MAT3006*: Real Analysis
Homework 1

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Page 24, Problem 44. Let p be a natural number greater than 1, and x a real umber, $0 < x < 1$. Show that there is a sequence $\{a_n\}$ of integers with $0 \le a_n < p$ for each n such that $x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$ and that this sequence is unique except when x is of the form q/p^n, in which case there are exactly two such sequences. Show that, conversely, if $\{a_n\}$ is any sequence of integers with $0 \le a_n < p$, the series $\sum_{n=1}^{\infty} \frac{a_n}{p^n}$ converges to a real number x with $0 \le x \le 1$.

Notice that p is integer at least 2, so given any $x \in (0,1)$, in the first step, we divide $(0,1)$ into p disjoint subintervals $I_{1,i}$, $i = 1, \dots, p$ with equal length $1/p$. Notice that each $I_{1,i}$ is open and if x is equal to one of two end points of $I_{1,i}$, then x must be of the form q/p for some integer $1 \le q \le p-1$. In this case we can either set $a_1 = q$ and set all other $a_n = 0$ or set $a_1 = q-1$ and set all other $a_n = p-1$. If x is none of the end points of those subintervals, then x must lie in $I_{1,i}$ for some i, and we let $a_1 = i-1$.

In the second step, we divide I_{1,a_1+1} into p disjoint subintervals $I_{2,i}$, $i = 1, \dots, p$ with equal length $1/p$. Then we repeat exactly the same thing as in step one, if x is one of end points of $I_{2,i}$, then it has a form of q/p^2 and all a_n will be defined in two different ways, so we can stop. If not, then we obtain a_2 and continue. Therefore, in the end, if x is the end point of any subinterval $I_{k,i}$ for $k = 1, 2, \dots$ and $i = 1, 2, \dots, p$, we can obtain two different sequences $\{a_n\}$. If not, then we can still obtain $\{a_n\}$ which satisfies

$$\frac{a_1}{p} + \cdots + \frac{a_n}{p^n} < x < \frac{a_1}{p} + \cdots + \frac{a_n}{p^n} + \frac{1}{p^n}$$

Since LHS and RHS are both nondecreasing and bounded by a geometric series, so they are both convergent and if we take the limit $n \to \infty$ on both sides, we can obtain $x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$.

Now it suffices to show that if we do not consider the case when there exists some n_0 such that $a_n = p-1$ for all $n \ge n_0$, which only appears when $x = q/p^n$, for each $x \in (0,1)$, the a_n we can obtain is unique. Suppose

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n} = \sum_{n=1}^{\infty} \frac{b_n}{p^n}$$

but $a_1 \ge b_1 + 1$, then

$$\sum_{n=1}^{\infty} \frac{b_n}{p^n} = \sum_{n=1}^{\infty} \frac{a_n}{p^n} \ge \frac{1}{p} + \frac{b_1}{p} + \sum_{n=2}^{\infty} \frac{a_n}{p^n} \Longrightarrow \sum_{n=2}^{\infty} \frac{b_n - a_n}{p^n} \ge \frac{1}{p}$$

However, we also have

$$\sum_{n=2}^{\infty} \frac{b_n - a_n}{p^n} \le \sum_{n=2}^{\infty} \frac{p-1}{p^n} = \frac{1}{p}$$

Therefore, $b_n - a_n = p-1$ for all $n \geq 2$, which shows $b_n = p-1$ and $a_n = 0$ for all $n \geq 2$. This means there exists some n_0 such that $a_n = p-1$ for all $n \geq n_0$, which is excluded by us. Similarly we can derive the same contradiction for $a_1 \leq b_1 - 1$. Thus, $a_1 = b_1$. Similarly, we can obtain $a_n = b_n$ for all $n \geq 1$.

