MAT3006*: Real Analysis
Homework 3

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Page 47, Problem 26. Let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collection of measurable sets. Prove that for any set A, $m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m^*(A \cap E_k)$.

Take a G_δ type set G such that $G \supset A \cap \bigcup_{k=1}^{\infty} E_k$ and $m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = m(G)$. Therefore, we can see

$$m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) \geq m\left(G \cap \bigcup_{k=1}^{\infty} E_k\right) = m\left(\bigcup_{k=1}^{\infty} (E_k \cap G)\right) = \sum_{k=1}^{\infty} m(E_k \cap G) \geq \sum_{k=1}^{\infty} m^*(E_k \cap A)$$

where the second equality uses the fact that $\{E_k \cap G\}_{k=1}^{\infty}$ are countable disjoint measurable sets. Since $m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(A \cap E_k)$ follows directly from σ-subadditivity of outer measure, we conclude that $m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m^*(A \cap E_k)$.

Extra Problem 1. Let E_k, $k \in \mathbb{N}^+$, be Lebesgue measurable, satisfying $\sum_{k=1}^{\infty} m(E_k) < \infty$. Prove that $m\left(\overline{\lim}_{k\to\infty} E_k\right) = 0$.

Since $\overline{\lim}_{k\to\infty} E_k = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$, denote $G = \overline{\lim}_{k\to\infty} E_k$ and $G_k = \bigcup_{n=k}^{\infty} E_n$, then G_k is decreasing. Also, for all $k \geq 1$, $m(G) \leq m(G_k) \leq \sum_{n=k}^{\infty} m(E_k)$. The RHS tends to 0 as $k \to \infty$ since the series $\sum_{k=1}^{\infty} m(E_k) < \infty$. Take $k \to \infty$ and we obtain $m(G) = 0$ as required.

Extra Problem 2. Give an example of an open set O such that the boundary of the closure of it has positive Lebesgue measure.

Consider the Cantor-like set with $p = 4$ defined in HW1, Denote $O = \bigcup_{k=1} E_{2k-1}$, where E_k's denote the union of all open intervals removed at step k. Also denote $G = \bigcup_{k=1} E_{2k}$. Then it is easy to see that $[0,1] = O \cup G \cup C_4$. Since O and G are disjoint open set, $\bar{O} \cap G = \varnothing$. We claim that $C_4 \subset \partial\bar{O}$, and if so, from HW2, we know $m(C_4) = \frac{1}{2}$, so $m(\partial\bar{O}) \geq \frac{1}{2} > 0$.

To prove $C_4 \subset \partial\bar{O}$, it suffices to prove for arbitrary $x \in C_4$, for all $\delta > 0$, $N_\delta(x) \cap G \neq \varnothing$ and $N_\delta(x) \cap O \neq \varnothing$. If so, since $G \cap \bar{O} = \varnothing$, $G \subset (\bar{O})^c$ and $N_\delta(x) \cap (\bar{O})^c \neq \varnothing$. Also, $N_\delta(x) \cap O \neq \varnothing$ implies that $N_\delta(x) \cap \bar{O} \neq \varnothing$. Therefore, x is the limit point of $\bar{O}$ and $(\bar{O})^c$, by definition of $\partial\bar{O}$, $x \in \partial\bar{O}$. Since x is arbitrary, $C_4 \in \partial\bar{O}$.

Now we prove $N_\delta(x) \cap G \neq \varnothing$ and $N_\delta(x) \cap O \neq \varnothing$ for all $\delta > 0$ for each fixed $x \in C_4$. Since $C_4 = \bigcup_{k=1}^{\infty} F_k$ where each F_k consists of disjoint closed interval with equal length. Since the length of each closed interval converges to zero, there exists a closed interval I such that $x \in I \subset N_\delta(x)$.

Then this closed interval I must contain open interval removed at both even and odd steps, so $I \cap G \neq \varnothing$ and $I \cap O \neq \varnothing$. This shows $N_\delta(x) \cap G \neq \varnothing$ and $N_\delta(x) \cap O \neq \varnothing$ for all $\delta > 0$.

