MAT2006: Elementary Real Analysis
Diagnostic Test

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Question 1. Set $x_n = \frac{\sin 1}{2} + \frac{\sin 2}{2^2} + \cdots + \frac{\sin n}{2^n}$ for $n = 1, 2, \cdots$. Prove that $\{x_n\}$ converges.

Let $a_k = \sin k$ and $b_k = 1/2^k$, then x_n can be regarded as a partial sum of $\sum a_k b_k$. Since b_k is obviously decreasing to zero as $k \to \infty$, we need to prove $\sum_{k=1}^n a_k$ is bounded for all n so as to apply Dirichlet's Test.

$$\begin{aligned}\sum_{k=1}^n a_k &= \sum_{k=1}^n \sin k \\ &= \frac{\sin \frac{1}{2} \sin 1 + \sin \frac{1}{2} \sin 2 + \cdots + \sin \frac{1}{2} \sin n}{\sin \frac{1}{2}} \\ &= \frac{\left(\cos \frac{1}{2} - \cos \frac{3}{2}\right) + \left(\cos \frac{3}{2} - \cos \frac{5}{2}\right) + \cdots + \left(\cos \frac{2n-1}{2} - \cos \frac{2n+1}{2}\right)}{2 \sin \frac{1}{2}} \\ &= \frac{\cos \frac{1}{2} - \cos \frac{2n+1}{2}}{2 \sin \frac{1}{2}}\end{aligned}$$

Thus we can see for all n it is bounded. Apply Dirichlet's Test, i.e., $\sum a_k b_k$ converges, which finishes the proof.

Question 2. Evaluate each of the following limits, showing all reasoning.

(a) $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}}$

The procedure is as follows

$$\begin{aligned}\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}} &= \exp\left\{\lim_{x \to 0} \left(\frac{1}{1-\cos x} \ln \frac{\sin x}{x}\right)\right\} && (1) \\ &= \exp\left\{\lim_{x \to 0} \left(\frac{1}{2\sin^2(x/2)} \ln \frac{\sin x}{x}\right)\right\} && (2) \\ &= \exp\left\{\lim_{x \to 0} \left(\frac{1}{2(x/2)^2} \ln \frac{\sin x}{x}\right)\right\} && (3) \\ &= \exp\left\{\lim_{x \to 0} \left(\frac{2}{x^2} \ln \frac{\sin x}{x}\right)\right\} && (4) \\ &= \exp\left\{\lim_{x \to 0} \frac{2}{x^2} \left(\frac{\sin x}{x} - 1\right)\right\} && (5) \\ &= \exp\left\{2 \lim_{x \to 0} \frac{\sin x - x}{x^3}\right\} && (6) \\ &= \exp\left\{2 \lim_{x \to 0} \frac{-\sin x}{6x}\right\} && (7) \\ &= e^{-1/3} && (8)\end{aligned}$$

Note that for (2) → (3) and (4) → (5) we apply Equivalent Infinitesimal, as $x \to 0$,

$$\sin\frac{x}{2} \sim \frac{x}{2} \qquad \ln\frac{\sin x}{x} \sim (\frac{\sin x}{x} - 1)$$

For (6) → (7) we apply L'Hôpital's rule twice.

(b) $\lim_{x\to 0}\int_0^1 \frac{x^n}{1+\sqrt{x}}\,dx$

Since $x \in [0,1]$, we have

$$0 \le \frac{x^n}{1+\sqrt{x}} \le \frac{x^n}{1+\sqrt{0}} = x^n$$

which means

$$0 \le \int_0^1 \frac{x^n}{1+\sqrt{x}}\,dx \le \int_0^1 x^n\,dx = \frac{1}{n+1}$$

By Squeeze Theorem, the integral will tend to zero since both sides has limit zero when $n \to \infty$, thus,

$$\lim_{x\to 0}\int_0^1 \frac{x^n}{1+\sqrt{x}}\,dx = 0$$

Question 3. Justify that e is an irrational number.

Suppose e is rational, then $e = p/q$, where $p, q \in \mathbb{Z}, q > 0$ and $\gcd(p,q) = 1$. We have

$$e = \frac{p}{q} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!} + \epsilon_q$$

which implies

$$p(q-1)! = q!\left(1 + \frac{1}{1!} + \cdots + \frac{1}{q!}\right) + q!\epsilon_q$$

It's obvious that $q!\epsilon_q$ is integer, because the other two terms above are both integers.

Now we prove that

$$\frac{1}{(n+1)!} < \epsilon_n < \frac{1}{n!n}$$

Since

$$\epsilon_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots$$

The L.H.S. of the inequality is true.

For R.H.S.,

$$\epsilon_n \le \frac{1}{(n+1)!}\left[1 + \frac{1}{n+2} + \frac{1}{(n+2)^2} + \cdots\right] = \frac{1}{(n+1)!}\frac{1-1/(n+2)^n}{1-1/(n+2)} \le \frac{n+2}{(n+1)!(n+1)} < \frac{1}{n!n}$$

Let $n = q$, we have

$$\frac{1}{(q+1)!} < \epsilon_q < \frac{1}{q!q} \Longrightarrow 0 < \frac{1}{q+1} < q!\epsilon_q < \frac{1}{q}$$

which means $q!\epsilon_q \notin \mathbb{Z}$. Thus, contradiction leads to the result that e is irrational.

Question 4. Every rational number x can be written in the form $x = p/q$, where $q > 0$, and p and q are integers without any common divisors. When $x = 0$, we take $q = 1$.

Define a function $f : R \to R$ to be $f(x) = 1/q$, when x is rational and $x = p/q$ (p and q are in the previous defined written form of rational number); $f(x) = 0$, when x is irrational.

Find out all points of continuity and discontinuity of the function $f(x)$. Prove your result.

We claim that continuous points are all irrational number, and discontinuous points are all rational number.

$\forall a \in \mathbb{R}$, let k be the closest integer to a, and then $a \in (k-1, k+1)$.

$\forall \epsilon > 0$, take N, s.t., $\frac{1}{N} < \epsilon$. In $(k-1, k+1)$, the number of p/q s.t., $0 < q \le N$ is finite.

For $0 < q \le N$, denote the closest p/q (not equal to a) to a as b, then let $\delta = |b - a|$. Then $\forall x$ s.t. $0 < |x - a| < \delta$, we have $|f(x)| < \frac{1}{N} < \epsilon$, which means $\lim_{x \to a} f(x) = 0$.

Therefore, the limit of every points is zero for funciton f. If $a \in \mathbb{Q}$, $f(a = p/q) = 1/q \ne 0$ or $f(0) = 1 \ne 0$, which means a is a removable discontinuous point. If $a \notin \mathbb{Q}$, then $f(a) = 0$, which means a is a continuous point.