MAT3006*: Real Analysis
Homework 10

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Extra Problem 1. Let $0 < p < 1$ and $q = \frac{p}{p-1}$. Assume that if $g = 0$ on E then $\|g\|_{L^q(E)} = 0$.

(i) Proved for f, g measurable on $E \in \mathcal{M}$ and $m(E) > 0$, we have the reversed Hölder's inequality, i.e., $\|fg\|_{L^1(E)} \geq \|f\|_{L^p(E)}\|g\|_{L^q(E)}$.

If $g = 0$ a.e. on E, then it is trivial to prove the inequality because both left and right sides are zero. Assume $|g| > 0$ on a subset of E with positive measure, and let $q = 1/p$, $q' = q/(q-1)$, and $u = |fg|^p$ and $v = |g|^{-p}$. Since $q > 1$, from Hölder's inequality, we have

$$
\begin{aligned}
\|uv\|_{L^1(E)} \leq \|u\|_{L^q(E)}\|v\|_{L^{q'}(E)} &\iff \int_E uv\,dx \leq \left(\int_E u^q\,dx\right)^{\frac{1}{q}}\left(\int_E v^{q'}\,dx\right)^{\frac{1}{q'}} \\
&\iff \int_E |f|^p\,dx \leq \left(\int_E |fg|\,dx\right)^p\left(\int_E |g|^{p'}\,dx\right)^{1-p} \\
&\iff \left(\int_E |f|^p\,dx\right)^{\frac{1}{p}} \leq \left(\int_E |fg|\,dx\right)\left(\int_E |g|^{p'}\,dx\right)^{-\frac{1}{p'}} \\
&\iff \left(\int_E |f|^p\,dx\right)^{\frac{1}{p}}\left(\int_E |g|^{p'}\,dx\right)^{\frac{1}{p'}} \leq \int_E |fg|\,dx
\end{aligned}
$$

where the last step is valid because $|g| > 0$ on a subset of E with positive measure, so the integral of $|g|^{p'}$ is positive. Therefore, we obtain the reversed Hölder inequality $\|f\|_{L^p(E)}\|g\|_{L^{p'}(E)} \leq \|fg\|_{L^1(E)}$ for all $f \in L^p(E)$ and $g \in L^{p'}(E)$.

(ii) Prove reversed Minkowski inequality, i.e., for measurable f, g s.t. $f \geq 0$, $g \geq 0$ on E, we have $\|f\|_{L^p(E)} + \|g\|_{L^p(E)} \leq \|f + g\|_{L^p(E)}$.

Notice that for $f \geq 0, g \geq 0$, we have $|f + g| = |f| + |g|$, and then we have

$$
\begin{aligned}
\int_E |f+g|^p &= \int_E |f||f+g|^{p-1} + \int_E |g||f+g|^{p-1} \\
&\geq \left(\int_E |f|^p\,dx\right)^{\frac{1}{p}}\left(\int_E |f+g|^p\,dx\right)^{\frac{p-1}{p}} + \left(\int_E |g|^p\,dx\right)^{\frac{1}{p}}\left(\int_E |f+g|^p\,dx\right)^{\frac{p-1}{p}} \\
&= \|f\|_{L^p}\|f+g\|_{L^p}^{p-1} + \|g\|_{L^p}\|f+g\|_{L^p}^{p-1}
\end{aligned}
$$

where the inequality follows from reversed Hölder inequality. If $f + g > 0$ on a subset with positive measure, then $\|f + g\|_{L^p}^{p-1} > 0$ and by cancelling that factor, we obtain the reversed Minkowski inequality. If $f + g = 0$ a.e., since $f \geq 0$ and $g \geq 0$, $f = 0$ a.e. and $g = 0$ a.e.. In this case $\|f\|_{L^p} = \|g\|_{L^p} = 0$ and $\|f + g\|_{L^p} = 0$, so the both sides are equal. In conclusion, the reversed Minkowski inequality holds for all $f, g \in L^p(E), f \geq 0, g \geq 0$.

(iii) Construct f and g s.t. $\|f\|_{L^p(E)} + \|g\|_{L^p(E)} < \|f+g\|_{L^p(E)}$.

