MAT2006: Elementary Real Analysis
Homework 7

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Due date: Today

Question 8.5-1. On the plane $\mathbb{R}^2$ with coordinates x and y a curve is defined by the relation $F(x,y)=0$, where $F \in \mathcal{C}^{(2)}(\mathbb{R}^2,\mathbb{R})$. Let (x_0,y_0) be a noncritical point of the function $F(x,y)$ lying on the curve.

a) Write the equation of the tangent to this curve at this point (x_0,y_0).

Since (x_0,y_0) is a noncritical point, at least one of the partial derivative of F at this point is nonzero. W.L.O.G., we suppose $\partial F(x_0,y_0)/\partial x \neq 0$, since $F(x_0,y_0)$, we apply IFT to F at (x_0,y_0). There exists function g, such that $x=g(y)$ and

$$g'(y) = -\frac{\partial F(x,y)/\partial y}{\partial F(x,y)/\partial x}$$

in some neighborhood of y_0. Then the tangent line at (x_0,y_0) to this curve is defined by

$$\frac{x-x_0}{y-y_0} = g'(y_0) = -\frac{F_y(x_0,y_0)}{F_x(x_0,y_0)}$$

which can be reformulated into

$$x = -\frac{F_y(x_0,y_0)}{F_x(x_0,y_0)}(y-y_0)+x_0$$

Similarly, if $\partial F(x_0,y_0)/\partial y \neq 0$, then the tangent line at (x_0,y_0) to this curve is defined by

$$y = -\frac{F_x(x_0,y_0)}{F_y(x_0,y_0)}(x-x_0)+y_0$$

b) Show that if (x_0,y_0) is a point of inflection of the curve, then the following equality holds,

$$\left(F''_{xx}{F'_y}^2 - 2F''_{xy}F'_xF'_y + F''_{yy}{F'_x}^2\right)(x_0,y_0) = 0$$

Since $F(x,y) \in \mathcal{C}^2$, we know $x=g(y)$ is also in $\mathcal{C}^2$. Therefore,

$$g''(y) = \left[-\frac{F_y(g(y),y)}{F_x(g(y),y)}\right]' = -\frac{[F_y(g(y),y)]'F_x(g(y),y) - F_y(g(y),y)[F_x(g(y),y)]'}{[F_x(g(y),y)]^2}$$

Since $x=g(y)$,

$$[F_y(g(y),y)]' = \frac{dF_y(g(y),y)}{dg(y)}g'(y) + \frac{dF_y(g(y),y)}{dy} = F_{yx}(g(y),y)g'(y) + F_{yy}(g(y),y)$$

and similarly,

$$[F_x(g(y),y)]' = F_{xx}g'(y) + F_{xy}(g(y),y)$$

We can find

$$g''(y) = -\frac{(F_{yx}(g(y),y)g'(y) + F_{yy}(g(y),y))F_x(g(y),y) - F_y(g(y),y)(F_{xx}g'(y) + F_{xy}(g(y),y))}{[F_x(g(y),y)]^2}$$

Subsititute $g'(y)$ into the above equation, we have

$$g''(y) = -\frac{F_{xx}F_y^2(g(y),y) - 2F_{xy}F_xF_y(g(y),y) + F_{yy}F_x^2(g(y),y)}{[F_x(g(y),y)]^3}$$

Since (x_0, y_0) is a point of inflection of the curve, $g''(y) = 0$. We yield that

$$\left(F_{xx}{F_y}^2 - 2F_{xy}F_xF_y + F_{yy}{F_x}^2\right)(x_0, y_0) = 0$$

Similarly, if $\partial F(x_0, y_0)/\partial y \neq 0$, let $f''(x) = 0$, and you will obtain exactly the same answer.

c) Find a formula for the curvature of the curve at the point (x_0, y_0).

Since the formula for curvature of the curve $(g(y), x)$ is given by

$$\kappa(y) = \frac{g''(y)}{[1 + (g'(x))^2]^{3/2}}$$

Using the same technique as part b), we can compute

$$\kappa = \frac{-F_x^2F_{yy}(x_0,y_0) + 2F_xF_yF_{xy}(x_0,y_0) - F_y^2F_{xx}(x_0,y_0)}{[F_x^2(x_0,y_0) + F_y^2(x_0,y_0)]^{3/2}}$$

Question 8.5-6. Show that the roots of the equation

$$z^n + c_1z^{n-1} + \cdots + c_n = 0$$

are smooth functions of the coefficients, at least when they are all distinct.

