MAT3006*: Real Analysis
Homework 14

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In this assignment, whenever (a, b) and $[a, b]$ are assumed, they are meant to be finite intervals.

Extra Problem 1. Let $f(x)$ be continuous and increasing on $[a, b]$. Prove $f \in \mathrm{AC}([a, b])$ if and only if for all $\epsilon > 0$, there exists $\delta > 0$ s.t. whenever $E \subset (a, b)$, $E \in \mathcal{M}$, $m(E) < \delta$, we have $m^*(f(E)) < \epsilon$.

For "if" part, for any $\epsilon > 0$, consider any finite number of disjoint open interval $(x_1, y_1), \ldots, (x_n, y_n)$. Take $E = \bigcup_{k=1}^n (x_k, y_k)$. By assumption, there exists $\delta > 0$ s.t. $m^*(f(E)) < \epsilon$ if $m(E) < \delta$. However, since (x_k, y_k) are disjoint, f is increasing and continuous,

$$f(E) = \bigcup_{k=1}^n f((x_k, y_k)) = \bigcup_{k=1}^n (f(x_k), f(y_k))$$

Thus, $m(f(E)) = \sum_{k=1}^n (f(y_k) - f(x_k)) < \epsilon$ as long as $m(E) = \sum_{k=1}^n (y_k - x_k) < \delta$. This shows $f \in \mathrm{AC}([a, b])$.

For "only if" part, in fact we don't need f to be increasing. Since $f \in \mathrm{AC}([a, b])$, for all $\epsilon > 0$, there exists δ s.t. for any disjoint open intervals $(x_1, y_1), \ldots, (x_n, y_n)$ with $\sum_{k=1}^n (y_k - x_k) < 2\delta$, we have $\sum_{k=1}^n |f(y_k) - f(x_k)| < \epsilon/2$. For any $E \subset (a, b)$, $E \in \mathcal{M}$, and $m(E) < \delta$, there exists an open $G \subset (a, b)$ s.t. $G \supset E$ with $m(G) < 2\delta$. Since G is open, we can write $G = \bigcup_{k=1}^\infty (a_k, b_k)$ with (a_k, b_k) pairwise disjoint. Since f is continuous, it attains its maximum and minimum on each $[a_k, b_k]$ at M_k and m_k respectively. Note that

$$m^*(f(E)) \leq m(f(G)) = m\left(\bigcup_{k=1}^\infty f((a_k, b_k))\right) \leq \sum_{k=1}^\infty (f(M_i) - f(m_i))$$

Observe that $\sum_{k=1}^\infty (M_k - m_k) \leq \sum_{k=1}^\infty (b_k - a_k) < 2\delta$, so for any finite n, we have $\sum_{k=1}^n (f(M_k) - f(m_k)) < \epsilon/2$. Take $n \to \infty$, we obtain

$$\sum_{k=1}^\infty (f(M_k) - f(m_k)) \leq \epsilon/2 < \epsilon$$

This implies that $m^*(f(E)) < \epsilon$.

Extra Problem 2. Let $f \in L^1(a, b)$ and $\int_a^b x^n f(x)\, dx = 0$ for all $n \geq 0$. Prove that $f(x) = 0$ a.e. on $[a, b]$.

First we prove if $f(x)$ is continuous on $[a, b]$, then the desired property holds. By Weierstrass approximation theorem, there exists polynomials $p_n(x)$ s.t. $p_n(x) \to f(x)$ uniformly on $[a, b]$. Since

$f(x)$ is continuous on compact set $[a,b]$, it is bounded, and hence $p_n f$ converges to f^2 uniformly on $[a,b]$. Therefore,

$$\int_a^b f^2(x)\,dx = \lim_{n\to\infty}\int_a^b p_n(x)f(x)\,dx = 0$$

where the last equality is due to linearity of integral and $\int_a^b x^n f(x)\,dx = 0$ for all $n \geq 0$. This implies that $f^2(x) = 0$ on $[a,b]$, so $f(x) = 0$ everywhere on $[a,b]$.