We first prove there exists subsets of E, e_1, e_2, such that $e_1 \cap e_2 = \varnothing$ and $m(e_1) = m(e_2) > 0$. Note that we have proved that $f(r) = m(B_r(\mathbf{0}) \cap E)$ is a continuous function on $[0, \infty)$. WLOG, suppose $m(E) < \infty$ (if $m(E) = \infty$ then take $F = E \cap B_1(\mathbf{0})$ and use F as E). Then there exists r s.t. $f(r) = \frac{m(E)}{2}$. Let $e_1 = B_r(\mathbf{0}) \cap E$ and $e_2 = E \setminus e_1$.

Let $f = \chi_{e_1}$ be the characteristic function of e_1, and $g = \chi_{e_2}$. Since $0 < p < 1$, $2^{1/p} > 2$. Let $m(e_1) = m(e_2) = l > 0$. Therefore, $\|f\|_{L^p} = \left(\int_{e_1} 1^p \, dx\right)^{1/p} = l^{1/p}$, and similarly, $\|g\|_{L^p} = l^{1/p}$. Also, since e_1 and e_2 are disjoint, $\|f+g\|_{L^p} = (2l)^{1/p}$. Therefore, $2l^{1/p} < 2^{1/p}l^{1/p} = (2l)^{1/p}$, meaning that the reversed strict Minkowski inequality holds.

Extra Problem 2. Let X be a normed space. Prove that if $\|x_k - x_\infty\| \to 0$ as $k \to \infty$, then $\|x_k\| \to \|x_\infty\|$. In $L^1(-1,1)$, construct a counterexample s.t. $\|f_k\|_{L^1} \to \|f_\infty\|_{L^1}$ but $f_k \not\to f_\infty$ in L^1.

By definition of norm, $\|x_k\| = \|x_\infty + (x_k - x_\infty)\| \le \|x_\infty\| + \|x_k - x_\infty\|$, thus we have $\|x_k\| - \|x_\infty\| \le \|x_k - x_\infty\|$. Similarly, we will obtain $\|x_\infty\| - \|x_k\| \le \|x_k - x_\infty\|$. Thus, $|\|x_k\| - \|x_\infty\|| \le \|x_k - x_\infty\| \to 0$, so $\|x_k\| \to \|x_\infty\|$. Consider $f_k(x) = I_{[-1,0]}(x)$ and $f_\infty(x) = I_{[0,1]}(x)$, then $\|f_k\|_{L^1} \equiv 1 = \|f_\infty\|_{L^1}$, so $\|f_k\|_{L^1} \to \|f_\infty\|_{L^1}$. However, $\|f_k - f_\infty\|_{L^1} = 2$ for all $k \ge 1$, so $f_k \not\to f_\infty$ in L^1.

Extra Problem 3. Let $E \subset \mathbb{R}^m$, $F \subset \mathbb{R}^n$, and $f(x,y)$ be measurable on $E \times F$, where $x \in E$, $y \in F$. For $1 \le p < \infty$, if $\int_F \|f(x,y)\|_{L^p_x(E)} \, dy < \infty$, prove

(i) For a.e. fixed $x \in E$, $f(x,y) \in L^1_y(F)$.

Notice that $\int_F |f(x,y)| \, dy$ is well defined (possibly infinity),

$$\begin{aligned}
\left\| \int_F |f(x,y)| \, dy \right\|^p_{L^p_x(E)} &= \int_E \left(\int_F |f(x,y)| \, dy \right)^p dx \\
&= \int_E \left(\int_F |f(x,z)| \, dz \right)^{p-1} \left(\int_F |f(x,y)| \, dy \right) dx \\
&= \int_E \left(\int_F |f(x,y)| \left(\int_F |f(x,z)| \, dz \right)^{p-1} dy \right) dx
\end{aligned}$$

