MAT3006*: Real Analysis
Homework 2

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Page 34, Problem 8. Let B be the set of rational numbers in the interval $[0,1]$, and let $\{I_k\}_{k=1}^n$ be a finite collection of open intervals that covers B. Prove that $\sum_{k=1}^n m^*(I_k) \geq 1$.

We first prove a lemma, i.e., $\overline{\bigcup_{i=1}^N E_i} = \bigcup_{i=1}^N \overline{E_i}$ for any finite $N \geq 1$. Since $\bigcup_{i=1}^N \overline{E_i}$ is a closed set (finite union of closd set is closed) containing $\bigcup_{i=1}^N E_i$, by definition of closure, $\overline{\bigcup_{i=1}^N E_i} \subset \bigcup_{i=1}^N \overline{E_i}$. If $x \in \bigcup_{i=1}^N \overline{E_i}$, then x is a limit point of some E_i, thus it is a limit point of $\bigcup_{i=1}^N E_i$, which shows $x \in \overline{\bigcup_{i=1}^N E_i}$. Therefore, $\overline{\bigcup_{i=1}^N E_i} \supset \bigcup_{i=1}^N \overline{E_i}$ and the claim is proved.

Take a sequence of I_k's that covers B, then $B \subset \bigcup_{k=1}^N I_k$, Take closure on both sides yields $[0,1] \subset \overline{\bigcup_{k=1}^N I_k} = \bigcup_{k=1}^N \overline{I_k}$. Therefore, $m^*([0,1]) \leq m^*(\cup_{k=1}^N \overline{I_k}) \leq \sum_{k=1}^N m^*(\overline{I_k})$. Since I_k's are open interval, so $m^*(I_k) = m^*(\overline{I_k})$ for all $k = 1, \ldots, N$. This is sufficient to show $\sum_{k=1}^n m^*(I_k) \geq 1$.

Page 34, Problem 9. Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.

Since $B \subset A \cup B$, by property 2 of outer measure, $m^*(B) \leq m^*(A \cup B)$. By property 3 of outer measure, $m^*(A \cup B) \leq m^*(A) + m^*(B) = m^*(B)$. Thus, $m^*(A \cup B) = m^*(B)$.

Page 34, Problem 10. Let A and B be bounded sets for which there is an $\alpha > 0$ such that $|a - b| \geq \alpha$ for all $a \in A, b \in B$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.

For all $a \in A$, let $N(a; \alpha/3)$ be open ball centered at a with radius $\alpha/3$, then $G = \bigcup_{a \in A} N(a; \alpha/3)$ is open set containing A. Similarly, denote $H = \bigcup_{b \in B} N(b; \alpha/3)$, and it is also an open set containing B. Now we claim that $G \cap H = \varnothing$. If there exists $c \in G \cap H$, then there exists $a_0 \in A$ and $b_0 \in B$ such that $|a_0 - c| < \alpha/3$ and $|b_0 - c| < \alpha/3$. Consider

$$|a_0 - b_0| \leq |a_0 - c| + |c - b_0| < \frac{2}{3}\alpha < \alpha$$

which contradicts to $|a - b| \geq \alpha$ for all $a \in A$ and $b \in B$. Therefore, by property 6 of outer measure, $m^*(A \cup B) = m^*(A) + m^*(B)$.

Extra Problem 1. Let $\mathcal{M}$ denote the collection of all Lebesgue measurable sets. Prove that if $E \in \mathcal{M}$, then $E^c \in \mathcal{M}$.

Since E is measurable, for all $n \in \mathbb{N}^+$, there exists open set G_n such that $m^*(G_n \setminus E) < \frac{1}{n}$. Since $F_n = G_n^c$ is closed, by property 4 of Lebesgue measure, $F_n \in \mathcal{M}$. Let $H = \bigcup_{n=1}^\infty F_n$, then H

is measurable by property 3 of Lebesgue measure. Note that $H \subset E^c$, so let $A = E^c \setminus H$, we tend to show $m^*(A) = 0$. This is true because for all $k \in \mathbb{N}^+$, $A = E^c \setminus H \subset E^c \setminus F_k = E^c \setminus G_k^c = G_k \setminus E$, which shows $m^*(A) \leq m^*(G_k \setminus E) < \frac{1}{k}$. Take $k \to \infty$, we conclude that $m^*(A) = 0$. Then by property 2 of Lebesgue measure, $A \in \mathcal{M}$, and by property 3 of Lebesgue measure, $E^c = A \cup H$ is measurable.

Extra Problem 2. If $E \in \mathcal{M}$, prove that for all $\epsilon > 0$, there exists closed subset $F \subset E$ such that $m^*(E \setminus F) < \epsilon$.

By Extra Problem 1, $E^c \in \mathcal{M}$, so for all $\epsilon > 0$, there exists an open set G such that $G \supset E^c$ and $m^*(G \setminus E^c) < \epsilon$. Consider $F = G^c$ is a closed set, and $F \subset E$. Note that $E \setminus F = E \setminus G^c = G \setminus E^c$, so $m^*(E \setminus F) = m^*(G \setminus E^c) < \epsilon$.

Extra Problem 3. If $E_k \in \mathcal{M}$ for $k = 1, 2, \ldots$, prove that $\bigcap_{k=1}^{\infty} E_k \in \mathcal{M}$.