By Fact 4 of absolute continuity, $F(x) = \int_a^x f(t)\,dt$ is absolutely continuous because $f(x)$ is in $L^1(a,b)$. Obviously x^n on a compact interval $[a,b]$ is Lipschitz, hence absolutely continuous. Therefore, using integration by parts, we have

$$\int_a^b F(x)x^n\,dx = \frac{1}{n+1}\int_a^b F(x)dx^{n+1} = \frac{1}{n+1}F(x)x^{n+1}\Big|_a^b - \frac{1}{n+1}\int_a^b x^{n+1}f(x)\,dx$$

By assumption, the second term is zero, and consider the first term,

$$\frac{1}{n+1}F(x)x^{n+1}\Big|_a^b = \frac{1}{n+1}F(b)b^{n+1} - \frac{1}{n+1}F(a)a^{n+1}$$

It is obvious that $F(a) = 0$, and also,

$$\frac{1}{n+1}F(b)b^{n+1} = \frac{b^{n+1}}{n+1}\int_a^b f(t)\,dt = 0$$

by assumption. We can conclude that $\int_a^b F(x)x^n\,dx = 0$ for $n \geq 0$.

Since $F(x)$ is continuous, by what we proved at the very beginning, $F(x) = 0$ everywhere on $[a,b]$. This shows $\int_c^d f(t)\,dt = 0$ for any subinterval $[c,d]$ of $[a,b]$ because

$$\int_c^d f(t)\,dt = \int_a^d f(t)\,dt - \int_a^c f(t)\,dt = F(d) - F(c) = 0 - 0 = 0$$

Now suppose $f > 0$ on $E \subset [a,b]$ where $m(E) > 0$. Then we can always find a closed subset $F \subset E$ s.t. $m(F) > 0$. This means $\int_F f(t)\,dt > 0$. Now let $U = [a,b] \setminus F$, since U is open in $[a,b]$, it can be written as disjoint union of open intervals in $[a,b]$, i.e., $U = \bigcup_{k=1}^\infty (a_k, b_k)$. Since $f \in L^1(a,b)$, we have

$$0 = \int_a^b f(t)\,dt = \sum_{k=1}^\infty \int_{a_k}^{b_k} f(t)\,dt + \int_F f(t)\,dt = \int_F f(t)\,dt > 0$$

This leads to contradiction. Similarly, suppose $f < 0$ on E, we can obtain nearly the same contradiction. This shows $f(t) = 0$ a.e. on $[a.b]$.

Extra Problem 3. Let f be increasing on $[a,b]$, satisfying $\int_a^b f'(x)\,dx = f(b) - f(a)$. Prove that f is absolutely continuous on $[a,b]$.

Since $f(x)$ is increasing, by Lebesgue's differentiation theorem for monotone function, $f'(x)$ exists a.e. in (a,b) and for all $a \leq x < y \leq b$, $\int_x^y f(t)\,dt \leq f(y) - f(x)$. Now suppose there exists $a \leq x' < y' \leq b$ s.t. $\int_{x'}^{y'} f(t)\,dt < f(y') - f(x')$. Since f is increasing on $[a,x']$ and $[y'b]$, we have $\int_a^{x'} f'(t)dt \leq f(x') - f(a)$ and $\int_{y'}^b f'(t)dt \leq f(b) - f(y')$. This implies

$$\begin{aligned}\int_a^b f'(t)\,dt &= \int_a^{x'} f'(t)dt + \int_{x'}^{y'} f(t)\,dt + \int_{y'}^b f'(t)dt \\ &< f(x') - f(a) + f(y') - f(x') + f(b) - f(y') \\ &= f(b) - f(a)\end{aligned}$$

This contradicts our assumption that $\int_a^b f'(x)\,dx = f(b) - f(a)$. Therefore, for all $a \le x < y \le b$, $\int_x^y f(t)\,dt = f(y) - f(x)$. For any finite number of disjoint open intervals $(x_1, y_1), \dots, (x_n, y_n)$ contained in $[a, b]$, we have

$$\sum_{k=1}^{n} |f(y_k) - f(x_k)| = \sum_{k=1}^{n} f(y_k) - f(x_k) = \int_{\bigcup_{k=1}^{n}(x_k, y_k)} f'(t)\,dt$$

