x^2 + y^2)$; $f_3(x, y) = \arctan(x/y)$ in the domain $y > 0$; $f_4(x, y) = xy$.

b) By Newton's law a particle of mass m at the point $0 \in \mathbb{R}^3$ attracts a particle of mass 1 at the point $x \in \mathbb{R}^3$ $(x \neq 0)$ with force $\vec{F} = -m|\vec{r}|^{-3}\vec{r}$, where $\vec{r}$ is the vector $\overrightarrow{Ox}$ (we have omitted the dimensional constant G_0). Show that the vector field $\vec{F}(x)$ in $\mathbb{R}^3 \backslash 0$ is a potential field.

c) Verify that masses m_i $(i = 1, \ldots, n)$ located at the points (ξ_i, η_i, ζ_i) $(i = 1, \ldots, n)$ respectively, create a Newtonian force field except at these points and that the potential is the function

$$U(x, y, z) = \sum_{i=1}^{n} \frac{m_i}{\sqrt{(x-\xi_i)^2 + (y-\eta_i)^2 + (z-\zeta_i)^2}}$$

d) Find the potential of the electrostatic field created by point charges q_i $(i = 1, \dots, n)$ located at the points (ξ_i, η_i, ζ_i) $(i = 1, \dots, n)$ respectively.

Question 8.3-6. *Homogeneous functions and Euler's identity.* A function $f : G \mapsto \mathbb{R}$ defined in some domain $G \subset \mathbb{R}^m$ is called *homogeneous* (resp. *positive-homogeneous*) *of degree* n if the equality

$$f(\lambda x) = \lambda^n f(x) \quad \Big(\text{resp. } f(\lambda x) = |\lambda|^n f(x)\Big)$$

holds for any $x \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}$ such that $x \in G$ and $\lambda x \in G$.

A function is *locally homogeneous* of degree n in the domain G if it is a homogeneous function of degree n in some neighborhood of each point of G.

a) Prove that in a convex domain every locally homogeneous function is homogeneous.

b) Let G be the plane $\mathbb{R}^2$ with the ray $L = \Big\{(x, y) \in \mathbb{R}^2 \mid x = 2 \wedge y \geq 0\Big\}$ removed. Verify that the function

$$f(x, y) = \begin{cases} y^4/x, & \text{if } x > 2 \wedge y > 0, \\ y^3, & \text{at other points of the domain,} \end{cases}$$

is locally homogeneous in G, but is not a homogeneous function in that domain.

c) Determine the degree of homogeneity or positive homogeneity of the following functions with their natural domains of definition,

$$f_1(x^1, \dots, x^m) = x^1x^2 + x^2x^3 + \dots + x^{m-1}x^m;$$

$$f_2(x^1, x^2, x^3, x^4) = \frac{x^1x^2 + x^3x^4}{x^1x^2x^3 + x^2x^3x^4};$$

$$f_3(x^1, \dots, x^m) = |x^1 \cdots x^m|^l$$

d) By differentiating the equality $f(tx) = t^n f(x)$ with respect to t, show that if a differentiable function $f : G \mapsto \mathbb{R}$ is locally homogeneous of degree n in a domain $G \subset \mathbb{R}^m$, it satisfies the following *Euler identity for homogeneous functions*,

$$x^1 \frac{\partial f}{\partial x^1}(x^1, \cdots, x^m) + \cdots + x^m \frac{\partial f}{\partial x^m}(x^1, \cdots, x^m) \equiv n f(x^1, \cdots, x^m)$$

e) Show that if Euler's identity holds for a differentiable function $f : G \mapsto \mathbb{R}$ in a domain G, then that function is locally homogeneous of degree n in G. (Verify that the function $\varphi(t) = t^{-n} f(tx)$ is defined for every $x \in G$ and is constant in some neighborhood of 1.)