If $E_k \in \mathcal{M}$ for all $k \in \mathbb{N}^+$, then by Extra Problem 1, $E_k^c \in \mathcal{M}$. Since $E_k^c \in \mathcal{M}$ for all $k \in \mathbb{N}^+$, then by property 3 of Lebesgue measure, $\bigcup_{k=1}^{\infty} E_k^c \in \mathcal{M}$. By Extra Problem 1, $(\cup_{k=1}^{\infty} E_k^c)^c \in \mathcal{M}$. Since $(\cup_{k=1}^{\infty} E_k^c)^c = \cap_{k=1}^{\infty} E_k$, we proved that $\bigcap_{k=1}^{\infty} E_k \in \mathcal{M}$.

Extra Problem 4. Let $E_k \in \mathcal{M}$ for $k = 1, 2, \ldots$, such that $E_k \cap E_j = \varnothing$ if $k \neq j$. Prove that $m(\cup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m(E_k)$.

We first prove a lemma, that is, if C_k, $k = 1, \ldots, K$, are pairwise disjoint compact subsets of $\mathbb{R}^n$, then $m(\cup_{k=1}^{K} C_k) = \sum_{k=1}^{K} m(C_k)$. Suppose $\text{dist}(C_i, C_j) = 0$, then there exists $a_n \in C_i$ and $b_n \in C_j$ such that $d(a_n, b_n) \to 0$, where d is the metric function. Since $C_i \times C_j$ is also compact, $d(x, y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is a continuous function, and continuous function on compact set attains its infimum, so there exists $a \in C_i$ and $b \in C_j$ such that $d(a, b) = 0$. However, $d(a, b) = 0$ implies that $a = b$, so $C_i \cap C_j \neq \varnothing$, contradiction. Therefore, $\text{dist}(C_i, C_j) > 0$. By remark of property 6 of outer measure, there exists open set G_i, G_j such that $G_i \supset C_i$ and $G_j \supset C_j$ and $G_i \cap G_j = \varnothing$. By property 6 of outer measure, $m^*(C_i \cup C_j) = m^*(C_i) + m^*(C_j)$. Using induction, it is easy to see $m^*(\cup_{k=1}^{K} C_k) = \sum_{k=1}^{K} m^*(C_k)$. Since compact set must be Lebesgue measurable, we obtain $m(\cup_{k=1}^{K} C_k) = \sum_{k=1}^{K} m(C_k)$.

Then we prove the desired statement is true when all E_k's are bounded. By property 3 of outer measure, $m(\cup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m(E_k)$. By Extra Problem 2, for all $\epsilon > 0$, there exists closed $F_k \subset E_k$ such that $m(E_k \setminus F_k) < \frac{\epsilon}{2^k}$. Then it is obvious that for all $k \geq 1$,

$$m(E_k) \leq m(F_k) + m(E_k \setminus F_k) < \frac{\epsilon}{2^k} + m(F_k)$$

Take summation on left, middle and right from $k = 1, 2, \ldots K$, we have

$$\sum_{k=1}^{K} m(E_k) \leq \sum_{k=1}^{K} (m(F_k) + m(E_k \setminus F_k)) < \epsilon + \sum_{k=1}^{K} m(F_k)$$

Since $F_k \subset E_k$ and F_k's are pairwise disjoint compact set, $\sum_{k=1}^{K} m(F_k) = m(\cup_{k=1}^{K} F_k) \leq m(\cup_{k=1}^{\infty} E_k)$. Take $K \to \infty$, we have $\sum_{k=1}^{\infty} m(E_k) \leq \epsilon + m(\cup_{k=1}^{\infty} E_k)$. Take $\epsilon \to \infty$, $\sum_{k=1}^{\infty} m(E_k) \leq m(\cup_{k=1}^{\infty} E_k)$. Therefore, we proved the desired statement for bounded E_k's.

For general E_k's, define B_j as the open ball in $\mathbb{R}^n$ centered at the origin with radius j for all $j \in \mathbb{N}^+$ and set $B_0 = \varnothing$. For $i, j \in \mathbb{N}^+$, define $E_{kj} = E_k \cap (B_j \setminus B_{j-1})$, then E_{kj}'s are measurable, pairwise disjoint and bounded. Therefore,

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = m\left(\bigcup_{k,j\in\mathbb{N}^+} E_{kj}\right) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} m(E_{kj}) = \sum_{k=1}^{\infty} m(E_k)$$

because $E_k = \bigcup_{j=1}^{\infty} E_{kj}$ and $m(E_k) = \sum_{j=1}^{\infty} m(E_{kj})$ because E_{kj}'s are pairwise disjoint and bounded. Therefore, the general case is also proved.

Extra Problem 5. For all $E, F \in \mathcal{M}$ such that $F \subset E$, prove that $m(E \setminus F) + m(F) = m(E)$. Furthermore, if $m(F) < \infty$, then $m(E \setminus F) = m(E) - m(F)$.

By Extra Problem 4, set $E_1 = E \setminus F$, $E_2 = F$, and $E_k = \varnothing$ for $k \geq 3$, then it is obvious that E_k's are pairwise disjoint measurable set, so $m(E_1) + m(E_2) = m(E_1 \cup E_2) = m(E)$, which shows $m(E \setminus F) + m(F) = m(E)$. If $m(F) < \infty$, then we can deduce a finite number $m(F)$ on both sides of the equation, and the equation still holds, i.e., $m(E \setminus F) = m(E) - m(F)$.