Since $f'(x) \ge 0$ a.e. on $[a, b]$, we can see $f' \in L^1(a, b)$, and so by absolute continuity of integral, for any $\epsilon > 0$, there exists $\delta > 0$ s.t. $\int_{\bigcup_{k=1}^{n}(x_k, y_k)} f'(t)\,dt < \epsilon$ when $m(\cup_{k=1}^{n}(x_k, y_k)) < \delta$. This is equivalent to say $\sum_{k=1}^{n} |f(y_k) - f(x_k)| < \epsilon$ if $\sum_{k=1}^{n}(y_k - x_k) < \delta$, and by definition, $f(x)$ is absolutely continuous on $[a, b]$.

Extra Problem 4. Suppose f is differentiable on $\mathbb{R}$ and $f, f' \in L^1(\mathbb{R})$. Prove that $\int_{\mathbb{R}} f'(x)\,dx = 0$.

We claim that there exists a sequence $a_n \to \infty$ as $n \to \infty$ s.t. $|f(a_n)| \to 0$. If such a_n does not exist, then there exists $K > 0$ s.t. for any $x > K$, $|f(x)| \ge C > 0$ for some constant C. This implies

$$\int_{\mathbb{R}} |f(x)|\,dx \ge \int_K^{\infty} |f(x)|\,dx \ge Cm([K, \infty)) = \infty$$

which contradicts $f \in L^1(\mathbb{R})$. Therefore, such sequence of a_n exists. Similarly, we can prove there exists $b_n \to -\infty$ as $n \to \infty$ s.t. $|f(b_n)| \to 0$. WLOG, assume $a_n \ge b_n$ for all $n \ge 1$.

Since $f' \in L^1(\mathbb{R})$ and $|I_{[b_n, a_n]}(x) f'(x)| \le |f'(x)|$; also $I_{[b_n, a_n]}(x) f'(x) \to f'(x)$ pointwisely on $\mathbb{R}$ as $n \to \infty$, we can apply DCT to obtain

$$\int_{\mathbb{R}} f'(x)\,dx = \lim_{n\to\infty} \int_{\mathbb{R}} I_{[b_n, a_n]}(x) f'(x)\,dx = \lim_{n\to\infty} \int_{b_n}^{a_n} f'(x)\,dx$$

Since f is differentiable on $\mathbb{R}$, we can see f is continuous on any closed bounded interval $[a, b]$, differentiable on (a, b), and $f' \in L^1(a, b)$. By Corollary 3, $f \in \mathrm{AC}([a, b])$ and $f(x) = f(a) + \int_a^x f'(t)\,dt$ for all $x \in [a, b]$. This implies that

$$\lim_{n\to\infty} \int_{b_n}^{a_n} f'(x)\,dx = \lim_{n\to\infty} (f(a_n) - f(b_n)) = 0 - 0 = 0$$

This shows $\int_{\mathbb{R}} f'(x)\,dx = 0$.

Extra Problem 5. Let $f_k(x)$ be increasing and absolutely continuous on $[a, b]$ for all $k \ge 1$. Suppose $\sum_{k=1}^{\infty} f_k(x)$ converges pointwise on $[a, b]$. Prove that $\sum_{k=1}^{\infty} f_k(x)$ is absolutely continuous on $[a, b]$.

Let $f(x) = \sum_{k=1}^{\infty} f_k(x)$, then $f(x)$ is finite on $[a, b]$. Since each f_k is increasing, f is also increasing. To prove $f \in \mathrm{AC}([a, b])$, by Extra Problem 3, we only need to prove $\int_a^b f'(x)\,dx = f(b) - f(a)$. By Fubini's differentiation theorem, $f'(x) = \sum_{k=1}^{\infty} f_k'(x)$ a.e. on $[a, b]$. Take integration on both sides,

$$\int_a^b f'(x)\,dx = \int_a^b \sum_{k=1}^{\infty} f_k'(x)\,dx$$

Notice that $f_k'(x) \ge 0$ for all $k \ge 1$, so using integration term by term (nonnegative version), we have

$$\int_a^b \sum_{k=1}^{\infty} f_k'(x)\,dx = \sum_{k=1}^{\infty} \int_a^b f_k'(x)\,dx$$