Question 8.4-1. Let $z = f(x, y)$ be a function of class $\mathcal{C}^{(1)}(G; \mathbb{R})$.

a) If $\frac{\partial f}{\partial y}(x, y) \equiv 0$ in G, can one assert that f is independent of y in G?

No. Let $G = \{(x, y) \,|\, x^2 + y^2 < 1 \text{ or } x^2 + (y-2)^2 < 1\}$. Define

$$f(x, y) = \begin{cases} 1 & x^2 + y^2 < 1 \\ 0 & x^2 + (y-2)^2 < 1 \end{cases}$$

Then one can see that $\frac{\partial f}{\partial y}(x, y) \equiv 0$ in G, but f is dependent on y, since

$$1 = f(0, 0) \neq f(0, 2) = 0$$

b) Under what condition on the domain G does the preceding question have an affirmative answer?

A sufficient condition is that if for all $x_0 \in \mathbb{R}$, the set

$$B = \{(x, y) \,|\, x = x_0\} \cap G$$

is a connected set, and $\frac{\partial f}{\partial y}(x, y) \equiv 0$ in G. Here connected is equivalent to pathwise-connected, because $B \subset \{x_0\} \times \mathbb{R}$, so B is connected if and only if it is an interval on straight line $x = x_0$, hence it must be pathwise-connected.

Suppose $\exists\, x_0 \in \mathbb{R}$, $y_1 \neq y_2$, such that $f(x_0, y_1) \neq f(x_0, y_2)$, $(x_0, y_1), (x_0, y_2) \in G$. Since B is an interval on straight line (hence convex), G contains all points on the segment between (x_0, y_1) and (x_0, y_2). By MVT (one-dimensional), there exists ξ between y_1 and y_2, s.t.

$$f(x_0, y_2) - f(x_0, y_1) = \frac{\partial f(x_0, \xi)}{\partial y}(y_2 - y_1)$$

However, the partial derivative on right hand side is zero, hence $f(x_0, y_2) - f(x_0, y_1) = 0$, which is a contradiction. Thus, for any fixed $x \in \mathbb{R}$, $f(x, y_1) \equiv f(x, y_2)$ for all y_1, y_2. Hence, f is independent of y in G.

Question 8.4-2.

a) Verify that for the function

$$f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}\,, & \text{if } x^2 + y^2 \neq 0, \\ 0\,, & \text{if } x^2 + y^2 \neq 0. \end{cases}$$

the following relations hold,

$$\frac{\partial^2 f}{\partial x \partial y}(0, 0) = 1 \neq -1 = \frac{\partial^2 f}{\partial y \partial x}(0, 0)$$

The first order derivative is

$$\frac{\partial f}{\partial x}(x, y) = \begin{cases} \dfrac{y[x^4 + 4x^2y^2 - y^4]}{(x^2 + y^2)^2}\,, & \text{if } x^2 + y^2 \neq 0, \\ 0\,, & \text{if } x^2 + y^2 \neq 0. \end{cases}$$

and

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} -\dfrac{x[y^4+4x^2y^2-x^4]}{(x^2+y^2)^2}, & \text{if } x^2+y^2 \neq 0, \\ 0, & \text{if } x^2+y^2 \neq 0. \end{cases}$$

Now, let us compute the two mixed second order partial derivatives.

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h} = \lim_{h\to 0} \frac{-h-0}{h} = -1$$

and

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1$$

b) Prove that if the function $f(x,y)$ has partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in some neighborhood U of the point (x_0,y_0), and if the mixed derivative $\frac{\partial^2 f}{\partial x \partial y}$ (or $\frac{\partial^2 f}{\partial y \partial x}$) exists in U and is continuous at (x_0,y_0), then the mixed derivative $\frac{\partial^2 f}{\partial y \partial x}$ (resp. $\frac{\partial^2 f}{\partial x \partial y}$) also exists at that point and the following equality holds,

$$\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0,y_0)$$

In the neighborhood U of point (x_0,y_0), we can find a closed rectangle with (x_0,y_0), (x_0+h_1,y_0), (x_0,y_0+h_2) and (x_0+h_1,y_0+h_2) as its four vertices, if $h_1>0, h_2>0$ is small enough. Define

$$F(h_1,h_2) = f(x_0+h_1,y_0+h_2) - f(x_0,y_0+h_2) - f(x_0+h_1,y_0) + f(x_0,y_0)$$