Suppose variable $\vec{z} = (z_1, \ldots, z_n)$, and variable $\vec{c} = (c_1, \ldots, c_n)$. Set $F(\vec{z}, \vec{c}) : \mathbb{R}^{2n} \mapsto \mathbb{R}^n$ with

$$F(\vec{z}, \vec{c}) = \begin{bmatrix} F_1(\vec{z}, \vec{c}) \\ F_2(\vec{z}, \vec{c}) \\ \vdots \\ F_n(\vec{z}, \vec{c}) \end{bmatrix} = \begin{bmatrix} z_1^n + c_1z_1^{n-1} + \ldots + c_n \\ z_2^n + c_1z_2^{n-1} + \ldots + c_n \\ \vdots \\ z_n^n + c_1z_n^{n-1} + \ldots + c_n \end{bmatrix}$$

For any coefficient vector $\vec{c}^0 = (c_1^0, \ldots, c_n^0)$, the equation $z^n + c_1^0z^{n-1} + \cdots + c_n^0 = 0$ has n distinct roots $z_1^0, z_2^0, \ldots, z_n^0$. Hence, we can find $\vec{z}^0 = (z_1^0, \ldots, z_n^0)$, such that $F(\vec{z}^0, \vec{c}^0) = \vec{0}$. Since all components of F are polynomials, so F is smooth function. Consider the partial derivative of F with respect to $\vec{z}$ at $(\vec{z}^0, \vec{c}^0)$,

$$\frac{\partial F(\vec{z}^0, \vec{c}^0)}{\partial \vec{z}} = \begin{bmatrix} \frac{\partial F_1}{\partial z_1} & \cdots & \frac{\partial F_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial z_1} & \cdots & \frac{\partial F_n}{\partial z_n} \end{bmatrix}(\vec{z}^0, \vec{c}^0) = \begin{bmatrix} \frac{\partial F_1(\vec{z}^0, \vec{c}^0)}{\partial z_1} & 0 & \cdots & 0 \\ 0 & \frac{\partial F_2(\vec{z}^0, \vec{c}^0)}{\partial z_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\partial F_n(\vec{z}^0, \vec{c}^0)}{\partial z_n} \end{bmatrix}$$

However, if a polynomial $P_n(x)$ of degree n has n distinct roots $x_1, \dots, x_n$, then we can write

$$P_n(x) = (x - x_i)R_{n-1}(x), \quad \text{where } R_n(x_i) \neq 0$$

which shows that

$$P_n'(x_i) = R_{n-1}(x_i) + (x_i - x_i)R_{n-1}'(x_i) = R_{n-1}(x_i) \neq 0$$

Hence, the determinant of $\partial F(\vec{z}^0, \vec{c}^0)/\partial \vec{z}$ is nonzero. In this case, we can apply IFT, and in a neighborhood of $(\vec{z}^0, \vec{c}^0)$, there exists a smooth function $\vec{z} = f(\vec{c})$. Since we choose $\vec{c}^0$ arbitrarily, we conclude that the roots are smooth functions of the coefficients, when the roots are distinct.

Question 8.6-1. Compute the Jacobian of the change of variable

$$\rho^{m-1} \sin^{m-2} \varphi_1 \sin^{m-3} \varphi_2 \cdots \sin \varphi_{m-2}$$

from polar coordinates to Cartesian coordinates in $\mathbb{R}^m$.

We prove it by induction, when $m = 2$, we have

$$x = \rho \cos \varphi_1, \quad y = \rho \sin \varphi_1$$

The Jacobian is our familiar one, which is r, satisfying the formula.