Since $f(x,y)$ is measurable on $E \times F$, $|f(x,y)|$ is nonnegative measurable, so we can apply Fubini's theorem to $|f(x,y)|$, $\int_F |f(x,z)| \, dz$ is measurable on E and thus $|f(x,y)| \left(\int_F |f(x,z)| \, dz\right)^{p-1}$ is nonegative measurable on E. Again, apply Fubini's theorem to $|f(x,y)| \left(\int_F |f(x,z)| \, dz\right)^{p-1}$, we obtain,

$$\begin{aligned}
\left\| \int_F |f(x,y)| \, dy \right\|^p_{L^p_x(E)} &= \int_E \left(\int_F |f(x,y)| \left(\int_F |f(x,z)| \, dz \right)^{p-1} dy \right) dx \\
&= \int_F \left(\int_E |f(x,y)| \left(\int_F |f(x,z)| \, dz \right)^{p-1} dx \right) dy \\
&\le \int_F \left(\|f(x,y)\|_{L^p_x(E)} \left\| \int_F |f(x,z)| \, dz \right\|^{p-1}_{L^p_x(E)} \right) dy \qquad \text{(Hölder's ineq.)}
\end{aligned}$$

Note that if $\left\|\int_F |f(x,z)|\,dz\right\|_{L^p_x(E)} = 0$, then $\left\|\int_F |f(x,y)|\,dy\right\|_{L^p_x(E)} \le \int_F \|f(x,y)\|_{L^p_x(E)}\,dy$ is trivial. If $0 < \left\|\int_F |f(x,z)|\,dz\right\|_{L^p_x(E)} < \infty$, we can cancel out it on both sides, and the same result is obtained. If $\left\|\int_F |f(x,z)|\,dz\right\|_{L^p_x(E)} = \infty$, then denote $E_k = E \cap B_k(\mathbf{0})$ and $F_k = F \cap B_k(\mathbf{0})$. Also define

$$f_k(x,y) = \begin{cases} f_k(x,y) & \text{if } |f_k(x,y)| \le k \\ k & \text{if } |f_k(x,y)| > k \end{cases}$$

Then $|f_k(x,y)| \to |f(x,y)|$ pointwisely. It is obvious that $|f_k(x,y)|$ is also nonnegative measurable on $E \times F$. Therefore, by Lemma 2 in lecture, $g_k(x,y) = |f_k(x,y)| I_{E_k}(x) I_{F_k}(y)$ is also nonnegative measurable on $E \times F$ and increases pointwisely to $g(x,y) = |f(x,y)| I_E(x) I_F(y)$. Note that $\left\|\int_F g_k(x,z)\,dz\right\|_{L^p_x(E)} < \infty$, so we always have $\left\|\int_F g_k(x,y)\,dy\right\|_{L^p_x(E)} \le \int_F \|g_k(x,y)\|_{L^p_x(E)}\,dy$. Take limit on both sides, for LHS,

$$\lim_{k\to\infty} \left\| \int_F g_k(x,y)\,dy \right\|_{L^p_x(E)} = \left\| \lim_{k\to\infty} \int_F g_k(x,y)\,dy \right\|_{L^p_x(E)} = \left\| \int_F |f(x,y)|\,dy \right\|_{L^p_x(E)}$$

For RHS, since $\|g_k(x,y)\|_{L^p_x(E)}$ is also nonnegative increasing in k,

$$\lim_{k\to\infty} \int_F \|g_k(x,y)\|_{L^p_x(E)}\,dy = \int_F \lim_{k\to\infty} \|g_k(x,y)\|_{L^p_x(E)}\,dy = \int_F \|f(x,y)\|_{L^p_x(E)}\,dy$$

Therefore, we still obtain the desired inequality, i.e., for any measurable $f(x,y)$ on $E \times F$, we always have

$$\left\| \int_F |f(x,y)|\,dy \right\|_{L^p_x(E)} \le \int_F \|f(x,y)\|_{L^p_x(E)}\,dy < \infty \tag{$*$}$$

Since $\int_F |f(x,y)|\,dy$ is in $L^p(E)$, it must be fintie a.e. on E, so $f(x,y) \in L^1_y(F)$ for almost all $x \in E$.

(ii) $\int_F f(x,y)\,dy$ is a measurable function of $x \in E$ and $\int_F f(x,y)\,dy \in L^p_x(E)$.