Conversely, since $0 \leq a_n \leq p-1$, we have

$$0 = \sum_{n=1}^{\infty} \frac{0}{p^n} \leq \sum_{n=1}^{\infty} \frac{a_n}{p^n} \leq \sum_{n=1}^{\infty} \frac{p-1}{p^n} = 1$$

where the last equality is because

$$\sum_{n=1}^{k} \frac{p-1}{p^n} = \sum_{n=1}^{k} \frac{1}{p^{n-1}} - \sum_{n=1}^{k} \frac{1}{p^n} = \sum_{n=0}^{k-1} \frac{1}{p^n} - \sum_{n=1}^{k} \frac{1}{p^n} = 1 - \frac{1}{p^k} \to 1$$

as $k \to \infty$. Thus, $\sum_{n=1}^{\infty} \frac{a_n}{p^n}$ is bounded by $[0,1]$. Notice that a_n is nonnegative, so the partial sum of this series is nondecreasing, which means the series converges to some number in $[0,1]$.

Extra Problem 1. Let A and B be sets. Suppose there exists injective mappings $f : A \mapsto B$ and $g : B \mapsto A$. Prove that $A \sim B$.

Denote $C = g(B)$ and $D = f(A)$. Let $E = B \setminus D$, and

$$S = g(E) \cup g[f \circ g(E)] \cup g[f \circ g \circ f \circ g(E)] \cup \cdots$$

Define $F : A \mapsto B$ by

$$F(a) = \begin{cases} f(a) & a \in A \setminus S \\ g^{-1}(a) & a \in S \end{cases}$$

Now we claim that F is bijective. First, we prove that F is surjective. Given $b \in B$, $g(b)$ is either in S or not in S. If $g(b) \in S$, then $F(g(b)) = g^{-1}(g(b)) = b$, which means such b can be attained by $g(b) \in A$. If $g(b) \notin S$, then $b \in D$. This means there exists $a \in A$ such that $f(a) = b$. Furthermore, $a \notin S$, because if yes, then $g \circ f(a) \in S$, which means $g(b) \in S$, contradiction. Thus, $a \in A \setminus S$, and $F(a) = f(a) = b$. This implies that F is surjective.

To prove it is injective, suppose not, then there exists $a_1 \in A \setminus S$ and $a_2 \in S$ such that $F(a_1) = F(a_2)$, i.e., $f(a_1) = g^{-1}(a_2)$, i.e., $g \circ f(a_1) = a_2$. Since $a_2 \in S$, $g \circ f(a_1) \in S$, and $S = \bigcup_{i=1}^{\infty} A_i$, where A_i denotes the i-th subset in the definition of S, e.g., $A_1 = g(E)$, and $A_2 = g[f \circ g(E)]$. It is obvious that $g \circ f(a_1)$ is not in A_1 because $f(a_1) \in D$ and $g(D)$ and $g(E)$ are disjoint. Then if $g \circ f(a_1)$ is in A_2, $a_1 \in g(E) \subset S$, which is contradiction to $a_1 \in A \setminus S$. Similarly, by induction, if $g \circ f(a_1)$ is in A_k, we will obtain $a_1 \in A_{k-1} \subset S$ for all $k \geq 2$, which is contradiction. Therefore, such a_1 and a_2 does not exist, which proves the injectivity of F.

Extra Problem 2. Let G_k $(k \in \mathbb{N}^+)$ be open and dense in $\mathbb{R}$. Prove that $\bigcap_{k=1}^{\infty} G_k$ is uncountable.

Since G_k is open and dense, its complement G_k^c is closed and nowhere dense. This is because G_k^c contains no open interval. Notice that if it contains an open interval I, then G_k is disjoint with I, then G_k cannot be dense because the middle point of I is not a limit point of G_k, which contradicts to the definition of dense set. Notice that $G = \bigcup_{k=1}^{\infty} G_k^c$ is of first category because it is countable union of nowhere dense set. Since $\mathbb{R}$ is of second category, so G^c is of second category. However, by De Morgan's Law,

$$G^c = \left(\bigcup_{k=1}^{\infty} G_k^c\right)^c = \bigcap_{k=1}^{\infty} (G_k^c)^c = \bigcap_{k=1}^{\infty} G_k$$

Therefore, $\bigcap_{k=1}^{\infty} G_k$ is of second category, but countable set must be of first category, so $\bigcap_{k=1}^{\infty} G_k$ is uncountable.