Now suppose when $m = n$, we have

$$x_1 \cos \varphi_1, \quad x_2 = \sin \varphi_1 \cos \varphi_2, \quad x_3 = \sin \varphi_1 \sin \varphi_2 \cos \varphi_3, \quad \cdots$$

$$x_{n-1} = \sin \varphi_1 \sin \varphi_2 \cdots \sin \varphi_{n-2} \cos \varphi_{n-1}, \quad x_n = \sin \varphi_1 \sin \varphi_2 \cdots \sin \varphi_{n-2} \sin \varphi_{n-1}$$

The Jacobian matrix is

$$\begin{bmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \varphi_1} & \cdots & \frac{\partial x_1}{\partial \varphi_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial r} & \frac{\partial x_n}{\partial \varphi_1} & \cdots & \frac{\partial x_n}{\partial \varphi_{n-1}} \end{bmatrix}$$

For simplicity, we denote it as

$$J_n = \begin{bmatrix} a_{x_1,r} & a_{x_1,\varphi_1} & \cdots & a_{x_1,\varphi_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{x_{n-1},r} & a_{x_{n-1},\varphi_1} & \cdots & a_{x_{n-1},\varphi_{n-1}} \\ a_{x_n,r} & a_{x_n,\varphi_1} & \cdots & a_{x_n,\varphi_{n-1}} \end{bmatrix}$$

We assume for this matrix, its determinant is

$$|J_n| = \rho^{n-1} \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \cdots \sin \varphi_{n-2}$$

Then, we consider when $m = n+1$, $x_1, \dots, x_{n-1}$ does not change, and the only two different variables are

$$x_n' = x_n \cos \varphi_n, \quad x_{n+1}' = x_n \sin \varphi_n$$

Hence, the new Jacobian is

$$J_{n+1} = \begin{bmatrix} a_{x_1,r} & a_{x_1,\varphi_1} & \cdots & a_{x_1,\varphi_{n-1}} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{x_{n-1},r} & a_{x_{n-1},\varphi_1} & \cdots & a_{x_{n-1},\varphi_{n-1}} & 0 \\ a_{x_n,r}\cos\varphi_n & a_{x_n,\varphi_1}\cos\varphi_n & \cdots & a_{x_n,\varphi_{n-1}}\cos\varphi_n & b_{x_n,\varphi_n} \\ a_{x_n,r}\sin\varphi_n & a_{x_n,\varphi_1}\sin\varphi_n & \cdots & a_{x_n,\varphi_{n-1}}\sin\varphi_n & b_{x_{n+1},\varphi_n} \end{bmatrix}$$

where

$$b_{x_n,\varphi_n} = \frac{\partial x'_n}{\partial \varphi_n} = -\rho\sin\varphi_1\cdots\sin\varphi_{n-1}\sin\varphi_n, \quad b_{x_{n+1},\varphi_n} = \frac{\partial x'_{n+1}}{\partial \varphi_n} = \rho\sin\varphi_1\cdots\sin\varphi_{n-1}\cos\varphi_n$$

We can expand the new Jacobian using the last column,

$$\begin{aligned} |J_{n+1}| &= b_{x_n,\varphi_n}(-1)^{2n+1}|M_{x_n,\varphi_n}| + b_{x_{n+1},\varphi_n}(-1)^{2n+2}|M_{x_{n+1},\varphi_n}| \\ &= \rho\sin\varphi_1\cdots\sin\varphi_{n-1}\sin\varphi_n|M_{x_n,\varphi_n}| + \rho\sin\varphi_1\cdots\sin\varphi_{n-1}\cos\varphi_n|M_{x_{n+1},\varphi_n}| \\ &= \rho\sin\varphi_1\cdots\sin\varphi_{n-1}\sin\varphi_n\sin\varphi_n|J_n| + \rho\sin\varphi_1\cdots\sin\varphi_{n-1}\cos\varphi_n\cos\varphi_n|J_n| \\ &= \rho\sin\varphi_1\cdots\sin\varphi_{n-1}|J_n| \\ &= (\rho\sin\varphi_1\cdots\sin\varphi_{n-1})(\rho^{n-1}\sin^{n-2}\varphi_1\sin^{n-3}\varphi_2\cdots\sin\varphi_{n-2}) \\ &= \rho^n\sin^{n-1}\varphi_1\sin^{n-2}\varphi_2\cdots\sin^2\varphi_{n-2}\sin\varphi_{n-1} \end{aligned}$$