Since we proved $f(x,y) \in L^1_y(F)$, $\int_F f(x,y)\,dy$ is well-defined and a.e. finite on E. Notice that by definition, $\int_F f(x,y)\,dy = \int_F f^+(x,y)\,dy - \int_F f^-(x,y)\,dy$. Since $f^+(x,y)$ and $f^-(x,y)$ are both nonnegative measurable function on $E \times F$, by Fubini's theorem (nonnegative version), $\int_F f^+(x,y)\,dy$ and $\int_F f^-(x,y)\,dy$ are both measurable on E. Therefore, $\int_F f(x,y)\,dy$ is also measurable on E. To prove $\int_F f(x,y)\,dy \in L^p_x(E)$, notice that

$$\left\| \int_F f(x,y)\,dy \right\|_{L^p_x(E)} \le \left\| \int_F |f(x,y)|\,dy \right\|_{L^p_x(E)} \le \int_F \|f(x,y)\|_{L^p_x(E)}\,dy < \infty \tag{$**$}$$

where the first inequality is because $\left|\int_F f(x,y)\,dy\right| \le \int_F |f(x,y)|\,dy$. Thus, $\int_F f(x,y)\,dy \in L^p_x(E)$.

(iii) $\left\|\int_F f(x,y)\,dy\right\|_{L^p_x(E)} \le \int_F \|f(x,y)\|_{L^p_x(E)}\,dy$.

Actually, in part (ii), when we prove $\int_F f(x,y)\,dy \in L^p_x(E)$ in equation $(**)$, we have already shown that

$$\left\| \int_F f(x,y)\,dy \right\|_{L^p_x(E)} \le \int_F \|f(x,y)\|_{L^p_x(E)}\,dy$$

so we are done.

Extra Problem 4. Let $1 < p < \infty$. For all $f \in L^p(0,\infty)$, define $Tf = \frac{1}{x}\int_0^x f(y)\,dy$ for $x \in (0,\infty)$. Prove that $\|Tf\|_{L^p(0,\infty)} \leq \frac{p}{p-1}\|f\|_{L^p(0,\infty)}$.

Let $z = \frac{y}{x}$, then by change of variable (since for each fixed x, $f \in L^1(0,x)$), we obtain

$$Tf(x) = \int_0^1 f(xz)\,dz$$

We claim that $f(xz) : (0,\infty)\times(0,1)$ is measurable, and since $f \in L^p_x(0,\infty)$, $\int_0^1 \|f(xz)\|_{L^p_x(E)}\,dz < \infty$, so we can apply Extra Problem 3,

$$\|Tf\|_{L^p(0,\infty)} \leq \int_0^1 \|f(xz)\|_{L^p_x(E)}\,dz = \int_0^1 \left(\int_0^\infty |f(xz)|^p\,dx\right)^{1/p} dz$$

Let $y = zx$, since for each fixed z, $|f(x)|^p \in L^1$ implies $|f(xz)|^p \in L^1$, by change of variable,

$$\int_0^\infty |f(xz)|^p\,dx = \frac{1}{z}\int_0^\infty |f(y)|^p\,dy = z^{-1}\|f\|^p_{L^p(0,\infty)}$$

Substitute it into the equation above it, we obtain

$$\|Tf\|_{L^p(0,\infty)} \leq \|f\|_{L^p(0,\infty)}\int_0^1 z^{-1/p}\,dz = \frac{p}{p-1}\|f\|_{L^p(0,\infty)}$$