Extra Problem 6. Supose $E_k \in \mathcal{M}$ for all $k = 1, 2, \ldots$, prove

(i) If $E_1 \subset E_2 \subset \cdots \subset E_k \subset E_{k+1} \subset \cdots$, then $\lim_{k\to\infty} m(E_k) = m(\lim_{k\to\infty} E_k)$.

Let $F_1 = E_1$, and $F_k = E_k \setminus E_{k-1}$ for all $k \geq 2$. Then F_k's are pairwise disjoint for all $k = 1, 2, \ldots$. Since all E_k's are measurable, by Extra Problem 1, E_k^c's are measurable, and by Extra Problem 3, F_k's are measurable. By Extra Problem 4, we have

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = m\left(\bigcup_{k=1}^{\infty} F_k\right) = \sum_{k=1}^{\infty} m(F_k) = \lim_{n\to\infty}\sum_{k=1}^{n} m(F_k) = \lim_{n\to\infty} m\left(\bigcup_{k=1}^{n} F_k\right) = \lim_{n\to\infty} m(E_n)$$

Therefore, $m(\lim_{k\to\infty} E_k) = m\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{n\to\infty} m(E_n)$.

(ii) If $E_1 \supset E_2 \supset \cdots \supset E_k \supset E_{k+1} \supset \cdots$ and there exists $k_0 \geq 1$ such that $m(E_{k_0}) < \infty$, then $\lim_{k\to\infty} m(E_k) = m(\lim_{k\to\infty} E_k)$.

WLOG, we only consider the smallest one among such k_0 and denote it as k_0. Let $F_k = E_k \setminus E_{k+1}$ for $k \geq k_0$, and denote $E = \cap_{k=k_0}^{\infty} E_k = \cap_{k=1}^{\infty} E_k$. Then E, F_k for all $k \geq k_0$ are all pairwise disjoint, and $E_n \setminus E = \cup_{k=n}^{\infty} F_n$ for $n \geq k_0$. Since $m(E_{k_0})$ is finite, $m(\cup_{k=n}^{\infty} F_n)$ is also finite for all $n \geq k_0$. Hence,

$$m(E_n) = m(E) + m\left(\bigcup_{k=n}^{\infty} F_k\right) = m(E) + \sum_{k=n}^{\infty} m(F_k) = m(E) + \sum_{k=k_0}^{\infty} m(F_k) - \sum_{k=k_0}^{n-1} m(F_k)$$

Since $\sum_{k=k_0}^{n-1} m(F_k)$ is increasing in n and $\sum_{k=k_0}^{\infty} m(F_k)$ is finite, take $n \to \infty$, we have

$$\lim_{n\to\infty} m(E_n) = m(E) + \sum_{k=k_0}^{\infty} m(F_k) - \sum_{k=k_0}^{\infty} m(F_k) = m(E) = m\left(\bigcap_{k=1}^{\infty} E_k\right)$$

Therefore, $\lim_{n\to\infty} m(E_n) = m(\lim_{n\to\infty} \cap_{k=1}^{n} E_k) = m(\lim_{n\to\infty} E_n)$.

(iii) Find a counter-example of (ii) if such k_0 in (ii) does not exist.

Let $E_k = [k, \infty)$, then $m(E_k) = \infty$ for all $k \geq 1$, so such k_0 does not exist. It is obvious that

$$\lim_{k\to\infty} m(E_k) = \infty \neq 0 = m(\varnothing) = m\left(\bigcap_{k=1}^{\infty} E_k\right) = m\left(\lim_{k\to\infty} \bigcap_{i=1}^{k} E_i\right) = m\left(\lim_{k\to\infty} E_k\right)$$

Therefore, (ii) is not true if such k_0 does not exist.

Extra Problem 7. Prove the Cantor set C is Lebesgue measurable and $m(C) = 0$.

Notice that $C = \bigcap_{k=1}^{\infty} F_k$ where F_k's satisfy $F_1 \supset F_2 \supset \cdots$ and since all $F_k \subset [0,1]$ are closed set, $m(F_k) \leq 1$. Therefore, we can apply Extra Problem 6 (ii), that is,

$$m(C) = m\left(\bigcap_{k=1}^{\infty} F_k\right) = m\left(\lim_{k\to\infty} F_k\right) = \lim_{k\to\infty} m(F_k)$$

Since F_k consists of 2^k disjoint closed intervals where each of the closed interval is of length 3^{-k}, we conclude that $m(C) = \lim_{k\to\infty} \left(\frac{2}{3}\right)^k = 0$.

Extra Problem 8. Let C_p be the Cantor-like set in HW1, Extra Problem 3. Prove that $C_p \in \mathcal{M}$ and compute $m(C_p)$.

Notice that $C_p = \bigcap_{k=1}^{\infty} F_k$ where F_k's consist of 2^n closed subintervals of equal length, and since closed set are measurable and finite union of measurable sets is measurable, $F_k \in \mathcal{M}$ for all $k \geq 1$. By Extra Problem 3, $C_p = \bigcap_{k=1}^{\infty} F_k \in \mathcal{M}$.

By HW1, Extra Problem 3, the total length of all open intervals removed is equal to $\frac{1}{p-2}$. Since every removed open intervals are disjoint, by σ-additivity, $m([0,1] \setminus C_p) = \frac{1}{p-2}$. Thus, $m(C_p) = \frac{p-3}{p-2}$ for all $p \geq 3$, p integer.