Since $f_k(x) \in \mathrm{AC}([a,b])$, by Fundamental Theorem of Calculus II,

$$\sum_{k=1}^{\infty}\int_a^b f_k'(x)\,dx = \sum_{k=1}^{\infty}(f_k(b)-f_k(a)) = \sum_{k=1}^{\infty} f_k(b) - \sum_{k=1}^{\infty} f_k(a) = f(b)-f(a)$$

Thus, we proved that $\int_a^b f'(x)\,dx = f(b)-f(a)$, and so $f \in \mathrm{AC}([a,b])$.

Extra Problem 6. Let $E \in \mathcal{M}$ be a subset of $[0,1]$ s.t. there exists constant $\alpha > 0$ satisfying $m(E \cap [a,b]) \geq \alpha(b-a)$ for all $0 \leq a < b \leq 1$. Prove that $m(E) = 1$.

Suppose $m(E) < 1$, then $m(E^c) > 0$ where $E^c = [0,1] \setminus E$. For any $x \in (0,1)$, there exists $h_0 > 0$ s.t. for all $0 < h < h_0$, $(x-h, x+h) \subset (0,1)$. By assumption, let $a = x-h$ and $b = x+h$,

$$\frac{m(E \cap [x-h, x+h])}{2h} \geq \alpha > 0, \quad \forall\, h \in (0, h_0)$$

Thus, by taking limit as $h \to 0+$ on both sides,

$$\lim_{h\to 0+} \frac{m(E \cap [x-h, x+h])}{2h} \geq \alpha \tag{1}$$

By Lebesgue density theorem,

$$\lim_{h\to 0+} \frac{m(E \cap (x-h, x+h))}{m((x-h, x+h))} = \lim_{h\to 0+} \frac{m(E \cap (x-h, x+h))}{2h} = 0 \tag{2}$$

for almost all $x \in E^c$, so let $A = \{x \in E^c \,|\, (2) \text{ holds}\}$, we have $m(A) > 0$. This implies we can find $x \in A \cap (0,1)$, and for such x, (1) and (2) both holds, which is contradiction. Thus, $m(E) = 1$.

Extra Problem 7. Let f be continuous on $[a,b]$ and differentiable at every $x \in (a,b) \setminus S$, where S is at most countable. Suppose $f'(x) \in L^1(a,b)$. Prove that

$$f(x) = f(a) + \int_a^x f'(t)\,dt, \quad \forall\, x \in [a,b] \tag{1}$$

Recall the Theorem which says if f is continuous on $[a,b]$ and f' exists a.e. on (a,b) s.t. $f' \in L^1(a,b)$ and $m(f(E)) = 0$ for any $E \subset [a,b]$ with $m(E) = 0$, then (1) holds. In this question, it suffices to show $m(f(E)) = 0$ for any $E \subset [a,b]$ with $m(E) = 0$. Take any E with $m(E) = 0$, since $E = (E \setminus S) \cup (E \cap S)$, we have $f(E) = f(E \setminus S) \cup f(E \cap S)$. Notice that $E \cap S$ is at most countable, $f(E \cap S)$ is also countable, so $m(f(E \cap S)) = 0$. By Lemma 1 in lecture, f is measurable on $[a,b]$ and $f'(x)$ exists for all $x \in (E \cap (a,b)) \setminus S$, then

$$m^*(f((E \cap (a,b)) \setminus S)) \leq \int_{(E\cap(a,b))\setminus S} |f'(x)|\,dx$$

Since $f' \in L^1(a,b)$ and $m((E \cap (a,b)) \setminus S) = 0$, $\int_{(E\cap(a,b))\setminus S} |f'(x)|\,dx = 0$, so $m(f((E \cap (a,b)) \setminus S)) = 0$. This shows $m(f(E \setminus S)) = 0$. Combined with $m(f(E \cap S)) = 0$, we have $m(f(E)) = 0$.