To prove the measurablity of $f(xz)$ on $(0,\infty)\times(0,1)$, define $S : (0,\infty)\times(0,1) \mapsto \mathrm{Im}(S)$ to be $S(x,z) = (xz, x)$. It is easy to check S is a diffeomorphism, so S^{-1} is well-defined and continuously differentiable. Notice that we can write $f(xz) = g(S(x,z))$ where $g(x,y) = f(x)$ is the projection-type function and its domain is $\mathrm{Im}(S)$. Since $\mathrm{Im}(S) = \{(x,y)\,|\,x > 0, y > 0, x < y\}$, it is an open set. It is easy to show $g(x,y)$ is measurable because for any Borel set $B \subset \mathbb{R}$, $g^{-1}(B) = (f^{-1}(B)\times\mathbb{R}) \cap \mathrm{Im}(S)$. Since $f^{-1}(B)$ is measurable and the product of two measurable sets is measurable, $g^{-1}(B)$ is measurable, hence g is a measurable function. Now we need to prove for any Borel set $B \subset \mathbb{R}$, $S^{-1}(g^{-1}(B))$ is measurable. Let $E_k = g^{-1}(B) \cap N_k$ where $N_k = \{(x,y)\,|\,k^{-2} \leq x^2 + y^2 \leq k^2\}$ is the annulus in $\mathbb{R}^2$. Since S^{-1} is continuously differentiable, it is Lipschitz on each E_k (can be proved by writing out the gradient of S^{-1} and using mean value theorem). Since E_k is measurable, $S^{-1}(E_k)$ is also measurable. Therefore, $S^{-1}(g^{-1}(B)) = \bigcup_{k=1}^\infty S^{-1}(E_k)$ is also measurable. This is enough to show that $f(xz)$ is measurable on $(0,\infty)\times(0,1)$.

Extra Problem 5. Let $f(x)$ be measurable on $\mathbb{R}^n$. Prove that $f(x-y)$ as a function of $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ is measurable. Also, prove that for all $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, $f * g \in L^p(\mathbb{R}^n)$ where $f * g = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy$. Furthermore, prove $\|f * g\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)}$.

Similar to Extra Problem 4, define $S : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n \times \mathbb{R}^n$ by $S(x,y) = (x-y, x)$. It is trivial that S is a diffeomorphism, so S^{-1} is well-defined and Lipschitz on $\mathbb{R}^n \times \mathbb{R}^n$. Notice that we can write $f(x-y) = g(S(x,y))$ where $g(x,y) = f(x)$ is defined the same as in Extra Problem 4, so $g(x,y)$ is measurable on $\mathbb{R}^n \times \mathbb{R}^n$. For any Borel set $B \subset \mathbb{R}^n$, $S^{-1}(g^{-1}(B))$ is measurable because $g^{-1}(B)$ is measurable and S^{-1} is Lipschitz. Thus, $f(x-y)$ is measurable on $\mathbb{R}^n \times \mathbb{R}^n$.

For $p = \infty$, since g is essentially bounded by some constant $\|g\|_{L^\infty(\mathbb{R}^n)}$,

$$\int_{\mathbb{R}^n} f(x-y)g(y)\,dy \leq \|g\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n} |f(x-y)|\,dy = \|g\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^1(\mathbb{R}^n)}$$

where the last equality is given by change of variable for each fixed x. Therefore, $f * g$ is also essentially bounded.

For $1 \le p < \infty$, let $h(x,y) = f(x-y)g(y)$ and $\phi(x,y) = f(y)g(x-y)$, then $h(x,y)$ and $\phi(x,y)$ are measurable. Consider

$$\begin{aligned}\int_{\mathbb{R}^n} \|\phi(x,y)\|_{L^p_x}\, dy &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(y)|^p |g(x-y)|^p\, dx\right)^{1/p} dy \\ &= \int_{\mathbb{R}^n} |f(y)| \left(\int_{\mathbb{R}^n} |g(x-y)|^p\, dx\right)^{1/p} dy \\ &= \int_{\mathbb{R}^n} |f(y)| \left(\int_{\mathbb{R}^n} |g(z)|^p\, dz\right)^{1/p} dy = \|g\|_{L^p}\|f\|_{L^1} < \infty\end{aligned}$$

Note that here we can use change of variable because $g(z) \in L^p$ implies $|g(z)|^p \in L^1$. Then by Extra Problem 3(i), for a.e. fixed $x \in E$, $\phi(x,y) \in L^1_y$. Therefore, for a.e. x, we can apply change of variable as below,

$$\int_{\mathbb{R}^n} |f(y)||g(x-y)|\, dy = \int_{\mathbb{R}^n} |f(x-y)||g(y)|\, dy \qquad y \to -y + x$$