We can see that when $m = n+1$, the Jacobian still satisfies our assumption, hence, we proved that for all $m \geq 2$, the Jacobian of the change of variable is

$$\rho^{m-1}\sin^{m-2}\varphi_1\sin^{m-3}\varphi_2\cdots\sin\varphi_{m-2}$$

Question 8.6-3. Let $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a smooth mapping satisfying the Cauchy-Riemann equations

$$\frac{\partial f^1}{\partial x^1} = \frac{\partial f^2}{\partial x^2}, \qquad \frac{\partial f^1}{\partial x^2} = -\frac{\partial f^2}{\partial x^1}$$

a) Show that the Jacobian of such a mapping is zero at a point if and only if $f'(x)$ is the zero matrix at that point.

Since f satisfies the Cauchy-Riemann equations

$$f'(x) = \begin{bmatrix} \frac{\partial f^1}{\partial x^1} & \frac{\partial f^1}{\partial x^2} \\ \frac{\partial f^2}{\partial x^1} & \frac{\partial f^2}{\partial x^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial f^1}{\partial x^1} & \frac{\partial f^1}{\partial x^2} \\ -\frac{\partial f^1}{\partial x^2} & \frac{\partial f^1}{\partial x^1} \end{bmatrix}$$

If the Jacobian of such a mapping is zero at a point x, then

$$\begin{vmatrix} \frac{\partial f^1}{\partial x^1} & \frac{\partial f^1}{\partial x^2} \\ -\frac{\partial f^1}{\partial x^2} & \frac{\partial f^1}{\partial x^1} \end{vmatrix} = \left(\frac{\partial f^1}{\partial x^1}\right)^2 + \left(\frac{\partial f^1}{\partial x^2}\right)^2 = 0$$

which means both of the above partial derivatives are zero. Again, using Cauchy-Riemann equation, we can obtain a zero Jacobian matrix. Thus, the Jacobian is zero at a point x if and only if $f'(x)$ is the zero matrix at x.

b) Show that if $f'(x) \neq 0$, then the inverse f^{-1} to the mapping f is defined in a neighborhood of f and also satisfies the Cauchy-Riemann equations.

Since $f'(x) \neq 0$, the Jacobian is nonzero, and $f'(x)$ is invertible. $f(x)$ is smooth, so we can apply inverse function theorem, and there exists the inverse of this function f^{-1} in a neighborhood of $f(x)$ and the derivative can be calculated by (set $y = f(x)$) $(f^{-1})'(y) = [f'(x)]^{-1}$, which is

$$(f^{-1})'(y) = \begin{bmatrix} \frac{\partial f^1}{\partial x^1} & \frac{\partial f^1}{\partial x^2} \\ -\frac{\partial f^1}{\partial x^2} & \frac{\partial f^1}{\partial x^1} \end{bmatrix}^{-1} = \frac{1}{\det(f'(x))} \begin{bmatrix} \frac{\partial f^1}{\partial x^1} & -\frac{\partial f^1}{\partial x^2} \\ \frac{\partial f^1}{\partial x^2} & \frac{\partial f^1}{\partial x^1} \end{bmatrix} = \begin{bmatrix} \frac{\partial (f^{-1})^1}{\partial x^1} & \frac{\partial (f^{-1})^1}{\partial x^2} \\ \frac{\partial (f^{-1})^2}{\partial x^1} & \frac{\partial (f^{-1})^2}{\partial x^2} \end{bmatrix}$$

Hence we have

$$\frac{\partial (f^{-1})^1}{\partial x^1} = \frac{\partial (f^{-1})^2}{\partial x^2}, \qquad \frac{\partial (f^{-1})^1}{\partial x^2} = -\frac{\partial (f^{-1})^2}{\partial x^1}$$

which shows the inverse of f also satisfies Cauchy-Riemann equations.

Question 8.6-5. Show that the rank of a smooth mapping $f : \mathbb{R}^m \mapsto \mathbb{R}^n$ is a lower semicontinuous function, that is rank $f(x) \geq$ rank $f(x_0)$ in a neighborhood of a point $x_0 \in \mathbb{R}^m$.