Extra Problem 9. A subset of $\mathbb{R}^n$ is said to be of F_σ-type if it is the countable union of closed subsets of $\mathbb{R}^n$. Similarly, a subset of $\mathbb{R}^n$ is said to be of G_δ-type if it is the countable intersection of open subsets of $\mathbb{R}^n$.

(i) Let $\{f_n(x)\}_{n=1}^{\infty}$ be continuous on $\mathbb{R}$. Prove that $\{x \in \mathbb{R} \mid \underline{\lim}_{n\to\infty} f_n(x) > 0\}$ is F_σ-type.

It is easy to see that $\{x \in \mathbb{R} \mid \underline{\lim}_{n\to\infty} f_n(x) > 0\} = \bigcup_{k\in\mathbb{N}^+}\{x \in \mathbb{R} \mid \sup_{m\geq 1} \inf_{n\geq m} f_n(x) \geq \frac{1}{k}\}$. Furthermore, we have $\{x \in \mathbb{R} \mid \sup_{m\geq 1} \inf_{n\geq m} f_n(x) \geq \frac{1}{k}\} = \bigcup_{m\geq 1} \bigcap_{n\geq m}\{x \in \mathbb{R} \mid f_n(x) \geq \frac{1}{k}\}$. Since $f_n(x)$ is continuous, $\{x \in \mathbb{R} \mid f_n(x) \geq \frac{1}{k}\}$ is a closed set because $[\frac{1}{k}, +\infty)$ is closed for all $k \in \mathbb{N}^+$. Therefore, we have $\{x \in \mathbb{R} \mid \underline{\lim}_{n\to\infty} f_n(x) > 0\} = \bigcup_{k\in\mathbb{N}^+, m\in\mathbb{N}^+} F_{m,k}$, where $F_{m,k} = \bigcap_{n\geq m}\{x \in \mathbb{R} \mid f_n(x) \geq \frac{1}{k}\}$ is closed because it is the intersection of closed set. Therefore, $\{x \in \mathbb{R} \mid \underline{\lim}_{n\to\infty} f_n(x) > 0\}$ is F_σ-type.

(ii) Let $f(x)$ be defined on $\mathbb{R}$. Prove that $\{x \in \mathbb{R} \mid \lim_{y\to x} f(y) < \infty\}$ is G_δ-type.

Define a function $W(x)$ on $\mathbb{R}$ as

$$W(x) = \inf_{\delta>0} \sup_{y,z\in N_\delta^o(x)} |f(y) - f(z)|$$

where $N_\delta^o(x)$ is the deleting neighborhood of x with radius δ. Then we claim that $W(x) = 0$ is equivalent to say $\lim_{y\to x} f(y) = L < \infty$. If this is true, then

$$\left\{x \in \mathbb{R} \,\middle|\, \lim_{y\to x} f(y) < \infty\right\} = \{x \in \mathbb{R} \,|\, W(x) = 0\} = \bigcap_{n=1}^{\infty} \left\{x \in \mathbb{R} \,|\, W(x) < \frac{1}{n}\right\}$$

Furthermore, consider

$$\left\{x \in \mathbb{R} \,\middle|\, W(x) < \frac{1}{n}\right\} = \bigcup_{x,\delta>0} \left\{N_\delta^o(x) \,\middle|\, \sup_{y,z\in N_\delta^o(x)} |f(y) - f(z)| < \frac{1}{n}\right\}$$

Note that LHS $\subset$ RHS is obvious, and for $u \in$ RHS, there exists x_0, δ_0 such that $u \in N_{\delta_0}^o(x_0)$. Since u is an interior point of $N_{\delta_0}^o(x_0)$, there exists δ_1 such that $N_{\delta_1}^o(u) \subset N_{\delta_0}^o(x_0)$. This shows

$$W(u) = \inf_{\delta>0} \sup_{y,z\in N_\delta^o(u)} |f(y) - f(z)| \leq \sup_{y,z\in N_{\delta_1}^o(u)} |f(y) - f(z)| \leq \sup_{y,z\in N_{\delta_0}^o(u)} |f(y) - f(z)| < \frac{1}{n}$$

Therefore, $u \in$ LHS, which verifies the equality above. Since $\left\{x \in \mathbb{R} \,\middle|\, W(x) < \frac{1}{n}\right\}$ is the union of open set, it is open. Hence, $\{x \in \mathbb{R} \,|\, \lim_{y\to x} f(y) < \infty\}$ is G_δ-type because it is the countable intersection of those open sets.

To prove our first claim, i.e.,

$$\left\{x \in \mathbb{R} \,\middle|\, \lim_{y\to x} f(y) < \infty\right\} = \{x \in \mathbb{R} \,|\, W(x) = 0\}$$

Note that by definition of function limit, it is easy to see LHS $\subset$ RHS. Now consider $x \in$ RHS, $W(x) = 0$. For $u \in$ RHS, we have for all $\epsilon > 0$, there exists $\delta > 0$, such that $\sup_{y,z\in N_\delta^o(x)} |f(y) - f(z)| < \epsilon$. Consider $x_n \to x$ and $x_n < x$, there exists N such that for all $n \geq m > N$, $|f(x_n) - f(x_m)| < \epsilon$. This shows $f(x_n)$ is a Cauchy sequence, hence convergent to L. Similarly, for $x_n \to x$ but $x_n > x$, $f(x_n) \to L'$. It is not hard to see $L' = L$, so the left limit of f at x is equal to the right limit, meaning that $\lim_{y\to x} f(y)$ exists.