Therefore, we have

$$\begin{aligned}\left\|\int_{\mathbb{R}^n} h(x,y)\, dy\right\|_{L^p_x} &= \left\|\int_{\mathbb{R}^n} |f(x-y)||g(y)|\, dy\right\|_{L^p_x} \\ &= \left\|\int_{\mathbb{R}^n} |f(y)||g(x-y)|\, dy\right\|_{L^p_x} \\ &\le \int_{\mathbb{R}^n} \|\phi(x,y)\|_{L^p_x}\, dy = \|g\|_{L^p}\|f\|_{L^1} < \infty\end{aligned}$$

This shows $f * g$ is in $L^p(\mathbb{R}^n)$.

Extra Problem 6. Let f be continuous on the interval $(0,1)$. Prove that $\|f\|_{L^\infty(0,1)} = \sup_{x\in(0,1)} |f(x)|$.

It is obvious that $\|f\|_{L^\infty(0,1)} \le \sup_{x\in(0,1)} |f(x)|$, since $\{x \,|\, f(x) > \sup_{x\in(0,1)} |f(x)|\}$ is empty set. If $\|f\|_{L^\infty(0,1)} = \infty$, then $\sup_{x\in(0,1)} |f(x)| \ge \infty$ implies that $\sup_{x\in(0,1)} |f(x)| = \infty$ and the desired equality holds.

If $\|f\|_{L^\infty(0,1)} < \infty$, and suppose $M = \|f\|_{L^\infty(0,1)} < \sup_{x\in(0,1)} |f(x)|$, then there exists small $\epsilon > 0$, s.t. $M + \epsilon < \sup_{x\in(0,1)} |f(x)|$ and $m(\{x \in (0,1) \,|\, f(x) > M + \epsilon\}) = 0$. Since f is continuous, $\{x \in (0,1) \,|\, f(x) > M + \epsilon\}$ is open set. Note that by definition of supremum, there must exists x_0 s.t. $f(x_0) > M + \epsilon$, so it is nonempty open set, whose measure must be positive. Therefore, contradiction shows that $\|f\|_{L^\infty(0,1)} = \sup_{x\in(0,1)} |f(x)|$.

Extra Problem 7. Let f be measurable on E and there exists $r > 0$ s.t. $f \in L^r(E)$. Prove that $\lim_{p\to\infty} \|f\|_{L^p(E)} = \|f\|_{L^\infty(E)}$.

If $\|f\|_{L^\infty(E)} = 0$, then $f(x) = 0$ a.e. on E, so $\|f\|_{L^p(E)} = 0$ and the desired conclusion follows immediately. If $\|f\|_{L^\infty(E)} > 0$, define $E_M = \{x \in E \,|\, |f(x)| \ge M\}$ for $0 < M < \|f\|_{L^\infty(E)}$, then $m(E_M) > 0$. Since $\|f\|_{L^r(E)} < \infty$, $m(E_M) < \infty$. Thus, we can consider

$$\|f\|_{L^p(E)} \ge \|f\|_{L^p(E_M)} \ge M(m(E_M))^{1/p}$$

which implies $\underline{\lim}_{p\to\infty}\|f\|_{L^p(E)} \geq M$. Since M is arbitrary, we have $\underline{\lim}_{p\to\infty}\|f\|_{L^p(E)} \geq \|f\|_{L^\infty(E)}$. If $\|f\|_{L^\infty(E)} = \infty$, then $\underline{\lim}_{p\to\infty}\|f\|_{L^p(E)} \geq \infty$ implies that $\lim_{p\to\infty}\|f\|_{L^p(E)} = \infty$.

If $\|f\|_{L^\infty(E)} < \infty$, then for any $p > r$, we have

$$\|f\|_{L^p(E)} = \left(\int_E |f|^{p-r}|f|^r \, dx\right)^{1/p} \leq \left(\int_E \|f\|^{p-r}_{L^\infty(E)}|f|^r \, dx\right)^{1/p} \leq \|f\|^{1-r/p}_{L^\infty(E)}\|f\|^{r/p}_{L^r(E)}$$

which implies $\overline{\lim}_{p\to\infty}\|f\|_{L^p(E)} \leq \|f\|_{L^\infty(E)}$. Therefore, we have $\lim_{p\to\infty}\|f\|_{L^p(E)} = \|f\|_{L^\infty(E)}$.

