MAT3006*: Real Analysis
Homework 12

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Extra Problem 1. Let $f(x)$ be increasing on $[a,b]$. Prove that the set of discontinuous points of f is at most countable.

Let A be the set of discontinuous points of f, then for any $x \in A$, $f(x+) > f(x-)$ because f is increasing. Thus, there exists $g(x) \in \mathbb{Q}$ s.t. $f(x+) > g(x) > f(x-)$. Notice that for $x_1 < x_2$, $f(x_1+) \leq f(x_2-)$ because f is increasing. Thus, $g(x_1) \neq g(x_2)$ if $x_1 \neq x_2$. This shows that g defines an injective function from A to $\mathbb{Q}$. Therefore, A is at most countable.

Extra Problem 2. Let $f(x) = x \sin \frac{1}{x}$ for $x \neq 0$ and $f(x) = 0$ for $x = 0$. Find Dini's derivative $D^{\pm} f(0)$ and $D_{\pm} f(0)$.

By definition,

$$D^{+} f(0) = \overline{\lim_{x \to 0+}} \frac{f(x) - f(0)}{x - 0} = \overline{\lim_{x \to 0+}} \sin \frac{1}{x}$$

Since we can find a sequence $x_k = (2k\pi + \pi/2)^{-1}$ s.t. $x_k \to 0+$ as $k \to \infty$,

$$1 \geq \overline{\lim_{x \to 0+}} \sin \frac{1}{x} \geq \lim_{k \to \infty} \sin \frac{1}{x_k} = 1$$

We can conclude that $D^{+} f(0) = 1$. Similarly,

$$D^{-} f(0) = \overline{\lim_{x \to 0-}} \frac{f(x) - f(0)}{x - 0} = \overline{\lim_{x \to 0-}} \sin \frac{1}{x}$$

and we can also find $x_k = (-2k\pi + \pi/2)^{-1}$ s.t. $x_k \to 0-$ as $k \to \infty$, thus,

$$1 \geq \overline{\lim_{x \to 0-}} \sin \frac{1}{x} \geq \lim_{k \to \infty} \sin \frac{1}{x_k} = 1$$

We can conclude that $D^{-} f(0) = 1$. Similarly,

$$D_{+} f(0) = \underline{\lim_{x \to 0+}} \frac{f(x) - f(0)}{x - 0} = \underline{\lim_{x \to 0+}} \sin \frac{1}{x}$$

and we can find $x_k = (2k\pi - \pi/2)^{-1}$ s.t. $x_k \to 0+$ as $k \to \infty$, thus,

$$-1 \leq \underline{\lim_{x \to 0+}} \sin \frac{1}{x} \leq \lim_{k \to \infty} \sin \frac{1}{x_k} = -1$$

We can conclude that $D_{+} f(0) = -1$. Similarly,

$$D_{-} f(0) = \underline{\lim_{x \to 0-}} \frac{f(x) - f(0)}{x - 0} = \underline{\lim_{x \to 0-}} \sin \frac{1}{x}$$

and we can find $x_k = (-2k\pi - \pi/2)^{-1}$ s.t. $x_k \to 0-$ as $k \to \infty$, thus,

$$-1 \leq \underline{\lim}_{x\to 0-} \sin\frac{1}{x} \leq \lim_{k\to\infty} \sin\frac{1}{x_k} = -1$$

We can conclude that $D_- f(0) = -1$.

Extra Problem 3. Let $f(x)$ be real-valued on (a, b). Define $E = \{x \in (a, b) \,|\, D^+ f(x) < D_- f(x)\}$. Prove that E is at most countable.

For $x \in E$, take $r_x \in \mathbb{Q}$ s.t. $D^+ f(x) < r_x < D_- f(x)$. Since

$$D^+ f(x) = \overline{\lim_{h\to 0+}} \frac{f(x+h) - f(x)}{h} < r_x$$

There must exist $t_x > x$ s.t. for all $y \in (x, t_x)$, $\frac{f(y)-f(x)}{y-x} < r_x$. If such t_x does not exist, then there exists a sequence t_n s.t. $t_n \to x+$ and $\frac{f(t_n)-f(x)}{t_n - x} \geq r_x$. This implies $D^+ f(x) \geq r_x$, which is a contradiction. Similar arguments show that there exists $s_x < x$ s.t. for all $y \in (s_x, x)$, $\frac{f(y)-f(x)}{y-x} > r_x$. Therefore, we can define a map $T : E \mapsto \mathbb{Q}^3$ by $Tx = (r_x, s_x, t_x)$.