Extra Problem 3. Let $3 \leq p < \infty$. The Cantor-like set is constructed as follows: On the interval $[0,1]$, first pick the middle point $1/2$ and remove the $1/p$ neighborhood of it. Denote the remaining part of $[0,1]$ by F_1. Now in the second stage, from each subterval in F_1, remove the $1/p^2$ neighborhood of its middle point. Denote the remaining part as F_2. Repeat this process we get F_n, which consists of 2^n closed subintervals of equal length. Define $C_p = \bigcap_{n=1}^{\infty} F_n$. Prove that

(i) C_p is nowhere dense;

For any $x \in C_p$, we want to show for all $\delta > 0$, $(x-\delta, x+\delta) \not\subset C_p$. Since $x \in C_p$, for all n, $x \in F_n$. Since F_n consists of closed and disjoint interval $I_{n,i}$ for $i = 1, \ldots 2^n$, we assume $x \in I_{n,i_n}$. Then we obtain a sequence of closed interval I_{n,i_n} whose length is decreasing to zero. Therefore, we can take n large such that $I_{n,i_n} \subset (x-\delta, x+\delta)$. However, when construct F_{n+1}, $1/p^{n+1}$-neighborhood of x in I_{n,i_n} is removed, so $(x-\delta, x+\delta) \not\subset F_{n+1}$, hence $(x-\delta, x+\delta) \not\subset C_p$.

(ii) C_p is a perfect set;

Since C_p is closed, we have $C_p' \subset C_p$. For all $x \in C_p$, $x \in F_n$ for all $n \geq 1$, so $x \in I_{n,i_n}$ which has length

$$\frac{1 - \sum_{k=1}^{n} \frac{1}{p^k}}{2^n} = \frac{p - 2 + p^{-n}}{2^n(p-1)} \to 0$$

Therefore, if we denote $x_n \in C_p$ be an end point of I_{n,i_n}, $d(x_n, x) \to 0$ as $n \to \infty$. This shows x is a limit point of C_p, so $x \in C_p'$, i.e., $C_p \subset C_p'$. Thus, $C_p = C_p'$ and C_p is a perfect set.

(iii) the total length of all open inverals removed is equal to $\frac{1}{p-2}$.

The total length of open inverals removed until step n is equal to

$$\frac{1}{p} + \frac{2^1}{p^2} + \frac{2^2}{p^3} + \cdots = \frac{1}{p}\left(1 + \left(\frac{2}{p}\right) + \left(\frac{2}{p}\right)^2 + \cdots\right) = \lim_{n\to\infty} \frac{1 \cdot \left(1 - \left(\frac{2}{p}\right)^n\right)}{p-2} = \frac{1}{p-2}$$

Thus, the total length of all open inverals removed is equal to $\frac{1}{p-2}$.

Extra Problem 4. Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of sets. Define

$$\overline{\lim_{n\to\infty}} E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n, \qquad \underline{\lim_{n\to\infty}} E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$$

(i) Prove $\overline{\lim}_{n\to\infty} E_n$ is equal to the set of points who belong to infinitely many E_n's, and

$$\underline{\lim_{n\to\infty}} E_n = \{x \,|\, \exists \text{ integer } n_x \geq 1, \text{ s.t. } x \in E_n \text{ whenever } n \geq n_x\}$$

If x belongs to infinitely many E_n's, then there exists a subsequence n_k such that $x \in E_{n_k}$ for all $k \geq 1$ and $n_k \to \infty$ as $k \to \infty$. Therefore, $x \in \bigcup_{n=n_k}^{\infty} E_n$ for all k. Since $n_k \geq k$, so for all $k \geq 1$, $x \in \bigcup_{n=n_k}^{\infty} E_n \subset \bigcup_{n=k}^{\infty} E_n$. This is sufficient to show that $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$. Conversely if x belongs to only finitely many E_n's, then denote the maximum n as n_0, for all $k \geq n_0 + 1$, $x \notin \bigcup_{n=k}^{\infty} E_n$, so $x \notin \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$. This proves that the set of points who belong to infinitely many E_n's is equal to $\overline{\lim}_{n\to\infty} E_n$.

