MAT3006*: Real Analysis
Homework 13

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Extra Problem 1. Let $\Delta_0 = \{a = x_0, x_1, x_2, x_3, b = x_4\}$. Then if a continuous function $f(x)$ defined on $[a,b]$ is increasing on $[a,x_1]$ and $[x_2,x_3]$, decreasing on $[x_1,x_2]$ and $[x_3,b]$, then $V_a^b(f) = v_{\Delta_0}$.

Notice that by definition $V_a^b(f) \geq v_{\Delta_0}$, where

$$v_{\Delta_0} = f(x_1) - f(a) + f(x_1) - f(x_2) + f(x_3) - f(x_2) + f(x_3) - f(b)$$

It suffices to show $V_a^b(f) \leq v_{\Delta_0}$. For any partition $\Delta = \{a, y_1, \ldots, y_n, b\}$, we can construct a new partition Δ_1 s.t. $\Delta_1 = \Delta \cup \Delta_0$. Then

$$\Delta_1 = \{a, y_1, \ldots, y_i, x_1, y_{i+1}, \ldots, y_j, x_2, y_{j+1}, \ldots, y_k, x_3, y_{k+1}, \ldots, y_n, b\}$$

Since $f(x)$ is increasing on $[a,x_1]$ and $[x_2,x_3]$,

$$|f(y_1) - f(a)| + \sum_{m=1}^{i-1} |f(y_{m+1}) - f(y_m)| + |f(x_1) - f(y_i)| = f(x_1) - f(a)$$

$$|f(y_{j+1}) - f(x_2)| + \sum_{m=j+1}^{k-1} |f(y_{m+1}) - f(y_m)| + |f(x_3) - f(y_k)| = f(x_3) - f(x_2)$$

Similarly, since $f(x)$ is decreasing on $[x_1,x_2]$ and $[x_3,b]$,

$$|f(y_{i+1}) - f(x_1)| + \sum_{m=i+1}^{j-1} |f(y_{m+1}) - f(y_m)| + |f(x_2) - f(y_j)| = -(f(x_2) - f(x_1))$$

$$|f(y_{k+1}) - f(x_3)| + \sum_{m=k+1}^{n-1} |f(y_{m+1}) - f(y_m)| + |f(b) - f(y_n)| = -(f(b) - f(x_3))$$

This implies that $v_{\Delta_1} = v_{\Delta_0}$ for any Δ. However, it is easy to see by triangle inequality that $v_\Delta \leq v_{\Delta_1}$, so $v_\Delta \leq v_{\Delta_0}$. Take supremum over all Δ on both sides, $V_a^b(f) = \sup_\Delta v_\Delta \leq v_{\Delta_0}$.

Extra Problem 2. Observe that $v_\Delta \leq v_{\Delta_1}$ if Δ_1 is a finer partition of $[a,b]$ than Δ. Use this observation to prove if f is real-valued on $[a,b]$ and $c \in (a,b)$, then $V_a^b(f) = V_a^c(f) + V_c^b(f)$.

Take arbitrary partition of $[a,c]$ and $[c,b]$, denoted as Δ_1 and Δ_2 respectively. Then $\Delta = \Delta_1 \cup \Delta_2$ is a partition of $[a,b]$. Furthermore, $v_{\Delta_1} + v_{\Delta_2} = v_\Delta \leq V_a^b(f)$. Take supremum over Δ_1 on both sides, $V_a^c(f) + v_{\Delta_2} \leq V_a^b(f)$. Again, take supremum over Δ_2 on both sides, $V_a^c(f) + V_c^b(f) \leq V_a^b(f)$. Conversely, for any partition of $[a,b]$, denoted as $\Delta = \{a, x_1, \ldots, x_{n-1}, b\}$ of $[a,b]$. Let $\Delta_0 = \{a, c, b\}$

and $\Delta' = \Delta \cup \Delta_0$. Then Δ' can be decomposed into Δ_1 and Δ_2, where Δ_1 is a partition of $[a, c]$ and Δ_2 is a partition of $[c, b]$. Furthermore, $v_{\Delta'} = v_{\Delta_1} + v_{\Delta_2} \le V_a^c(f) + V_c^b(f)$. Since Δ' is finer than Δ, $v_\Delta \le v_{\Delta'} \le V_a^c(f) + V_c^b(f)$. Take supremum over Δ on both sides, $V_a^b(f) \le V_a^c(f) + V_c^b(f)$. This shows that $V_a^b(f) = V_a^c(f) + V_c^b(f)$.