Suppose there exists $s_x < x_1 < x_2 < t_x$ s.t. $\frac{f(y)-f(x_1)}{y-x_1} < r_x$ for all $y \in (s_x, x_1)$, $\frac{f(y)-f(x_1)}{y-x_1} > r_x$ for all $y \in (x_1, t_x)$; $\frac{f(y)-f(x_2)}{y-x_2} < r_x$ for all $y \in (s_x, x_2)$, $\frac{f(y)-f(x_2)}{y-x_2} > r_x$ for all $y \in (x_2, t_x)$. Then since $x_2 \in (x_1, t_x)$, we obtain

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > r_x$$

Since $x_1 \in (s_x, x_2)$, we obtain

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} < r_x$$

This is a contradiction. Hence, for given (r_x, s_x, t_x), we cannot find two different x_1, x_2 s.t. $Tx_1 = Tx_2 = (r_x, s_x, t_x)$, i.e. T is injective. Since $\mathbb{Q}^3$ is countable, E is at most countable.

Extra Problem 4. Let $f(x)$ be increasing on (a, b). Let $E \subset (a, b)$ s.t. $E \in \mathcal{M}$ and for all $\epsilon > 0$, there exists open $G \subset (a, b)$, $G \supset E$ s.t. $\sum_i (f(b_i) - f(a_i)) < \epsilon$, where $G = \bigcup_i (a_i, b_i)$. Prove that $f'(x) = 0$ for a.e. $x \in E$.

Since $f(x)$ is increasing on (a, b), by Lebesgue differentiation theorem, $f'(x) \geq 0$ a.e. on (a, b). Fix any $\epsilon > 0$, we have

$$\int_E f'(x)\,dx \leq \int_G f'(x)\,dx \leq \sum_i \int_{a_i}^{b_i} f'(x)\,dx \leq \sum_i (f(b_i) - f(a_i)) < \epsilon$$

where the third inequality is due to Lebesgue differentiation theorem. Take $\epsilon \to 0$, we obtain $\int_E f'(x)\,dx = 0$. Since $f'(x) \geq 0$ a.e. on E, $f'(x) = 0$ a.e. on E.

Extra Problem 5. Suppose $f(x)$ is continuous on I. Prove that it is impossible that $D^+ f(x) > c > D_- f(x)$ for all $x \in I$, where c is a constant and I is an interval.

Suppose $D^+ f(x) > c > D_- f(x)$, then since cx is differentiable, we have $D^+(f(x) - cx) > 0$. Since $f(x) - cx$ is a continuous function on I, $f(x) - cx$ is increasing by Example 3 in lecture note. Similarly, $D_-(f(x) - cx) < 0$ implies $f(x) - cx$ is decreasing function on I. Thus, $f(x) - cx = C$

for some constant C on I. Then, $f(x) = cx + C$ is a linear function on I. However, we know $f'(x) = c$, so $f(x)$ is differentiable and $D^+f(x) = D_-f(x) = c$, which is a contradiction. Therefore, it is impossible that $D^+f(x) > c > D_-f(x)$ for all $x \in I$.

Extra Problem 6. Find a function $f(x)$ that is strictly increasing on $\mathbb{R}$, discontinuous at and only at every $q \in \mathbb{Q}$, and $f'(x) = 0$ a.e. on $\mathbb{R}$.

List all rational number as $\{q_n\}_{n=1}^{\infty}$, and let $f_n(x) = \frac{1}{2^n} I_{x \geq q_n}(x)$. Define $f(x) = \sum_{n=1}^{\infty} f_n(x)$, then we claim that $f(x)$ is the desired function. First, $f(x)$ is increasing because each $f_n(x)$ is increasing. Also, for each $x_1 < x_2$, there exists a rational number q_n s.t. $x_1 < q_n < x_2$, so

$$f(x_2) - f(x_1) \geq f_n(x_2) - f_n(x_1) = \frac{1}{2^n} > 0$$

This implies that $f(x)$ is strictly increasing. Note that $f(x)$ is discontinuous at each q_n because

$$f(q_n+) - f(q_n-) \geq f_n(q_n+) - f_n(q_n-) = \frac{1}{2^n} > 0$$

Thus, each q_n is a jump discontinuous point. To see $f(x)$ is continuous at each irrational point, consider each f_n, it is trivial that $f_n(x)$ is continuous at x. Furthermore,

$$|f(x)| \leq \sum_{n=1}^{\infty} |f_n(x)| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} < 1$$

By M-test, it implies that the partial sum $S_k(x) = \sum_{n=1}^{k} f_n(x)$ converges to $f(x)$ uniformly. However, $S_k(x)$ is continuous at x, so $f(x)$ is continuous at x. Finally, since f_n is increasing on any bounded closed interval $[a, b]$ and $\sum_{n=1}^{\infty} f_n(x)$ is convergent for all $x \in [a, b]$, by Fubini's differentiation theorem, $f(x)$ is differentiable a.e. on (a, b) and $f'(x) = \sum_{n=1}^{\infty} f_n'(x)$. Since on irrational number set, $f_n'(x) = 0$, $f'(x) = 0$ on irrational number set. This shows $f'(x) = 0$ a.e. on (a, b). Since (a, b) is arbitrary and differentiation is a local property, $f'(x) = 0$ a.e. on $\mathbb{R}$.