We first prove that for arbitrary small h_1,h_2, there exists a point (x^*,y^*) in the interior of the closed rectangle, such that

$$\frac{F(h_1,h_2)}{h_1h_2} = \frac{\partial^2 f}{\partial x \partial y}(x^*,y^*)$$

In the rectangle, for any fixed x, by MVT (one-dimensional), there exists $\xi \in (y_0, y_0+h_2)$, such that

$$\frac{g(x,y_0+h_2) - g(x,y_0)}{(y_0+h_2)-y_0} = \frac{\partial g}{\partial y}(x,\xi)$$

where

$$g(x,y) = f(x+h_1,y) - f(x,y)$$

Thus,

$$\begin{aligned} \frac{F(h_1,h_2)}{h_2} &= \frac{f(x_0+h_1,y_0+h_2) - f(x_0,y_0+h_2)}{h_2} - \frac{f(x_0+h_1,y_0) - f(x_0,y_0)}{h_2} \\ &= \frac{g(x_0,y_0+h_2) - g(x_0,y_0)}{h_2} \\ &= \frac{\partial g}{\partial y}(x_0,\xi) \\ &= \frac{\partial f}{\partial y}(x_0+h_1,\xi) - \frac{\partial f}{\partial y}(x_0,\xi) \end{aligned}$$

Since ξ is fixed, consider MVT again, there exists $\eta \in (x_0, x_0+h_1)$, such that

$$\frac{F(h_1,h_2)}{h_1h_2} = \frac{f_y(x_0+h_1,\xi) - f_y(x_0,\xi)}{h_1} = \frac{\partial f_y}{\partial x}(\eta,\xi) = \frac{\partial^2 f}{\partial x \partial y}(\eta,\xi)$$

Since point (η, ξ) lies in the interior of the rectangle, the proof of our claim was finished.

Then, we can prove our main statement. Since $\frac{\partial^2 f}{\partial x \partial y}$ is continuous at (x_0, y_0), for any $\epsilon > 0$, there exists small enough h_1, h_2, such that for all points (x, y) in the rectangle, we have

$$\left| \frac{\partial^2 f}{\partial x \partial y}(x, y) - \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \right| < \epsilon$$

Also, (η, ξ) always lies in the rectangle, so

$$\left| \frac{F(h_1, h_2)}{h_1 h_2} - \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \right| = \left| \frac{\partial^2 f}{\partial x \partial y}(\eta, \xi) - \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \right| < \epsilon$$

Fix h_2 and let $h_1 \to 0$, we can obtain

$$\left| \frac{f_x(x_0, y_0 + h_2) - f_x(x_0, y_0)}{h_2} - \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \right| < \epsilon$$

Let $h_2 \to 0$, we can obtain

$$\left| \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) - \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \right| < \epsilon$$

Since ϵ is arbitrary, we have

$$\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$$

Question 8.4-3. Let $x^1, \dots, x^m$ be Cartesian coordinates in $\mathbb{R}^m$. The differential operator

$$\Delta = \sum_{i=1}^{m} \frac{\partial^2}{\partial {x^i}^2}$$

acting on functions $f \in \mathcal{C}^{(2)}(G; \mathbb{R})$ according to the rule

$$\Delta f = \sum_{i=1}^{m} \frac{\partial^2 f}{\partial {x^i}^2}(x^1, \dots, x^m)$$

is called the *Laplacian*.