Extra Problem 3. Find $V_0^{2\pi}(\sin 2x)$ by using Extra Problem 2.

By a slightly generalized version of Extra Problem 2,

$$V_0^{2\pi}(\sin 2x) = V_0^{\pi/4}(\sin 2x) + V_{\pi/4}^{3\pi/4}(\sin 2x) + V_{3\pi/4}^{5\pi/4}(\sin 2x) + V_{5\pi/4}^{7\pi/4}(\sin 2x) + V_{7\pi/4}^{2\pi}(\sin 2x)$$

By Example 1 in lecture, if f is monotone on $[a, b]$, then $V_a^b(f) = |f(b) - f(a)|$. Thus, we have

$$V_0^{2\pi}(\sin 2x) = |1 - 0| + |-1 - 1| + |1 - (-1)| + |-1 - 1| + |0 - (-1)| = 8$$

Therefore, $V_0^{2\pi}(\sin 2x) = 8$.

Extra Problem 4. Let $f_k(x) \in \mathrm{BV}([a, b])$ for all $k \ge 1$. Suppose $V_a^b(f_k) \le M$ for all $k \ge 1$, and $f_k \to f$ pointwise on $[a, b]$ as $k \to \infty$. Prove $f \in \mathrm{BV}([a, b])$ and $V_a^b(f) \le M$.

First fixed any partition $\Delta = \{a = x_0, x_1, \ldots, x_n, x_{n+1} = b\}$ of $[a, b]$. Since $f_k \to f$ pointwise, for any $\epsilon > 0$, there exists K_i s.t. $|f_k(x_i) - f(x_i)| < \frac{\epsilon}{2n+2}$ for all $k \ge K_i$. Take $K = \max_{i=0}^{n+1} K_i$, then for all $i = 1, \ldots, n+1$, we have

$$\begin{aligned} |f(x_i) - f(x_{i-1})| &\le |f(x_i) - f_K(x_i)| + |f_K(x_i) - f_K(x_{i-1})| + |f_K(x_{i-1}) - f(x_{i-1})| \\ &\le |f_K(x_i) - f_K(x_{i-1})| + \frac{\epsilon}{n+1} \end{aligned}$$

Sum both sides up from $i = 1$ to $i = n + 1$, we have

$$\sum_{i=1}^{n+1} |f(x_i) - f(x_{i-1})| \le \sum_{i=1}^{n+1} |f_K(x_i) - f_K(x_{i-1})| + \epsilon$$

which implies

$$v_\Delta(f) \le v_\Delta(f_K) + \epsilon \le V_a^b(f_K) + \epsilon \le M + \epsilon$$

Take $\epsilon \to 0$, we have $v_\Delta(f) \le M$. Take supremum over all Δ on both sides, $V_a^b(f) \le M$. This shows $f \in \mathrm{BV}([a, b])$.

Extra Problem 5. Denote $\gamma : [0, 1] \mapsto \mathbb{C}$ by $\gamma(t) = x(t) + iy(t)$, where $x(t)$ and $y(t)$ are real-valued continuous functions on $[0, 1]$. A curve γ is rectifiable if $V_0^1(\gamma) < \infty$. In this case, the length of γ is defined to be $V_0^1(\gamma)$. Prove that if $x(t)$ and $y(t)$ are continuously differentiable on $[0, 1]$, then $V_0^1(\gamma) = \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$.

Take any partition $\Delta = \{0 = t_0, t_1, \ldots, t_n, t_{n+1} = 1\}$, then

$$\begin{aligned} v_\Delta &= \sum_{i=1}^{n+1} |\gamma(t_i) - \gamma(t_{i-1})| = \sum_{i=1}^{n+1} \left| \int_{t_{i-1}}^{t_i} \gamma'(t)\, dt \right| \\ &\le \sum_{i=1}^{n+1} \int_{t_{i-1}}^{t_i} |\gamma'(t)|\, dt = \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \end{aligned}$$

Take supremum over Δ on both sides, we obtain $V_0^1(\gamma) \leq \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$.

