MAT2006: Elementary Real Analysis
Quiz 1

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Question 1. Let $A = (-1, 1)$ and $B = [-1, 1]$. Define $f(x) = x$ and $g(x) = \frac{x}{2}$. Then $f : A \mapsto B$ and $g : B \mapsto A$ are both one-to-one. Construct an one-to-one onto mapping between A and B from f and g.

Consider $f : A \mapsto B$ where $A = (-1, 1)$, $B = [-1, 1]$, $D = f(A) = (-1, 1)$, and $B \setminus D = \{-1, 1\}$. Apply Schröder-Bernstein,

$$S = g \circ f(B \setminus D) \cup g \circ f \circ g \circ f(B \setminus D) \cup \cdots = \left\{-\frac{1}{2}, \frac{1}{2}\right\} \cup \left\{-\frac{1}{4}, \frac{1}{4}\right\} \cup \cdots = \pm\frac{1}{2^n},\ n \in \mathbb{N}^+$$

$$F(x) = \begin{cases} f(x) & x \in A \setminus S \\ g^{-1}(x) & x \in S \end{cases}$$

Therefore, the bijective mapping is

$$F(x) = \begin{cases} x & x \in (-1, 1) \setminus \{\pm 2^{-n} : n \in \mathbb{N}^+\} \\ 2x & x = \pm 2^{-n}\ (n \in \mathbb{N}^+) \end{cases}$$

Question 2. Evaluate the following limits

(a) $\lim_{x\to 0+} x^x$

See Assignment 1 Question 3.2-3 a).

(b) $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}}$

See Diagnostic Test Question 2(a).

Question 3. Suppose that $a_n \geq 0$ for all n, and $\sum_{n=1}^{\infty} a_n$ converges. Then the series $\sum_{n=1}^{\infty} A_n$, where $A_n = \sqrt{\sum_{k=n}^{\infty} a_k} - \sqrt{\sum_{k=n+1}^{\infty} a_k}$, also converges, and $a_n = o(A_n)$ as $n \to \infty$.

See Assignment 1 Question 3.2-6 c).

Question 4. Prove that the subset Q of all rational numbers in $\mathbb{R}$ is not the countable intersection of open sets. (Hint: Use Baire Category Theorem.)

Suppose Q is the countable intersection of open sets E_k's, then E_k^c is closed and

$$Q = \bigcap_{k=1}^{\infty} E_k = \left(\bigcup_{k=1}^{\infty} E_k^c\right)^c \implies Q^c = \bigcup_{k=1}^{\infty} E_k^c$$

Notice that E_k^c is no where dense because if not, then $\overline{E_k^c} = E_k^c$ will contain an open set, in which there exists a neighborhood containing both rationals and irrationals. This is impossible because E_k^c only contains irrational numbers, so E_k^c is nowhere dense. This shows that Q^c is of first category. It is easy to show that Q, as a countable subset of $\mathbb{R}$, is of first category. Thus, $\mathbb{R} = Q \cup Q^c$ is of first category, which is a contradiction to the fact that $\mathbb{R}$ is of second category. Therefore, we can conclude that Q is not the countable intersection of open sets.

Question 5. Suppose that $f : \mathbb{R} \mapsto \mathbb{R}$ is uniformly continuous. Is $\lim_{x\to\infty} \frac{f(x)}{x^2} = 0$? Justify your answer.

Since $f : \mathbb{R} \mapsto \mathbb{R}$ is uniformly continuous, for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in \mathbb{R}$, $|x - y| < \delta$, and $|f(x) - f(y)| < \epsilon$. Take $\epsilon = 1$, there exists $\delta_0 > 0$ such that for all x, y satisfying $|x - y| < \delta_0$, $|f(x) - f(y)| < 1$.

Let $y = 0$, we have for all $|x| < \delta_0$, $|f(x) - f(0)| < 1$, which means $|f(x)| < 1 + |f(0)|$. Let $y = \frac{1}{2}\delta_0$, for all $|x - \frac{1}{2}\delta_0| < \delta_0$, $|f(x) - f(\frac{1}{2}\delta_0)| < 1$. Therefore, for all $x \in (-\frac{1}{2}\delta_0, \frac{3}{2}\delta_0)$,

$$|f(x)| < 1 + \left|f\left(\frac{1}{2}\delta_0\right)\right| < 1 + 1 + |f(0)| = 2 + |f(0)|$$

Similarly, let $y = \delta_0$, for all $x \in (0, 2\delta_0)$, we can obtain $|f(x)| < 3 + |f(0)|$. By induction, let $y = \frac{n-1}{2}\delta_0$, for all $x \in (\frac{n-3}{2}\delta_0, \frac{n+1}{2}\delta_0)$, we have $|f(x)| < n + |f(0)|$. Since $x > \frac{n-3}{2}\delta_0$, $n < \frac{2x}{\delta_0} + 3$. Therefore,

$$\lim_{x\to\infty} \frac{|f(x)|}{x^2} \leq \lim_{x\to\infty} \frac{n + |f(0)|}{x^2} \leq \lim_{x\to\infty} \left(\frac{2}{\delta_0 x} + \frac{3 + |f(0)|}{x^2}\right) \to 0$$

In conclusion, $\lim_{x\to\infty} \frac{f(x)}{x^2} = 0$.