If there exists $n_x \geq 1$ such that $x \in E_n$ for all $n \geq n_x$, then $x \in \bigcap_{n=n_x}^{\infty} E_n$, and thus $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$. Conversely, if $x \in \underline{\lim}_{n\to\infty} E_n$, then there must exists a k_0 such that $x \in \bigcap_{n=k_0}^{\infty} E_n$, which further means for all $n \geq k_0$, $x \in E_n$. Thus, we prove the required statement.

(ii) Suppose $E_1 \subset E_2 \subset E_3 \subset \cdots \subset E_n \subset E_{n+1} \subset \cdots$, find $\underline{\lim}_{n\to\infty} E_n$ and $\overline{\lim}_{n\to\infty} E_n$.

For each $k \geq 1$, $\bigcap_{n=k}^{\infty} E_n = E_k$, therefore, $\underline{\lim}_{n\to\infty} E_n = \bigcup_{k=1}^{\infty} E_k$. For each $k \geq 2$, $\bigcup_{n=k}^{\infty} E_n = E_k = \bigcup_{n=k-1}^{\infty} E_n$, therefore, $\overline{\lim}_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n$. This shows that $\underline{\lim}_{n\to\infty} E_n = \overline{\lim}_{n\to\infty} E_n$.

(iii) Suppose $E_n \cap E_m = \varnothing$, if $n \neq m$. Find $\underline{\lim}_{n\to\infty} E_n$ and $\overline{\lim}_{n\to\infty} E_n$.

Since each E_n is pairwise disjoint, for all $k \geq 1$, $\bigcap_{n=k}^{\infty} E_n = \varnothing$, thus $\underline{\lim}_{n\to\infty} E_n = \bigcup_{k=1}^{\infty} \varnothing = \varnothing$. We claim that $\overline{\lim}_{n\to\infty} E_n = \varnothing$. Suppose there exists $x \in \overline{\lim}_{n\to\infty} E_n$, then for all $k \geq 1$, $x \in \bigcup_{n=k}^{\infty} E_n$. This implies that there exists a unique $n_k \geq k$ such that $x \in E_{n_k}$ for all $k \geq 1$. However, if $k = 1$, there exists unique $n_1 \geq 1$ such that $x \in E_{n_1}$, but if we take $k = n_1 + 1$, then there exists unique $n_2 > n_1$ such that $x \in E_{n_2}$, which contradicts to the uniqueness of n_1. Therefore, there is no such x, that is to say $\overline{\lim}_{n\to\infty} E_n = \varnothing$.

(iv) Let all $E_n \subset \mathbb{R}^N$. Prove that

$$\left(\overline{\lim_{n\to\infty}} E_n\right)^c = \underline{\lim_{n\to\infty}} (E_n)^c, \qquad \left(\underline{\lim_{n\to\infty}} E_n\right)^c = \overline{\lim_{n\to\infty}} (E_n)^c$$

Notice that De Morgan's Law can be generalized into infinite number of sets, so

$$\left(\overline{\lim_{n\to\infty}} E_n\right)^c = \left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n\right)^c = \bigcup_{k=1}^{\infty} \left(\bigcup_{n=k}^{\infty} E_n\right)^c = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n^c = \underline{\lim_{n\to\infty}} (E_n)^c$$

$$\left(\underline{\lim_{n\to\infty}} E_n\right)^c = \left(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n\right)^c = \bigcap_{k=1}^{\infty} \left(\bigcap_{n=k}^{\infty} E_n\right)^c = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n^c = \overline{\lim_{n\to\infty}} E_n^c$$