Extra Problem 10. Let F_k for $k \in \mathbb{N}^+$ be nonempty closed subsets of $\mathbb{R}^n$ s.t. $\text{dist}(x_0, F_k) \to \infty$ as $k \to \infty$ for a fixed point $x_0 \in \mathbb{R}^n$. Prove that $\overline{\bigcup_{k=1}^\infty F_k} = \bigcup_{k=1}^\infty F_k$.

The fact that $\overline{\bigcup_{k=1}^\infty F_k} \supset \bigcup_{k=1}^\infty F_k$ is trivial, so we only need to prove $\overline{\bigcup_{k=1}^\infty F_k} \subset \bigcup_{k=1}^\infty F_k$. For arbitrary $x \in \overline{\bigcup_{k=1}^\infty F_k}$, if $x \notin \bigcup_{k=1}^\infty F_k$, then there exists a sequence $\{x_n\} \subset \bigcup_{k=1}^\infty F_k$ such that $x_n \to x$. If all x_n's lie in finitely many F_k's then there exists k_0 such that F_{k_0} contains infinitely many x_n's, i.e., there exists a subsequence of x_n, denoted as x_{n_j} such that for all $j \in \mathbb{N}^+$, $x_{n_j} \in F_{k_0}$. Note that $x_{n_j} \to x$, so x is a limit point of F_{k_0}, but F_{k_0} is closed, so $x \in F_{k_0} \subset \bigcup_{k=1}^\infty F_k$ which is a contradiction to $x \notin \bigcup_{k=1}^\infty F_k$. This means all x_n's cannot lie in finitely many F_k's, so there exists a subsequence F_{k_j} of F_k such that there exists at least one $x_{n_j} \in F_{k_j}$. Then since $\text{dist}(x_0, F_k) \to \infty$, $\text{dist}(x_0, F_{k_j}) \to \infty$ and thus $d(x_0, x_{n_j}) \geq \text{dist}(x_0, F_{k_j}) \to \infty$. This is impossible because

$$d(x_0, x_{n_j}) \leq d(x_0, x) + d(x, x_{n_j}) \to d(x_0, x)$$

as $j \to \infty$, and $d(x_0, x)$ is a finite constant which cannot tend to infinity. This implies if $x \in \overline{\bigcup_{k=1}^\infty F_k}$, we must have $x \in \bigcup_{k=1}^\infty F_k$. Therefore, $\overline{\bigcup_{k=1}^\infty F_k} \subset \bigcup_{k=1}^\infty F_k$ and we are done.

Extra Problem 11. Prove that $\frac{1}{4}$ is in Cantor set C.

Notice that

$$\frac{1}{4}=\frac{0}{3}+\frac{2}{3^2}+\frac{0}{3^3}+\frac{2}{3^4}+\cdots+\frac{0}{3^{2k-1}}+\frac{2}{3^{2k}}+\cdots=\sum_{k=1}^{\infty}\frac{a_k}{3^k}$$

which shows that in the ternary expression, $\frac{1}{4}=0.020202\cdots_{(3)}$. Notice that any number in $[0,1]$ that only contains digits 0 and 2 under ternary system are in Cantor set, so $\frac{1}{4}\in C$.

Another way to prove $\frac{1}{4}\in C$ is by using the formula

$$C=[0,1]\setminus\bigcup_{n=1}^{\infty}\bigcup_{k=0}^{3^{n-1}-1}\left(\frac{3k+1}{3^n},\frac{3k+2}{3^n}\right)$$

Suppose $\frac{1}{4}\notin C$, then there exists $n\in\mathbb{N}^+$ and $k\in\mathbb{N}$, $k\le 3^{n-1}-1$ such that $\frac{1}{4}\in(\frac{3k+1}{3^n},\frac{3k+2}{3^n})$. Therefore, we have $12k+4<3^n<12k+8$, which means there are only three possible cases: $3^n=12k+5$, $3^n=12k+6$ and $3^n=12k+7$. However, 3^n is an odd integer but $12k+6$ is even for all k, so $3^n\neq 12k+6$. Also, $3^n\equiv 0 \pmod 3$, but $12k+5\equiv 2 \pmod 3$ and $12k+7\equiv 1 \pmod 3$, so $3^n\neq 12k+5$ and $3^n\neq 12k+7$. Therefore, such n and k does not exist, which means $\frac{1}{4}\in C$.

Extra Problem 12. Let $E\subset\mathbb{R}$ with finite $m^*(E)>0$. Prove that $\forall\, a\in(0,m^*(E))$, there exists $A\subset E$ such that $m^*(A)=a$.

Define $f(r)=m^*(E\cap(-r,r))$, then we claim that $f(0)=0$, $f(r)\to m^*(E)$ as $r\to\infty$ and $f(r)$ is continuous on $[0,\infty)$. If all of these claims are true, then by intermediate value theorem, for all $a\in(0,m^*(E))$ there exists $r>0$ such that $f(r)=a$ and we can take $A=E\cap(-r,r)$. Then $A\subset E$ and $m^*(A)=a$.

Let us prove all those claims. First, it is trivial that $f(0)=0$ because $m^*(\varnothing)=0$.

