MAT2006: Elementary Real Analysis
Quiz 2

李肖鹏 (116010114)

Question 1. Let $f(x,y) = e^x \cos y$, $(x,y) \in \mathbb{R}^2$.

(i) Compute the differential $\mathrm{D}f(x,y)$.

It is easy to compute the partial derivatives $f_x = e^x \cos y$ and $f_y = -e^x \sin y$. Thus,

$$\mathrm{D}f(x,y) = \begin{bmatrix} f_x & f_y \end{bmatrix} = \begin{bmatrix} e^x \cos y & -e^x \sin y \end{bmatrix}$$

(ii) Compute the Hessian $\mathrm{D}^2 f(x,y)$.

Also compute the second order partial derivatives $f_{xx} = e^x \cos y$ and $f_{xy} = f_{yx} = -e^x \sin y$ and $f_{yy} = -e^x \cos y$. Thus,

$$\mathrm{D}^2 f(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ -e^x \sin y & -e^x \cos y \end{bmatrix}$$

(iii) Compute the Taylor's formula for f around $(0,0)$ to the second order.

The Taylor's formula for f around $(0,0)$ to the second order is

$$\begin{aligned} f(h_1,h_2) &= f(0,0) + f_x(0,0)h_1 + f_y(0,0)h_2 \\ &\quad + \frac{1}{2!}[f_{xx}(0,0)h_1^2 + f_{xy}(0,0)h_1h_2 + f_{yx}(0,0)h_2h_1 + f_{yy}(0,0)h_2^2] + o(h_1^2 + h_2^2) \\ &= 1 + h_1 + \frac{1}{2}(h_1^2 - h_2^2) + o(h_1^2 + h_2^2) \end{aligned}$$

Therefore, the Taylor's formula for f around $(0,0)$ is $f(x,y) = 1 + x + \frac{1}{2}(x^2 - y^2) + o(x^2 + y^2)$.

Question 2. Find all critical points of $f(x,y) = (y - x^2)(y - 3x^2)$ and determine whether f has a (local) maximum, (local) minimum, or saddle at each of these critical points.

Notice that $f(x,y) = 3x^4 - 4x^2y + y^2$, so consider the first order conditions, we have

$$\begin{cases} f_x = 12x^3 - 8xy = 0 \\ f_y = -4x^2 + 2y = 0 \end{cases} \implies \begin{cases} x = 0 \\ y = 0 \end{cases}$$

Therefore, $(0,0)$ is the only critical points of f. However, $(0,0)$ is not local maximum or minimum because on the curve $y=2x^2$, $f(x,2x^2)=-x^4$, where $x=0$ is a strict local maximum. On the curve $y=4x^2$, $f(x,4x^2)=3x^4$, where $x=0$ is a strict local minimum.

This shows that on any neighborhood of $(0,0)$, there are some points on $y=2x^2$ which have smaller function value than $f(0,0)$; there are also some points on $y=4x^2$ which have larger function value than $f(0,0)$. Therefore, $f(0,0)$ is neither local minimum nor maximum, and $(0,0)$ is a saddle point.

Question 3. Let $F(r,\theta)=f(g(r,\theta))$, where $g(r,\theta)=(x,y)$ with $x=r\cos\theta$, $y=r\sin\theta$, and $f(x,y)=e^{xy}$. Compute $\mathrm{D}F(r,\theta)$.

By chain rule,

$$\mathrm{D}F(r,\theta)=\mathrm{D}_{(r,\theta)}f(g(r,\theta))=\mathrm{D}_{(x,y)}f(g(r,\theta))\cdot\mathrm{D}_{(r,\theta)}g(r,\theta)=\mathrm{D}_{(x,y)}f(x,y)\cdot\mathrm{D}_{(r,\theta)}g(r,\theta)$$

Compute the differential,

$$\mathrm{D}_{(x,y)}f(x,y)=\begin{bmatrix}f_x & f_y\end{bmatrix}=\begin{bmatrix}ye^{xy} & xe^{xy}\end{bmatrix}$$

$$\mathrm{D}_{(r,\theta)}g(r,\theta)=\begin{bmatrix}x_r & x_\theta\\ y_r & y_\theta\end{bmatrix}=\begin{bmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{bmatrix}$$

Therefore, we can compute

$$\mathrm{D}F(r,\theta)=\begin{bmatrix}ye^{xy} & xe^{xy}\end{bmatrix}\cdot\begin{bmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{bmatrix}=e^{r^2\sin\theta\cos\theta}\begin{bmatrix}r\sin 2\theta & r^2\cos 2\theta\end{bmatrix}$$

Question 4. Let $f(x,y)\in\mathcal{C}^1(D;\mathbb{R})$ where D is a (connected) domain in $\mathbb{R}^2$. Suppose that $\frac{\partial f}{\partial y}\equiv 0$ in D.