(v) Let $f(x)$, $\{f_n(x)\}_{n=1}^{\infty}$ be defined on a set $E \subset \mathbb{R}^N$. Prove that

$$Z \triangleq \{x \in E \mid f_n(x) \not\to f(x) \text{ as } n \to \infty\} = \bigcup_{l=1}^{\infty} \left(\overline{\lim_{k\to\infty}} E_l^k\right)$$

where $E_l^k = \left\{x \in E \mid |f_k(x) - f(x)| \geq \frac{1}{l}\right\}$.

Given $x \in E$, $f_n(x) \not\to f(x)$ is equivalent to say there exists $l \in \mathbb{N}^+$ such that for all $N \in \mathbb{N}^+$, there exists $k \geq N$, such that $|f_k(x) - f(x)| \geq \frac{1}{l}$. In other words, it means that there exists $l \in \mathbb{N}^+$ such that $x \in E_l^k$ for infinitely many k, which by part (i), is equivalent to that $x \in \overline{\lim}_{k\to\infty} E_l^k$. The existence of l is equivalent to $x \in \bigcup_{l=1}^{\infty} \overline{\lim}_{k\to\infty} E_l^k$, so we prove the desired statement.

Extra Problem 5. Let E be a bounded closed subset of $\mathbb{R}^n$. Suppose $\{f_k\}_{k=1}^{\infty}$ are continuous on E and $f_k \to f$ uniformly for some f as $k \to \infty$. Prove that

$$f(E) = \bigcap_{j=1}^{\infty} \left(\overline{\bigcup_{k=j}^{\infty} f_k(E)} \right)$$

For $y \in f(E)$, there exists $x \in E$ such that $f(x) = y$. Since f_k is uniformly convergent to f, so $f_k(x) \to f(x) = y$, which means y is a limit point of $\bigcup_{k=j} f_k(E)$ for all $j \geq 1$. Therefore, $y \in \bigcap_{j=1}^{\infty} \left(\overline{\bigcup_{k=j}^{\infty} f_k(E)} \right)$.

For the other direction, if $y \in \bigcap_{j=1}^{\infty} \left(\overline{\bigcup_{k=j}^{\infty} f_k(E)} \right)$, then there exists a sequence $a_k \subset \mathbb{N}^+$ such that $f_{a_k}(x_{a_k}) \to y$ as $k \to \infty$, where $x_{a_k} \in E$. Since E is bounded subset of $\mathbb{R}^n$, by Bolzano-Weierstrass, there exists a subsequence of x_{a_k} which converges in $\mathbb{R}^n$ and by closedness of E, $x_{a_{k_p}} \to x \in E$. We claim that $f(x) = y$, and then $y \in f(E)$. Since f_k is continuous and uniformly convergent to f, so f is continuous. Also, by the definition of uniform convergence, $\sup_{x\in E} |f_k(x) - f(x)| \to 0$ as $k \to \infty$. Consider

$$\begin{aligned} |f_{a_{k_p}}(x_{a_{k_p}}) - f(x)| &\leq |f_{a_{k_p}}(x_{a_{k_p}}) - f(x_{a_{k_p}})| + |f(x_{a_{k_p}}) - f(x)| \\ &\leq \sup_{x\in E} |f_{a_{k_p}}(x) - f(x)| + |f(x_{a_{k_p}}) - f(x)| \end{aligned}$$

Since $a_{k_p} \to \infty$ and $x_{a_{k_p}} \to x$ as $p \to \infty$, with the continuity of f, we conclude that the above two terms both converge to zero, i.e., $f_{a_{k_p}}(x_{a_{k_p}}) \to f(x)$. However, since $f_{a_{k_p}}(x_{a_{k_p}})$ is a subsequence of $f_{a_k}(a_k)$ and $f_{a_k}(a_k) \to y$, these two limits must be equal, that is, $f(x) = y$.