Extra Problem 3. Suppose $E, F \subset \mathbb{R}$ and $E, F \in \mathcal{M}$. If $m(E) > 0$ and $m(F) > 0$, then $E + F$ contains an interval.

First, since $E = \bigcup_{n\in\mathbb{N}} E \cap [-n, n]$ and $F = \bigcup_{n\in\mathbb{N}} F \cap [-n, n]$, we can find out n_0 and n_1 such that $m(E \cap [-n_0, n_0]) > 0$ and $m(F \cap [-n_1, n_1]) > 0$. Denote $E' = E \cap [-n_0, n_0]$, $F' = F \cap [-n_1, n_1]$ and $n = n_1 + n_2$. Then $E' + F' \subset E + F$ and $E', F', E' + F'$ are all subset of $[-n, n]$. Thus, we only need to show $E' + F'$ contains an interval.

Then, we claim that convolution of $f, g \in L^2(-n, n)$ is continuous on $[-n, n]$. Denote the indicator function of E' and F' as $I_{E'}(x)$ and $I_{F'}(x)$. Note $f(x) = \int_{[-n,n]} I_{E'}(x - t) I_{F'}(t)\, dm(t)$ is nonnegative and continuous. Hence, the set $G = \{x \in [-n, n] \,|\, f(x) > 0\}$ is open. Since $\int_{[-n,n]} f(x)\, dm(x) = m(E') \cdot m(F') > 0$, $m(G) > 0$, G is nonempty and contains an open non-empty interval U. Thus, we only need to show $G \subset E' + F'$. If $x \in [-n\ n] \setminus (E' + F')$, then for such x, if $t \in F'$, $x - t \notin E'$, so $f(x) = 0$. Therefore, if $f(x) > 0$, $x \in E' + F'$, so $G \subset E' + F'$.

For the proof of our claim, since there exists $f_n \to f$ and $g_n \to g$ converging in $L^2(-n, n)$ with f_n, g_n continuous function on $[-n, n]$. It is trivial that $f_n * g_n$ is continuous and $f * g$ is the uniform limit of $f_n * g_n$, so it is also continuous. The uniform convergence is given by

$$\sup_{[-n,n]} |f * g - f_n * g| \leq \|f_n - f\|_{L^2} \|g\|_{L^2} \to 0$$

$$\sup_{[-n,n]} |f_n * g - f_n * g_n| \leq \|f_n\|_{L^2} \|g - g_n\|_{L^2} \to 0$$

Therefore, we finish the whole proof.

Extra Problem 4. Let f be continuous on $[0, 1]$. Prove that the graph Γ of $y = f(x)$, as a subset of $\mathbb{R}^2$, has Lebesgue measure 0.

Let $\epsilon > 0$, since f is continuous on $[0, 1]$, it is uniformly continuous. Therefore, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Let $P = \{x_0 = 0, x_1, \ldots, x_{n-1}, x_n = 1\}$ be a partition of $[0, 1]$ such that $|x_i - x_{i-1}| < \delta$ are of the same length for all $i = 1, \ldots, n$.

The graph $\Gamma = \{(x, f(x)) \,|\, x \in [0, 1]\}$ satisfies $\Gamma \subset \bigcup_{i=1}^n [x_{i-1}, x_i] \times [m_i, M_i]$ where $m_i = \min_{[x_{i-1},x_i]} f(x)$ and $M_i = \max_{[x_{i-1},x_i]} f(x)$. Since $|x_i - x_{i-1}| < \delta$ for all i, $M_i - m_i \leq \epsilon$ for all i. Thus,

$$m^*(\Gamma) \leq \sum_{i=1}^n m([x_{i-1}, x_i]) \cdot m([m_i, M_i]) \leq \epsilon \sum_{i=1}^n m([x_{i-1}, x_i]) = \epsilon$$

Therefore, take $\epsilon \to 0$, we obtain $m^*(\Gamma) = 0$, which means Γ is measurable with zero measure.