Extra Problem 8. Suppose $f \in \mathrm{AC}([a,b])$ and $f(0) = 0$. Prove that

$$\int_0^1 |f(x)f'(x)|\,dx \leq \frac{1}{\sqrt{2}} \int_0^1 (f'(x))^2\,dx$$

Since $f \in \mathrm{AC}([0,1])$, by Fundamental Theorem of Calculus II, $f' \in L^1(0,1)$ and

$$f(x) = f(0) + \int_0^x f'(t)\, dt = \int_0^x f'(t)\, dt$$

If $(f'(x))^2$ is not in $L^1(0,1)$, i.e., $\|f'\|_{L^2(0,1)} = \infty$, then the desireed inequality holds trivially, because $f \in \mathrm{AC}([0,1])$ implies f is bounded by some constant M on $[0,1]$, so

$$\int_0^1 |f(x)f'(x)|\, dx \le M \int_0^1 |f'(x)|\, dx = M\|f'\|_{L^1(0,1)} < \infty = \frac{1}{\sqrt{2}} \int_0^1 (f'(x))^2\, dx$$

If $f'(x) \in L^2(0,1)$, then by Cauchy Schwarz inequality,

$$|f(x)| = \left|\int_0^x f'(t)\, dt\right| \le \left(\int_0^x 1^2\, dt\right)^{1/2} \left(\int_0^x (f'(t))^2\, dt\right)^{1/2} \le \sqrt{x}\|f'\|_{L^2(0,1)}$$

Apply Cauchy Schwarz inequality again,

$$\begin{aligned}\int_0^1 |f(x)f'(x)|\, dx &\le \|f'\|_{L^2(0,1)} \int_0^1 \sqrt{x}|f'(x)|\, dx \\ &\le \left(\int_0^1 (\sqrt{x})^2\, dx\right)^{1/2} \left(\int_0^1 (f'(x))^2\, dx\right)^{1/2} \|f'\|_{L^2(0,1)} \\ &= \frac{1}{\sqrt{2}} \int_0^1 (f'(x))^2\, dx\end{aligned}$$

Thus, we obtain the desired inequality.

Extra Problem 9. Let $\{g_k\}_{k=1}^{\infty} \subset \mathrm{AC}([a,b])$. Assume

- $|g_k'(x)| \le F(x)$ a.e. on (a,b) for all $k \ge 1$, where $F \in L^1(a,b)$.
- there exists $c \in [a,b]$ s.t. $\lim_{k\to\infty} g_k(c)$ exists as a finite number.
- $\lim_{k\to\infty} g_k'(x)$ exists and equal to some finite $f(x)$ a.e. on (a,b).

Prove

(i) $\lim_{k\to\infty} g_k(x)$ exists and equal to some finite $g(x)$ for every $x \in [a,b]$.

For all $x \in [a,c]$, since $g_k \in \mathrm{AC}([a,c])$, by Fundamental Theorem of Calculus II (FTC2), we have

$$g_k(c) - g_k(x) = \int_x^c g_k'(t)\, dt$$

Take limit as $k \to \infty$ on both sides, since $|g_k'(t)| \le F(t) \in L^1(a,b)$, and $g_k \to f$ a.e. on (a,b), we can use DCT to obtain

$$\lim_{k\to\infty} \int_x^c g_k'(t)\, dt = \int_x^c f(t)\, dt$$

Since $g_k \in \mathrm{AC}([a,c])$, $g_k' \in L^1(a,c)$ and so $f \in L^1(a,c)$ because DCT implies $g_k' \to f$ in $L^1(a,c)$ and L^1 space is Banach. Since by assumption (ii), $\lim_{k\to\infty} g_k(c)$ exists as finite number,

$$\hat{g}(x) = \lim_{k\to\infty} g_k(x) = \lim_{k\to\infty} g_k(c) - \int_x^c f(t)\, dt = \lim_{k\to\infty} g_k(c) + \int_a^x f(t)\, dt - \int_a^c f(t)\, dt$$

is a finite number.