Second, by corollary of property 5 of outer measure, there exists a G_δ set G such that $G\supset E$ and $m(G)=m^*(E)$, so

$$\begin{aligned}f(r)&=m^*(E\cap(-r,r))\ge m^*(E)-m^*(E\cap(-r,r)^c)\\&\ge m^*(E)-m^*(G\cap(-r,r)^c)=m^*(E)-m(G\cap(-r,r)^c)\end{aligned}$$

By Extra Problem 6 (ii), since $m(G)=m^*(E)<\infty$, $m(G\cap(-n,n))<\infty$ for all $n\in\mathbb{N}^+$,

$$\begin{aligned}\lim_{r\to\infty}f(r)&=\lim_{n\to\infty}f(n)\ge\lim_{n\to\infty}[m^*(E)-m(G\cap(-n,n)^c)]\\&=m^*(E)-m\left(G\cap\lim_{n\to\infty}(-n,n)^c\right)=m^*(E)-m(G\cap\varnothing)=m^*(E)\end{aligned}$$

Since $f(r)\le m^*(E)$ is trivial, we can conclude that $f(r)\to m^*(E)$ as $r\to\infty$.

To prove the continuity, first it is obvious that $f(r_1)\ge f(r_2)$ if $r_1\ge r_2$. For all $\epsilon>0$, fix any $r_0\in[0,\infty)$, take $\delta=\epsilon/2$, for all $0<r-r_0<\delta$, we have

$$\begin{aligned}f(r)-f(r_0)&=m^*(E\cap(-r,r))-m^*(E\cap(-r_0,r_0))\\&\le m^*(E\cap(-r,-r_0))+m^*(E\cap(-r_0,r_0))+m^*(E\cap(r_0,r))-m^*(E\cap(-r_0,r_0))\\&=m^*(E\cap(-r,-r_0))+m^*(E\cap(r_0,r))\le 2(r-r_0)<2\delta<\epsilon\end{aligned}$$

Combined with $f(r) - f(r_0) \geq 0$, we can conclude that $f(r)$ is right continuous at any point $r_0 \in [0, \infty)$. Similarly, for all $0 < r_0 - r < \delta$, we can prove $0 \leq f(r_0) - f(r) < \epsilon$, which means $f(r)$ is left continuous at any point $r_0 \in (0, \infty)$. Therefore, $f(r)$ is continuous at $[0, \infty)$.

Extra Problem 13. Let $A_1, A_2 \subset \mathbb{R}^n$, $A_1 \subset A_2$, $A_1 \in \mathcal{M}$, $m(A_1) = m^*(A_2) < \infty$. Prove that $A_2 \in \mathcal{M}$.

By definition of outer measure, for all $\epsilon > 0$, there exists a collection of open rectangles $\{I_n\}_{n=1}^{\infty}$ such that $m^*(A_2) + \epsilon > \sum_{n=1}^{\infty} m^*(I_n)$. Since $m(A_1) = m^*(A_2)$ and I_n's are measurable, we have $m(A_1) + \epsilon > \sum_{n=1}^{\infty} m(I_n)$. Consider $G = \bigcup_{n=1}^{\infty} I_n \supset A_2 \supset A_1$, we have

$$m^*(G \setminus A_2) \leq m\left(G \setminus A_1\right) = m\left(\bigcup_{n=1}^{\infty} I_n\right) - m(A_1) < \sum_{n=1}^{\infty} m(I_n) - \left(\sum_{n=1}^{\infty} m(I_n) - \epsilon\right) = \epsilon$$

where the first equality is because G (open set) and A_1 are both measurable and the last equality is because $\sum_{n=1}^{\infty} m(I_n)$ is bounded above by $m^*(A_2) + \epsilon < \infty$. Therefore, we proved that for all $\epsilon > 0$, there exists an open set G covering A_2 and $m^*(G \setminus A_2) < \epsilon$, which means $A_2 \in \mathcal{M}$.

Extra Problem 14. Prove that $E \in \mathcal{M}$ if and only if $\forall\, T \subset \mathbb{R}^n$, $m^*(T) = m^*(T \cap E) + m^*(T \cap E^c)$.

We first prove the "only if" part. It is obvious that $m^*(T) \leq m^*(T \cap E) + m^*(T \cap E^c)$. Again, by corollary of property 5 of outer measure, there exists G_δ set G such that $G \supset T$ and $m(G) = m^*(T)$. Notice that $G = (G \cap E) \cup (G \setminus E)$, since G and E are measurable,

$$m^*(T) = m(G) = m(G \cap E) + m(G \setminus E) \geq m^*(T \cap E) + m^*(T \setminus E)$$

Therefore, $m^*(T) = m^*(T \cap E) + m^*(T \cap E^c)$.

Then we prove the "if" part. Note that $E = \bigcup_{k=1}^{\infty} E_k$ where $E_k = E \cap B(0; k)$ for $k \in \mathbb{N}^+$, so we only need to prove $E_k \in \mathcal{M}$. For each $k \in \mathbb{N}^+$ and all $\epsilon > 0$, there exists sequence of open rectangles $\{I_n^k\}_{n=1}^{\infty}$ such that $m^*(E_k) + \epsilon > \sum_{n=1}^{\infty} m^*(I_n^k) \geq m^*\left(\bigcup_{n=1}^{\infty} I_n^k\right)$. Denote $U_k = \bigcup_{n=1}^{\infty} I_n^k$ and $V_k = U_k \cap B_k$ where $B_k = B(0; k)$. Note that V_k is bounded and since U_k and B_k are open, so is V_k. Now V_k is open and $U_k \supset V_k \supset E_k$, so we only need to prove $m^*(V_k \setminus E_k) < \epsilon$. By assumption, take $T = V_k$, $m^*(V_k) = m^*(V_k \setminus E) + m^*(V_k \cap E) < \infty$.