Conversely, since $\gamma'(t)$ is continuous on $[0,1]$, it is uniformly continuous on $[0,1]$. Thus, for any $\epsilon > 0$, there exists $\delta > 0$ s.t. if $|t_1 - t_2| < \delta$, $|\gamma(t_1) - \gamma(t_2)| < \epsilon$ for all $t_1, t_2 \in [0,1]$. Let $\Delta = \{0 = t_0, t_1, \dots, t_n, t_{n+1} = 1\}$ be a partition of $[0,1]$ with $\|\Delta\| < \delta$. Thus, if $t \in [t_{i-1}, t_i]$, we have $|\gamma'(t)| \leq |\gamma'(t_i)| + \epsilon$. Hence,

$$\begin{aligned}
\int_a^b |\gamma'(t)|\, dt &= \sum_{i=1}^{n+1} \int_{t_{i-1}}^{t_i} |\gamma'(t)|\, dt \ \leq \sum_{i=1}^{n+1} (|\gamma'(t_i)|(t_i - t_{i-1}) + \epsilon(t_i - t_{i-1})) \\
&= \sum_{i=1}^{n+1} \left| \int_{t_{i-1}}^{t_i} (\gamma'(t) + \gamma'(t_i) - \gamma'(t))\, dt \right| + \epsilon(b-a) \\
&\leq \sum_{i=1}^{n+1} \left| \int_{t_{i-1}}^{t_i} \gamma'(t)\, dt \right| + \sum_{i=1}^{n+1} \left| \int_{t_{i-1}}^{t_i} (\gamma'(t_i) - \gamma'(t))\, dt \right| + \epsilon(b-a) \\
&< \sum_{i=1}^{n+1} |\gamma(t_i) - \gamma(t_{i-1})| + 2\epsilon(b-a) = v_\Delta + 2\epsilon(b-a)
\end{aligned}$$

Take $\epsilon \to 0$, we can obtain $\int_a^b |\gamma'(t)|\, dt \leq v_\Delta$, i.e., $v_\Delta \geq \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$. Take supremum over all Δ on both sides, we conclude $V_0^1(\gamma) \geq \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$. Combine the two inequalities, we can see $V_0^1(\gamma) = \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$.

Extra Problem 6. Suppose $f \in \mathrm{BV}([0,1])$. Define $F(x) = \frac{1}{x}\int_0^x f(t)\, dt$ for $x \in (0,1]$ and $F(0) = 2020$. Prove that $F \in \mathrm{BV}([0,1])$ and $\lim_{x\to 0+} F(x)$ exists as a finite number.

By Jordan decomposition theorem, $f = g - h$ where g, h are two increasing real-valued function on $[0,1]$. Then, f, g, h are all bounded function, so

$$F(x) = \frac{1}{x}\int_0^x f(t)\, dt = \frac{1}{x}\int_0^x g(t)\, dt - \frac{1}{x}\int_0^x h(t)\, dt = G(x) - H(x)$$

for $x \in (0,1]$. We claim that $G(x)$ and $H(x)$ are increasing on $(0,1]$. If so, then consider any partition of $[0,1]$, denoted as $\Delta = \{0 = x_0, x_1, \dots, x_n, x_{n+1} = 1\}$, we can compute

$$\begin{aligned}
v_\Delta(F) &= |F(x_1) - F(x_0)| + \sum_{i=2}^{n+1} |F(x_i) - F(x_{i-1})| \\
&\leq |F(x_1) - 2020| + \sum_{i=1}^{n+1} |G(x_i) - G(x_{i-1})| + \sum_{i=2}^{n+1} |H(x_i) - H(x_{i-1})| \\
&= |F(x_1) - 2020| + G(1) - G(x_1) + H(1) - H(x_1)
\end{aligned}$$

Notice that G, H are both bounded on $(0,1]$ and F is bounded on $[0,1]$, so $v_\Delta(F) \leq M$ for some constant M. Take supremum over all Δ on both sides, we have $V_0^1(f) \leq M$, so $F \in \mathrm{BV}([0,1])$.