The equation $\Delta f = 0$ for the function f in the domain $G \subset \mathbb{R}^m$ is called *Laplace's equation*, and its solutions are called *harmonic functions in the domain G*.

a) Show that if $x = (x^1, \dots, x^m)$ and

$$\|x\| = \sqrt{\sum_{i=1}^{m} (x^i)^2},$$

then for $m > 2$ the function

$$f(x) = \|x\|^{2-m}$$

is harmonic in the domain $\mathbb{R}^m \backslash 0$, where $0 = (0, \dots, 0)$.

Notice that

$$\frac{\partial f}{\partial x^i} = \frac{\partial}{\partial x^i} \left(\sum_{i=1}^{m} (x^i)^2 \right)^{\frac{2-m}{2}} = (2-m) x^i \left(\sum_{i=1}^{m} (x^i)^2 \right)^{-\frac{m}{2}}$$

Also, the second derivative

$$\frac{\partial^2 f}{\partial {x^i}^2} = (2-m)\left(\sum_{i=1}^{m}(x^i)^2\right)^{-\frac{m}{2}} - (2-m)m(x^i)^2\left(\sum_{i=1}^{m}(x^i)^2\right)^{-\frac{m+2}{2}}$$

Therefore, we can show

$$\begin{aligned}
\Delta f &= \sum_{i=1}^{m}\frac{\partial^2 f}{\partial {x^i}^2} \\
&= \sum_{i=1}^{m}(2-m)\left(\sum_{i=1}^{m}(x^i)^2\right)^{-\frac{m}{2}} - \sum_{i=1}^{m}(2-m)m(x^i)^2\left(\sum_{i=1}^{m}(x^i)^2\right)^{-\frac{m+2}{2}} \\
&= m(2-m)\left(\sum_{i=1}^{m}(x^i)^2\right)^{-\frac{m}{2}} - (2-m)m\left(\sum_{i=1}^{m}(x^i)^2\right)\left(\sum_{i=1}^{m}(x^i)^2\right)^{-\frac{m+2}{2}} \\
&= m(2-m)\left(\sum_{i=1}^{m}(x^i)^2\right)^{-\frac{m}{2}} - (2-m)m\left(\sum_{i=1}^{m}(x^i)^2\right)^{-\frac{m}{2}} \\
&= 0
\end{aligned}$$

Hence the function f is harmonic.

Note that the original function is not harmonic, there must be some typo in the textbook.

b) Verify that the function

$$f(x^1, \ldots, x^m, t) = \frac{1}{(2a\sqrt{\pi t})^m}\cdot\exp\left(-\frac{\|x\|^2}{4a^2 t}\right),$$

which is defined for $t > 0$ and $x = (x^1, \ldots, x^m) \in \mathbb{R}^m$, satisfies the *heat equation*

$$\frac{\partial f}{\partial t} = a^2 \Delta f,$$

that is, verify that $\frac{\partial f}{\partial t} = a^2 \sum\limits_{i=1}^{m} \frac{\partial^2 f}{\partial {x^i}^2}$ at each point of the domain of definition of the function.

It is not hard to compute that

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{1}{(2a\sqrt{\pi})^m}t^{-3m/2}\left(-\frac{m}{2}\right)\exp\left(-\frac{\|x\|^2}{4a^2 t}\right) + \frac{1}{(2a\sqrt{\pi t})^m}\exp\left(-\frac{\|x\|^2}{4a^2 t}\right)\frac{\|x\|^2}{4a^2 t^2} \\
&= \frac{1}{(2a\sqrt{\pi t})^m}\exp\left(-\frac{\|x\|^2}{4a^2 t}\right)\left(-\frac{m}{2t} + \frac{\|x\|^2}{4a^2 t^2}\right)
\end{aligned}$$

Also, we can compute

$$\frac{\partial f}{\partial x^i} = \frac{1}{(2a\sqrt{\pi t})^m}\exp\left(-\frac{\|x\|^2}{4a^2 t}\right)\left(-\frac{x^i}{2a^2 t}\right)$$