Notice that $V_k \setminus E = U_k \cap B_k \cap E^c$, and

$$\begin{aligned} V_k \setminus E_k &= (U_k \cap B_k) \cap (E \cap B_k)^c = (U_k \cap B_k) \cap (E^c \cup B_k^c) \\ &= (U_k \cap B_k \cap E^c) \cup (U_k \cap B_k \cap B_k^c) = U_k \cap B_k \cap E^c \end{aligned}$$

Therefore, $m^*(V_k \setminus E) = m^*(V_k \setminus E_k)$ and since $V_k \cap E = U_k \cap E_k = E_k$, we have $m^*(V_k) = m^*(V_k \setminus E_k) + m^*(E_k)$. Combined with $m^*(E_k) + \epsilon > m^*(U_k) \geq m^*(V_k)$, we obtain $m^*(V_k \setminus E_k) < \epsilon$. Therefore $E_k \in \mathcal{M}$ and thus $E \in \mathcal{M}$.

Extra Problem 15. Let $A \in \mathcal{M}$, $B \subset \mathbb{R}^n$ with $m^*(B) < \infty$. Prove $m^*(A \cup B) + m^*(A \cap B) = m^*(A) + m^*(B)$.

Since $A \in \mathcal{M}$, by Extra Problem 14, for all $T \subset \mathbb{R}^n$, $m^*(T) = m^*(T \cap A) + m^*(T \cap A^c)$. Let $T = B$ and $A \cup B$, we have

$$m^*(B) = m^*(B \cap A) + m^*(B \cap A^c) \tag{1}$$

$$m^*(A \cup B) = m^*(A \cup B \cap A) + m^*(A \cup B \cap A^c) = m^*(A) + m^*(B \cap A^c) \tag{2}$$

Since $m^*(B) < \infty$, $m^*(B \cap A^c) \le m^*(B) < \infty$ and $m^*(A \cap B) \le m^*(B) < \infty$, (2) − (1) yields

$$m^*(A \cup B) - m^*(B) = m^*(A) - m^*(B \cap A) \Longrightarrow m^*(A \cup B) + m^*(A \cap B) = m^*(A) + m^*(B)$$

Notice that add or subtract a finite number on both sides of a equation will not change the equality.

Extra Problem 16. Suppose $m^*(E) < \infty$. If $m^*(E) = \sup\{m(F) \mid F \subset E, F \text{ closed}\}$, then $E \in \mathcal{M}$.

By definition, for all $n \in \mathbb{N}^+$, there exists F_n closed and $F_n \subset E$ such that $m^*(E) < m^*(F_n) + \frac{1}{n}$. Take $F = \bigcup_{n=1}^\infty F_n$, so F is measurable and $m^*(F) \ge m(F_n)$. This impiles that $m^*(E) < m^*(F) + \frac{1}{n}$ for all $n \in \mathbb{N}^+$. Take $n \to \infty$, we have $m^*(E) \le m^*(F)$, and since $m^*(E) \ge m^*(F)$ by $F \subset E$, we have $m^*(E) = m^*(F) = m(F)$. Apply Extra Problem 13, since $F \in \mathcal{M}$ and $m(F) = m^*(E) < \infty$, we conclude that $E \in \mathcal{M}$.

Extra Problem 17. Prove that if $E_k \in \mathcal{M}$ for $k \in \mathbb{N}^+$.

(i) $m\left(\underline{\lim}_{k\to\infty} E_k\right) \le \underline{\lim}_{k\to\infty} m(E_k)$.

Let $F_n = \bigcap_{k=n}^\infty E_k$, then $F_n \in \mathcal{M}$ and they form an increasing sequence. By Extra Problem 6(i), $m(\cup_{n=1}^\infty F_n) = \lim_{n\to\infty} m(F_n)$. Also, for all $k \ge n$, $m(F_n) \le m(E_k)$, so $m(F_n) \le \inf_{k\ge n} m(E_k)$. Therefore,

$$m\left(\underline{\lim}_{k\to\infty} E_k\right) = m\left(\bigcup_{n=1}^\infty \bigcap_{k=n}^\infty E_k\right) = \lim_{n\to\infty} m(F_n) \le \lim_{n\to\infty} \inf_{k\ge n} m(E_k) = \underline{\lim}_{k\to\infty} m(E_k)$$

This shows $m\left(\underline{\lim}_{k\to\infty} E_k\right) \le \underline{\lim}_{k\to\infty} m(E_k)$.

(ii) If there exists $k_0 \ge 1$ such that $m\left(\cup_{k=k_0}^\infty E_k\right) < \infty$, then $m\left(\overline{\lim}_{k\to\infty} E_k\right) \ge \overline{\lim}_{k\to\infty} m(E_k)$.

Again, Let $F_n = \bigcup_{k=n}^\infty E_k$, then $F_n \in \mathcal{M}$ and they form an decreasing sequence satisfying that there exists k_0 such that $m(F_{k_0}) < \infty$. Therefore, by Extra Problem 6(ii), $m(\cap_{n=1}^\infty F_n) = \lim_{n\to\infty} m(F_n)$. Since $m(F_n) \ge m(E_k)$ for all $k \ge n$, we have $m(F_n) \ge \sup_{k\ge n} m(E_k)$. Thus,

$$m\left(\overline{\lim}_{k\to\infty} E_k\right) = m\left(\bigcap_{n=1}^\infty \bigcup_{k=n}^\infty E_k\right) = \lim_{n\to\infty} m(F_n) \ge \lim_{n\to\infty} \sup_{k\ge n} E_k = \overline{\lim}_{k\to\infty} m(E_k)$$

This shows $m\left(\overline{\lim}_{k\to\infty} E_k\right) \ge \overline{\lim}_{k\to\infty} m(E_k)$.