Similarly, for $x \in [c, b]$, since $g_k \in \text{AC}([c, b])$, by FTC2, we have

$$g_k(x) - g_k(c) = \int_c^x g_k'(t)\, dt$$

Take limit as $k \to \infty$ on both sides, for the same reason we can use DCT to obtain

$$\lim_{k\to\infty} (g_k(x) - g_k(c)) = \int_c^x f(t)\, dt$$

By the same reason, $\lim_{k\to\infty} g_k(x)$ exists and

$$\tilde{g}(x) = \lim_{k\to\infty} g_k(x) = \lim_{k\to\infty} g_k(c) + \int_c^x f(t)\, dt$$

is a finite number. Therefore, for every $x \in [a, b]$,

$$g(x) = \lim_{k\to\infty} g_k(x) = \begin{cases} \hat{g}(x) & \text{if } x \in [a, c] \\ \tilde{g}(x) & \text{if } x \in [c, b] \end{cases}$$

where $g(x)$ is well-defined because $\hat{g}(c) = \tilde{g}(c) = \lim_{k\to\infty} g_k(c)$.

(ii) Show $g \in \text{AC}([a, b])$ and $g' = f$ a.e. on (a, b).

Notice that $\lim_{k\to\infty} g_k(c)$ and $\int_a^c f(t)\, dt$ is a finite constant, so it must be in $\text{AC}([a, c])$. Since $f \in L^1(a, c)$, $\int_a^x f(t)\, dt$ is in $\text{AC}([a, c])$ by using Fact 4 of absolute continuity, hence $\hat{g}(x)$ is in $\text{AC}([a, c])$. Similarly, $\tilde{g}(x) \in \text{AC}([c, b])$. Furthermore, by FTC1, we can see $\hat{g}'(x) = f(x)$ a.e. on $[a, c]$ and $\tilde{g}'(x) = f(x)$ a.e. on $[c, b]$. This shows that $g'(x) = f(x)$ a.e. on $[a, b]$.

To see $g(x)$ is in $\text{AC}([a, b])$, for any disjoint open intervals $(x_1, y_1), \ldots, (x_n, y_n)$ (in ascending order), if some intervals contains c, then there exists unqiue interval, (x_K, y_K) that contains c and (x_k, y_k) on left hand side of c for $k < K$, on the right hand side of c for $k > K$. Note that

$$\begin{aligned}\sum_{k=1}^{n} |g(y_k) - g(x_k)| &\leq \sum_{k=1}^{K-1} |g(y_k) - g(x_k)| + |g(y_K) - g(c)| \\ &\qquad + |g(c) - g(x_K)| + \sum_{k=K+1}^{n} |g(y_k) - g(x_k)| \\ &= \sum_{k=1}^{K-1} |\hat{g}(y_k) - \hat{g}(x_k)| + |\hat{g}(y_K) - \hat{g}(c)| \\ &\qquad + |\tilde{g}(c) - \tilde{g}(x_K)| + \sum_{k=K+1}^{n} |\tilde{g}(y_k) - \tilde{g}(x_k)|\end{aligned}$$

Since $\hat{g} \in \text{AC}([a, c])$, there exists a small δ_1 s.t. for any finite number of disjoint open intervals with total length less than δ_1, the total variation of $\hat{g}$ on them is less than $\epsilon/2$. Similarly, since $\tilde{g} \in \text{AC}([c, b])$, for any finite number of disjoint open intervals with total length less than δ_2, the total variation of $\tilde{g}$ on them is less than $\epsilon/2$. Let $\hat{\delta} = \min\{\delta_1, \delta_2\}$. Consider any disjoint open intervals $(x_1, y_1), \ldots, (x_n, y_n)$ (in ascending order) containing c, if $\sum_{k=1}^{n}(y_k - x_k) < \hat{\delta}$, then on $[a, c]$, the total length of disjoint open intervals $(x_1, y_1), \ldots, (x_{K-1}, y_{K-1}), (x_K, c)$ is obviously less than $\sum_{k=1}^{n}(y_k - x_k)$, hence less than δ_1, so we have

$$\sum_{k=1}^{K-1} |\hat{g}(y_k) - \hat{g}(x_k)| + |\hat{g}(y_K) - \hat{g}(c)| < \frac{\epsilon}{2}$$