Then the second derivative is given by

$$\frac{\partial^2 f}{\partial {x^i}^2} = \frac{1}{(2a\sqrt{\pi t})^m}\exp\left(-\frac{\|x\|^2}{4a^2 t}\right)\left(-\frac{1}{2a^2 t} + \frac{(x^i)^2}{4a^4 t^2}\right)$$

Hence, the Laplacian is given by

$$\Delta f = \sum_{i=1}^{m} \frac{\partial^2 f}{\partial x^{i2}} = \frac{1}{(2a\sqrt{\pi t})^m} \exp\left(-\frac{\|x\|^2}{4a^2 t}\right)\left(-\frac{m}{2a^2 t} + \frac{\|x\|^2}{4a^4 t^2}\right)$$

Compare Δf and $\frac{\partial f}{\partial t}$, we can conclude that

$$\frac{\partial f}{\partial t} = a^2 \sum_{i=1}^{m} \frac{\partial^2 f}{\partial x^{i2}}$$

Question 8.4-6. Prove the following *generalization of Rolle's theorem for functions of several variables.*

If the function f is continuous in a closed ball $\overline{B}(0;r)$, equal to zero on the boundary of the ball, and differentiable in the open ball $B(0;r)$, then at least one of the points of the open ball is a critical point of the function.

We only consider function defined on finite dimensional Euclidean space (at least the closed ball should be compact), say $\mathbb{R}^n$. By Theorem 4.15 in Rudin's book, since f is continuous on $\overline{B}(0;r)$, $f\left(\overline{B}(0;r)\right)$ is closed and bounded. Therefore, the maximum and minimum value of f must be assumed in $\overline{B}(0;r)$. Suppose M is the maximum value and m is the minimum value, then we can find $p, q \in \overline{B}(0;r)$ such that $f(p) = m, f(q) = M$. If neither p nor q is in $B(0;r)$, then $f(p) = f(q)$, which indicates f is a constant function, hence any interior point of $B(0;r)$ is a critical point.

Suppose at least one of p or q are in $B(0;r)$, W.O.L.G., we assume p is in $B(0;r)$. Then, since f is differentiable at p and p is the global minimum point, so by the first order necessary condition, the gradient of f at p must be zero, meaning that p is a critical point.

Question 8.4-7. Verify that the function

$$f(x, y) = (y - x^2)(y - 3x^2)$$

does not have an extremum at the origin, even though its restriction to each line passing through the origin has a strict local minimum at that point.

Consider f on two curves C_1, C_2 going through the origin, where $C_1 : y = 2x^2$ and $C_2 : y = 4x^2$. Any neighborhood of the origin contains infinite points on C_1 and C_2. On C_1, $f(x, y) = -x^4$, so $x = 0$ is a strict local maximum. On C_2, $f(x, y) = 3x^4$, so $x = 0$ is a strict local minimum. However, this just means in any neighborhood of the origin, there are some points whose function value is strictly less than the value at the origin, and some other points whose function value is strictly larger than the value at the origin. Thus it is not an extreme point.

If the line passing through the origin is $x = 0$, then $f(x, y) = y^2$ on it. It is obvious that $(0, 0)$ is a strict local minimum. Otherwise, the line passing through the origin is $y = kx$, then $f(x, kx) = 3x^4 - 4kx^3 + k^2x^2$. Compute the first derivative $f'(x, kx) = 12x^3 - 12kx^2 + 2k^2x$. Then compute the second derivative, we have $f''(x, kx) = 36x^2 - 24kx + 2k^2$. If $k \neq 0$, then

$f''(0,0) = 2k^2 > 0$ and $f'(0,0) = 0$. Hence, $(0,0)$ must be a strict local minimum (since the function is continuous at $(0,0)$). If $k = 0$, $f(x, kx) = 3x^4$, it is obvious that $x = 0$ is a local minimum (also global minimum) point, which is exactly the point $(0,0)$. In conclusion, any line passing through the origin has a strict local minimum at that point.