For any x_0, if rank $f(x_0)$ is zero, then we are done. If not, suppose rank $f(x_0)$ is $k \geq 1$, then we consider the rank of the differential matrix $f'(x)$ of $f(x)$, because this is the definition of rank of a function. Denote this matrix as $A(x_0)$, and all entries of $A(x_0)$ are in the form of $\partial f_i(x_0)/\partial x^j$, where $i = 1, \cdots, n$ and $j = 1, \cdots, m$. Since $A(x_0)$ is of rank k, there exists a nonsingular $k \times k$ submatrix of $A(x_0)$, called $A^k(x_0)$, we have det $A^k(x_0) \neq 0$. Since f is smooth mapping, all partial derivatives are continuous, and the determinant of $A^k(x)$ is just a composite function of $\partial f_i/\partial x^j$ for all x, so the determinant of $A^k(x)$ is continuous. Therefore, there exists a neighborhood of x_0, such that for all x in this neighborhood, det $A^k(x) \neq 0$. This shows that for all x in this neighborhood, $A(x)$ has at least a $k \times k$ (may be larger than k) nonsingular submatrix $A^k(x)$, meaning that the rank of $A(x)$ is larger than or equal to k. Hence rank $f(x) \geq$ rank $f(x_0)$ in this neighborhood. So rank of a smooth mapping is indeed lower-semicontinuous.

Question 8.6-6.

a) Give a direct proof of Morse's lemma for functions $f : \mathbb{R} \mapsto \mathbb{R}$.

We first restate what we need to prove,

> ***Morse lemma.*** If $f : G \mapsto \mathbb{R}$ is a function of class $\mathcal{C}^{(3)}(G; \mathbb{R})$ defined on an open set $G \subset \mathbb{R}$ and $x_0 \in G$ is a nondegenerate critical point of f, then there exists a diffeomorphism $g : V \mapsto U$ of some neighborhood of 0 onto a neighborhood U of x_0 such that
>
> $$(f \circ g)(y) = f(x_0) \pm y^2$$
>
> for all $y \in V$. (Actually, the sign in front of y^2 depends only the sign of $f''(x_0) \neq 0$). □

By linear transformation we can reduce the problem to the case when $x_0 = 0$ and $f(x_0) = 0$. Then by Taylor expansion at $x = 0$,

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + o(x^2) \\ &= f(0) + \frac{1}{2}f''(0)x^2 + o(x^2) \\ &= f(0) \pm g^2(x) \end{aligned}$$

where $y = g(x)$ is defined by

$$y = g(x) = x\sqrt{\frac{|f''(0)|}{2} + o(1)}$$

Let's check the first derivative of $g(x)$,

$$\left.\frac{dy}{dx}\right|_{x=0} = \left.\sqrt{\frac{|f''(0)|}{2} + o(1)}\right|_{x=0} + \left.\frac{x[o(1)]'}{2\sqrt{\frac{|f''(0)|}{2} + o(1)}}\right|_{x=0}$$

Since when $x = 0$, $o(1) = 0$, and $f''(0) \neq 0$, we have

$$\left.\frac{dy}{dx}\right|_{x=0} = \sqrt{\frac{|f''(0)|}{2}} \neq 0$$

Since here $g(x)$ is obviously smooth function, by Inverse Function Theorem, there exists a neighborhood of $U(0)$ of 0 and a neighborhood $V(0)$ of 0 such that $g : U(0) \mapsto V(0)$ is a smooth diffeomorphism. Thus, $x = h(y) = g^{-1}(y)$, and

$$(f \circ h)(y) = f(x) = f(0) \pm y^2$$

b) Determine whether Morse's lemma is applicable at the origin to the following functions,

$$f(x) = x^3; \qquad f(x) = x \sin\frac{1}{x}; \qquad f(x) = e^{-1/x^2}\sin^2\frac{1}{x};$$
$$f(x, y) = x^3 - 3xy^2; \qquad f(x, y) = x^2$$

We assume the definition of all function above at origin is just the limit of that function at origin. The first one is obviously not applicable because $f''(0) = 0$, the critical point is degenerate. The second one is also not applicable because $f(x)$ is not continuously differentiable. The third one is also not applicable because though it is smooth function, any order derivative of it at $x = 0$ will be 0, hence the critical point at $x = 0$ is degenerate. The fifth one is obviously not applicable, because $f(x, y) = x^2$ is independent of y, meaning that any partial derivative containing y will be zero, so the only nonzero entry of its hessian is in the first row, first column. Hence, $x = 0$ is also degenerate critical point.