Extra Problem 8. Let $f \in L^2(0,1)$ and $\int_0^1 f(x)x^n \, dx = 0$, $\forall\, n \in \mathbb{N}$. Prove $f(x) = 0$ a.e. on $(0,1)$.

If we can prove $\int_0^1 f^2 \, dx = 0$, then $f^2(x) = 0$ a.e. on $[0,1]$, and $f(x) = 0$ a.e. on $[0,1]$. Recall we have proved polynomials are dense in $L^2(0,1)$, so there exists polynomial $p_k(x)$ s.t. $\|p_k - f\|_{L^2(0,1)} \to 0$ as $k \to \infty$. By Cauchy Schwarz,

$$\|p_k f - f^2\|_{L^1(0,1)} \leq \|f\|_{L^2(0,1)}\|p_k - f\|_{L^2(0,1)} \to 0$$

since $f \in L^2(0,1)$. This implies that

$$\int_0^1 p_k f \, dx \to \int_0^1 f^2 \, dx$$

but for all k, $\int_0^1 p_k f \, dx$ is a finite linear combination of $\int_0^1 f x^n \, dx$, so $\int_0^1 p_k f \, dx$ is always zero. This shows exactly $\int_0^1 f^2 \, dx = 0$.

Extra Problem 9. Let f be positive and measurable on $(0,1)$. Prove that $1 \leq \left(\int_0^1 f(x) \, dx\right)\left(\int_0^1 \frac{1}{f(x)} \, dx\right)$.

Let $g(x) = \sqrt{f(x)} > 0$, then $g(x)$ and $1/g(x)$ are both measurable. By Cauchy Schwarz,

$$1 = 1^2 = \|g \cdot (1/g)\|^2_{L^1(0,1)} \leq \|g\|^2_{L^2(0,1)}\|1/g\|^2_{L^2(0,1)}$$

Notice that

$$\|g\|^2_{L^2(0,1)} = \int_0^1 |g|^2 \, dx = \int_0^1 f(x) \, dx, \quad \|1/g\|^2_{L^2(0,1)} = \int_0^1 \frac{1}{f(x)} \, dx$$

Therefore, the desired inequality holds.

Extra Problem 10. Suppose $f_k \to f$ a.e. on $(0,1)$ and for some $r \in (0,\infty)$, $\int_0^1 |f_k(x)|^r \, dx \leq M$ for constant M and for all $k \geq 1$. Prove that for all $0 < p < r$, $\int_0^1 |f_k(x) - f(x)|^p \, dx \to 0$ as $k \to \infty$.

By Egorov's theorem, $f_k \to f$ a.u., so for all $\delta > 0$, there exists E_δ s.t. $m(E_\delta) < \delta$ and $f_k \to f$ uniformly on $(0,1) \setminus E_\delta$. Therefore,

$$\int_{(0,1)\setminus E_\delta} |f_k - f|^p \, dx \to 0$$

It suffices to show that $\int_{E_\delta} |f_k - f|^p \, dx \to 0$. By Hölder's inequality,

$$\||f_k - f|^p\|_{L^1(E_\delta)} \leq \||f_k - f|^p\|_{L^{r/p}(E_\delta)}\|1\|_{L^{r/(r-p)}(E_\delta)}$$

This implies that

$$\int_{E_\delta} |f_k - f|^p \, dx \leq \delta^{1-p/r}\left(\int_{E_\delta} |f_k - f|^r \, dx\right)^{p/r}$$

By Fatou's lemma,

$$\int_0^1 |f(x)|^r \, dx \leq \varliminf_{k\to\infty} \int_0^1 |f_k(x)|^r \, dx \ \leq M$$

Therefore, we obtain

$$\int_{E_\delta} |f_k - f|^r \, dx \leq \int_{E_\delta} 2^r(|f_k|^r + |f|^r) \, dx \leq 2^{r+1} M$$

This implies that $\int_{E_\delta} |f_k - f|^p \, dx \to 0$.