Similarly, on $[c,b]$, the total length of disjoint open intervals $(c,y_K),(x_{K+1},y_{K+1}),\dots,(x_n,y_n)$ is less than $\sum_{k=1}^n(y_k-x_k)$, hence less than δ_2. This shows

$$|\tilde{g}(c)-\tilde{g}(x_K)|+\sum_{k=K+1}^{n}|\tilde{g}(y_k)-\tilde{g}(x_k)|<\frac{\epsilon}{2}$$

The above argument shows that $\sum_{k=1}^n|g(y_k)-g(x_k)|<\epsilon$.

If $(x_1,y_1),\dots,(x_n,y_n)$ (in ascending order) does not contain c, then there exists K s.t. (x_1,y_1), $\dots$,(x_K,y_K) lies on LHS of c and $(x_{K+1},y_{K+1}),\dots,(x_n,y_n)$ lies on RHS of c, then

$$\sum_{k=1}^{n}|g(y_k)-g(x_k)|=\sum_{k=1}^{K}|\hat{g}(y_k)-\hat{g}(x_k)|+\sum_{k=K+1}^{n}|\tilde{g}(y_k)-\tilde{g}(x_k)|$$

Similarly, on $[a,c]$, the total length of $(x_1,y_1),\dots,(x_K,y_K)$ is obviously less than δ_1, so $\sum_{k=1}^K|\hat{g}(y_k)-\hat{g}(x_k)|<\frac{\epsilon}{2}$; on $[c,b]$, the total length of $(x_{K+1},y_{K+1}),\dots,(x_n,y_n)$ is less than δ_2, so $\sum_{k=K+1}^n|\tilde{g}(y_k)-\tilde{g}(x_k)|<\frac{\epsilon}{2}$. This shows $\sum_{k=1}^n|g(y_k)-g(x_k)|<\epsilon$. In conclusion, for any $\epsilon>0$, no matter $(x_1,y_1),\dots,(x_n,y_n)$ contains c or not, we can find $\hat{\delta}$ s.t. as long as $\sum_{k=1}^n(y_k-x_k)<\hat{\delta}$, we have $\sum_{k=1}^n|g(y_k)-g(x_k)|<\epsilon$. This shows $g\in \mathrm{AC}([a,b])$.

Extra Problem 10. Let $f\in \mathrm{BV}([a,b])$. Define $v(x)=V_a^x(f)$. Prove that $f\in\mathrm{AC}([a,b])$ if and only if $v\in\mathrm{AC}([a,b])$.

For "only if" part, if $f\in\mathrm{AC}([a,b])$, then by Fundamental Theorem of Calculus II, $f'\in L^1(a,b)$ and $f(u)=f(a)+\int_a^u f'(t)\,dt=f(a)+g(u)$. Notice that $v(x)=V_a^x(f)=V_a^x(g)=\int_a^x|f'(t)|\,dt$ by a theorem in lecture. However, since $|f'|\in L^1(a,b)$, by Fact 4 of absolute continuity, $v(x)\in\mathrm{AC}([a,b])$.

For "if" part, given $v\in\mathrm{AC}([a,b])$, then for all $\epsilon>0$, there exists $\delta>0$ s.t. for all disjoint open intervals $(x_1,y_1),\dots,(x_n,y_n)$ with $\sum_{k=1}^n(y_k-x_k)<\delta$, $\sum_{k=1}^n|v(y_k)-v(x_k)|<\epsilon$. However,

$$\sum_{k=1}^{n}|f(y_k)-f(x_k)|\le\sum_{k=1}^{n}V_{x_k}^{y_k}(f)=\sum_{k=1}^{n}(V_a^{y_k}-V_a^{x_k})=\sum_{k=1}^{n}|v(y_k)-v(x_k)|$$

This implies that $\sum_{k=1}^n|f(y_k)-f(x_k)|<\epsilon$, so $f\in\mathrm{AC}([a,b])$.