To see our claim is true, we take $G(x)$ for example, and $H(x)$ is exactly the same.

$$\begin{aligned}
G(x_2) = \frac{1}{x_2}\int_0^{x_2} g(t)\, dt &\geq \frac{1}{x_2}\left(\int_0^{x_1} g(t)\, dt + (x_2 - x_1)g(x_1)\right) \\
&= \frac{1}{x_2}\left(\int_0^{x_1} g(t)\, dt + \frac{x_2 - x_1}{x_1} x_1 g(x_1)\right) \\
&\geq \frac{1}{x_2}\left(\int_0^{x_1} g(t)\, dt + \frac{x_2 - x_1}{x_1}\int_0^{x_1} g(t)\, dt\right) = G(x_1)
\end{aligned}$$

for any $0 < x_1 < x_2 \leq 1$. Therefore, our claim is true.

Since g, h are increasing on $[0, 1]$ and they are real-valued, $g(0+) = \lim_{x\to 0+} g(x)$ and $h(0+) = \lim_{x\to 0+} h(x)$ exists as a finite number. Notice that

$$G(x) = \frac{1}{x}\int_0^x g(t)\,dt \geq \frac{1}{x}\int_0^x g(0+)\,dt = g(0+)$$

This implies $G(0+) = \lim_{x\to 0+} G(x) \geq g(0+)$. However,

$$xg(x) \geq \int_0^x g(t)\,dt \Longrightarrow g(x) \geq G(x) \Longrightarrow \lim_{x\to 0+} g(x) \geq \lim_{x\to 0+} G(x) \Longrightarrow g(0+) \geq G(0+)$$

Therefore, $G(0+) = g(0+)$. Similarly, we can prove $H(0+) = h(0+)$. Therefore,

$$\lim_{x\to 0+} F(x) = G(0+) - H(0+) = g(0+) - h(0+) = f(0+)$$

This implies $F(0+) = \lim_{x\to 0+} F(x)$ exists as a finite number.

Extra Problem 7. Let $f(x)$ be real-valued on $[a, b]$, satisfying that for all $\epsilon > 0$, $V_{a+\epsilon}^b(f) \leq M$, where M is a constant. Prove that $f \in \mathrm{BV}([a, b])$.

First we prove $f(x)$ is bounded on $[a, b]$. We claim that there exists small $\epsilon_0 > 0$ s.t. f is bounded on $[a, a+\epsilon_0]$. If not, then there exists $a_n \to a$ as $n \to \infty$ where $a_n > a$ is a sequence s.t. $f(a_n) \to \infty$. Since for all n,

$$\sum_{k=2}^{\infty} |f(a_k) - f(a_{k-1})| \geq f(a_n) - f(a_1) \to \infty$$

there exists K s.t. $\sum_{k=2}^{K} |f(a_k) - f(a_{k-1})| > M$. There exists $\epsilon_0 > 0$ s.t. $a_1, \ldots, a_K \in [a+\epsilon_0, b]$. Then consider the partition $\Delta = \{a_1, \ldots, a_K\} \cup \{a+\epsilon_0, b\}$ of $[a+\epsilon_0, b]$, $v_\Delta \geq \sum_{k=2}^{K} |f(a_k) - f(a_{k-1})| > M$. This shows $V_{a+\epsilon_0}^b \geq v_\Delta > M$, which is a contradiction. Thus, there exists some $\epsilon_0 > 0$ s.t. $|f(x)| \leq N_1$ for $x \in [a, a+\epsilon_0]$.

Note that $V_{a+\epsilon_0}^b \leq M$, so by a basic fact in lecture, $f(x)$ is bounded on $[a+\epsilon_0, b]$, so denote $|f(x)| \leq N_2$. Then let $N = \max\{N_1, N_2\}$, $|f(x)| \leq N$ for all $x \in [a, b]$. Then for any partition $\Delta = \{a, x_1, \ldots, x_n, b\}$ of $[a, b]$, take $\epsilon = x_1 - a$, we have

$$v_\Delta = |f(a) - f(x_1)| + \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq 2N + V_{a+\epsilon}^b(f) < 2N + M$$

Hence, by taking supremum over Δ on both sides, $V_a^b(f) \leq 2N + M$, so $f \in \mathrm{BV}([a, b])$.