(i) Is f independent of the variable y in D if D is a convex domain?

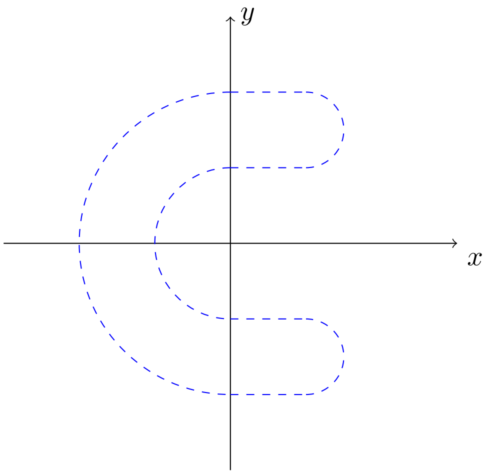
Yes, if D is convex domain, then the set $E=\{(x,y)\,|\,x=x_0\}\cap D$ is connected for any fixed x_0. Suppose f is dependent on y in D, then there exists (x_1,y_1) and (x_1,y_2) where $y_1\neq y_2$ such that $f(x_1,y_1)\neq f(x_1,y_2)$. Since E is one dimensional and connected, it is an interval, so we can apply MVT

$$f(x_1,y_1)-f(x_1,y_2)=f_y(x_1,\xi)\left(\begin{bmatrix}x_1\\ y_1\end{bmatrix}-\begin{bmatrix}x_1\\ y_2\end{bmatrix}\right)$$

where ξ lies between y_1,y_2. We know LHS $\neq 0$ but $f_y(x,\xi)=0$, so RHS $=0$. This is a contradiction, and hence f is independent on y in D.

(ii) Does the answer in (i) change if D is an arbitrary domain?

Yes. Suppose D is a domain drawn (blue region) in the graph below:



and the function f is defined by

$$f(x,y) = \begin{cases} -e^{-1/x^2} & \text{if } x > 0 \text{ and } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

For such a function $f(x,y)$, $f(x,y) \in \mathcal{C}^1(D;\mathbb{R})$, and $f_y \equiv 0$. However, for all $x > 0$, there exists $y_1 < 0$ and $y_2 > 0$ such that $f(x,y_1) \neq f(x,y_2)$. Therefore, the answer in (i) changes if D is nonconvex.

Question 5. Let $f : \mathbb{R}^2 \mapsto \mathbb{R}^3$ be given by $f(x,y) = (x+y^3, xy, y+y^2)$. Can the range of f be "straightened out" or "flattened out" near $(0,0)$?

First compute the differential

$$\mathrm{D}f(x,y) = \begin{bmatrix} 1 & 3y^2 \\ y & x \\ 0 & 1+2y \end{bmatrix}, \quad \mathrm{D}f(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

It is trivial that rank of $\mathrm{D}f(0,0)$ is 2. Since $\det(M_2) = 1+2y$, where M_2 is the matrix obtained by removing the second row of $\mathrm{D}f(x,y)$, we can see that in a small neighborhood of $(0,0)$, $\det(M_2) \neq 0$, so $\mathrm{D}f(x,y)$ will have at least rank 2 inside this neighborhood. However, $\mathrm{D}f(x,y)$ has at most rank 2, so it will have constant rank 2 inside this neighborhood of $(0,0)$. Given that f is a function such that $f(0,0) = (0,0,0)$, $f \in \mathcal{C}^\infty(\mathbb{R}^2;\mathbb{R}^3)$, and f has constant rank 2 near $(0,0)$, by Rank Theorem, there exists two diffeomorphisms $u = \phi(x,y)$ near $(0,0)$ amd $v = \psi(a,b,c)$ near $(0,0,0)$ such that $v = \psi \circ f \circ \phi^{-1}(u_1,u_2)$ has the representation $(u_1,u_2) \mapsto (u_1,u_2,0)$ in the neighborhood of $\phi(0,0)$. Therefore, the range of f can be "straightened out" near $(0,0)$.