The fourth one is still not applicable, because

$$\frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial y^2} = -6x, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -6y$$

Hence the hessian at $x = 0$ is a zero matrix, so $x = 0$ is degenerate critical point.

c) Show that nondegenerate critical points of a function $f \in \mathcal{C}^{(3)}(\mathbb{R}^m, \mathbb{R})$ are isolated (each of then has a neighborhood in which it is the only critical point of f).

This is trivial, because if $x^0 = (x_1^0, \ldots, x_m^0)$ is a nondegenerate critical point, then we can apply Morse lemma at x^0, and we will have a new coordinate system $y_1, y_2, \cdots, y_m$ instead of $x_1, x_2, \ldots, x_m$ in $U(x^0)$ such that

$$h(y) = (f \circ g)(y) = f(x^0) - (y_1)^2 - \cdots - (y_k)^2 + (y_{k+1})^2 + \cdots + (y_m)^2$$

Thus, $x = g(y) \in U(x^0)$ and we have

$$h'(y) = f'(g(y)) \cdot g'(y) = \begin{bmatrix} -2y_1 & \cdots & -2y_k & 2y_{k+1} & \cdots & 2y_m \end{bmatrix}$$

Hence, $h'(y) = 0$ if and only if all $y_i = 0$. Thus in the new coordinate, there is only one critical point, which is the origin. In this case, if f has critical point in $U(x^0)$, then $f'(g(y)) = 0$, and this yields the unique solution $y = (y_1, \ldots, y_m) = (0, \ldots, 0)$. Thus $g(0)$ is the only critical point of f in $U(x^0)$, and we have known that x^0 is a critical point, so $x^0 = g(0)$, and the proof is finished.

d) Show that the number k of negative squares in the canonical representation of a function in the neighborhood of a nondegenerate critical point is independent of the reduction method, that is, independent of the coordinate system in which the function has canonical form. This number is called the *index of the critical point*.

Recall that when we prove Morse lemma, we first write f into a quadratic form

$$f(x_1, \ldots, x_m) = \sum_{i,j=1}^{m} x_i x_j h_{ij}(x_1, \ldots, x_m)$$

where $h_{ij} = h_{ji}$. Then we need to reduce it into diagonal form (canonical form), and we need to prove the index of critical point is independent of this procedure. No matter what h_{ij} you choose, it must satisfy

$$h_{ij}(x^0) = \frac{\partial^2 f}{\partial x_i \partial x_j}(x^0)$$

But the hessian matrix $H_f(x^0)$ is nonsingular at x^0, meaning that all of its eigenvalue is nonzero. Suppose k of them are positive, and $m-k$ of them are negative, and matrix $A(x) = [h_{ij}(x)]$ has the same eigenvalue as $H_f(x^0)$ at $x = x^0$. We claim that the eigenvalue of a matrix is a continuous function of the elements in this matrix. This can be verified by the following procedure:

- the characteristic polynomial $p(\lambda)$ of a matrix is given by det $(A - \lambda I)$, and by the definition of determinant, the coefficients of $p(\lambda)$ is a continuous function of all entries of this matrix;
- in Question 8.5-6, we prove that the roots of a equation is a smooth function of all its coefficients. Consider the roots of equation $p(\lambda) = 0$, which is just the eigenvalues of that matrix, hence all eigenvalues are a continuous function of all entries of this matrix.

Therefore, in a neighborhood of $A(x_0)$, the eigenvalue will preserve its sign (the positive one at x_0 will always be positive throughout the neighborhood). At each point of this neighborhood, applying Sylvester's law of inertia (Advanced Algebra), the coefficient matrix of a quadratic form has a unique canonical form, i.e., the number of negative elements in the diagonal of D, where $D = SA(x)S^t$ is always the same (independent of S, which is an invertible matrix representing the change of coordinate).

The negative squares in the canonical representation of a function in Morse lemma is exactly those negative one in D, so the independence of the number of negative squares k on reduction method (S) is proved throughout a proper neighborhood.