Extra Problem 18. Let $E_k \subset [0,1]$, $E_k \in \mathcal{M}$, $m(E_k) = 1$ for all $k \in \mathbb{N}^+$. Prove $m(\cap_{k=1}^\infty E_k) = 1$.

Since $E_k \subset [0,1]$, $E_k \in \mathcal{M}$, for all $k \in \mathbb{N}^+$, we have

$$m([0,1] \setminus E_k) = m([0,1]) - m(E_k) = 1 - m(E_k) = 0$$

Consider

$$m([0,1] \setminus \cap_{k=1}^{\infty} E_k) = m(\cup_{k=1}^{\infty}([0,1] \setminus E_k)) \le \sum_{k=1}^{\infty} m([0,1] \setminus E_k) = 0$$

Therefore, $m([0,1] \setminus \cap_{k=1}^{\infty} E_k) = 0$ and we can prove $m(\cap_{k=1}^{\infty} E_k) = m([0,1]) - m([0,1] \setminus \cap_{k=1}^{\infty} E_k) = 1$.

Extra Problem 19. Let $E_i \subset [0,1]$, $E_i \in \mathcal{M}$ for all $i = 1, \dots, k$, and $\sum_{i=1}^{k} m(E_i) > k-1$. Prove that $m(\cap_{i=1}^{k} E_i) > 0$.

Since $E_i \in \mathcal{M}$ and $E_i \subset [0,1]$ for all $i = 1, \dots, k$, we have

$$m([0,1] \setminus E_i) = m([0,1]) - m(E_i) = 1 - m(E_i)$$

Suppose $m(\cap_{i=1}^{k} E_i) = 0$, then

$$1 = m([0,1] \setminus \cap_{i=1}^{k} E_i) = m(\cup_{i=1}^{k}([0,1] \setminus E_i)) \le \sum_{i=1}^{k} m([0,1] \setminus E_i) = k - \sum_{i=1}^{k} m(E_i)$$

This shows that $\sum_{i=1}^{k} m(E_i) \le k-1$ which is a contradiction to $\sum_{i=1}^{k} m(E_i) > k-1$. Therefore, $m(\cap_{i=1}^{k} E_i) > 0$.

Extra Problem 20. Let $E \subset \mathbb{R}$ and define outer Jordan content of E by

$$J_*(E) = \inf \left\{ \sum_{i=1}^{N} |I_i| \;\middle|\; I_i \text{ intervals}, \bigcup_{i=1}^{N} I_i \supset E \right\}$$

(i) Prove that $J_*(E) = J_*(\bar{E})$.

It is obvious that $J_*(E) \le J_*(\bar{E})$. To show the converse, consider any I_k's such that $G = \bigcup_{k=1}^{N} I_k \supset E$, it is easy to see $\bar{G} \supset \bar{E}$. Since $\bar{G} = \bigcup_{k=1}^{N} \bar{I}_k$ by the lemma proved in Page 34, Problem 8, and $|I_k| = |\bar{I}_k|$, we can conclude that for each $\{I_k\}_{k=1}^{N}$ such that $\bigcup_{k=1}^{N} I_k \supset E$, we can find $\{\bar{I}_k\}_{k=1}^{\infty}$ such that $\bigcup_{k=1}^{N} \bar{I}_k \supset \bar{E}$ and $\sum_{k=1}^{N} |I_k| = \sum_{k=1}^{N} |\bar{I}_k|$. This shows that $J_*(E) \ge J_*(\bar{E})$. Therefore, $J_*(E) = J_*(\bar{E})$.

(ii) Find a countable set $E \subset [0,1]$ such that $J_*(E) = 1$, and $m^*(E) = 0$.

Consider the countable set $E = [0,1] \cap \mathbb{Q}$. Since E is countable, we can enumerate E as $\{q_n\}_{n=1}^{\infty}$. For all $\epsilon > 0$, the set $U = \bigcup_{n=1}^{\infty}(q_n - \epsilon/2^{n+1}, q_n + \epsilon/2^{n+1})$. Then U is obviously an open set covering E, so

$$m^*(E) \le m^*(U) \le \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon$$

Take $\epsilon \to 0$, we can conclude that $m^*(E) = 0$. Since $\bar{E} = [0,1]$, by part (i), we have $J_*(E) = J_*(\bar{E}) = J_*([0,1])$. It is obvious that $J_*([0,1]) \le 1$. Since intervals are rectangles in one dimensional case, we can still apply Fact 2 of volume of rectangles, i.e., if $[0,1] \subset \bigcup_{k=1}^{N} I_k$, then $1 \le \sum_{k=1}^{N} |I_k|$. By definition of infimum, for $\epsilon > 0$, there exists I_k's such that $J_*([0,1]) + \epsilon > \sum_{k=1}^{N} |I_k| \ge 1$. Therefore, take $\epsilon \to 0$, we have $J_*([0,1]) \ge 1$, which means $J_*([0,1]) = 1$. Therefore, E is a countable set such that $J_*(E) = 1$, and $m^*(E) = 0$.