Extra Problem 5. Let $A, B \subset \mathbb{R}^n$ with finite outer measure. Prove $|m^*(A) - m^*(B)| \leq m^*(A \triangle B)$.

Since $m^*(A)$ and $m^*(B)$ are finite, it suffices to show

$$m^*(A) \leq m^*(B) + m^*(A \triangle B), \qquad m^*(B) \leq m^*(A) + m^*(A \triangle B)$$

Apply the equivalent definition of symmetric difference, we have $A \triangle B = (A \cup B) \setminus (A \cap B)$. Then it is easy to see that $(A \triangle B) \cup B = A \cup B$ and $(A \triangle B) \cup A = A \cup B$. Therefore,

$$m^*(A) \leq m^*(A \cup B) = m^*((A \triangle B) \cup B) \leq m^*(B) + m^*(A \triangle B)$$

$$m^*(B) \leq m^*(A \cup B) = m^*((A \triangle B) \cup A) \leq m^*(A) + m^*(A \triangle B)$$

Therefore, we proved the desire inequality $|m^*(A) - m^*(B)| \leq m^*(A \triangle B)$.

Extra Problem 6. Does there exists a closed proper subset F of $[0, 1]$ such that $m(F) = 1$?

Suppose yes, then $E = [0, 1] \setminus F$ is a nonempty open set. Then it must contains an open interval which has measure $k > 0$, so $m(E) \geq k > 0$. Since $m([0, 1]) = m(E) + m(F) = 1$, we have $m(F) = 1 - k < 1$, contradiction. Therefore, such F does not exist.

Extra Problem 7. Let $E \in \mathcal{M}$ with $m(E) > 0$. Prove that there exists $x \in E$ such that for all $\delta > 0$, $m(E \cap B_\delta(x)) > 0$, where $B_\delta(x)$ is the ball centered at x with radius $\delta > 0$.

Suppose not, then for all $x \in E$, there exists δ_x such that $m(E \cap B_{\delta_x}(x)) = 0$. Notice that $\bigcup_{x \in E} B_{\delta_x}(x) \supset E$, so by Lindelöf covering theorem, there exists a countable open subcover $\{B_{\delta_{x_n}}(x_n)\}_{n=1}^{\infty}$ of E. Therefore, $\bigcup_{n=1}^{\infty}(E \cap B_{\delta_{x_n}}(x_n)) \supset E$ and

$$m(E) \leq m\left(\bigcup_{n=1}^{\infty}(E \cap B_{\delta_{x_n}}(x_n))\right) \leq \sum_{n=1}^{\infty} m(E \cap B_{\delta_{x_n}}(x_n)) = 0$$

since each $m(E \cap B_{\delta_{x_n}}(x_n)) = 0$. However, we assume $m(E) > 0$, so contradiction shows that there exists $x \in E$ such that for all $\delta > 0$, $m(E \cap B_\delta(x)) > 0$.

Extra Problem 8. Let $E \subset \mathbb{R}^n$. Prove that there exists G_δ set $G \supset E$ such that for all $A \in \mathcal{M}$, we have $m^*(E \cap A) = m(G \cap A)$.

Since $A \in \mathcal{M}$, by Carathéodory property, we have $m^*(E) = m^*(E \cap A) + m^*(E \cap A^c)$. Take a G_δ set such that $G \supset E$, $m^*(E) = m(G)$. Also, $m(G) = m(G \cap A) + m(G \cap A^c)$. If $m^*(E) < \infty$,

$$m^*(E \cap A) - m^*(G \cap A) = m^*(G \cap A^c) - m^*(E \cap A^c) \geq 0$$

However, by monotonicity, $m^*(E \cap A) \leq m^*(G \cap A)$, so $m^*(E \cap A) = m^*(G \cap A)$.

If $m^*(E) = \infty$, then let $C_k = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, |x_i| \leq k, \ \forall 1 \leq i \leq n\}$. Denote $E_k = E \cap C_k$, so $m^*(E_k) < \infty$. Therefore, we can find G_δ set $G_k \supset E_k$ such that $m(G_k) = m^*(E_k)$ and $m(G_k \cap A) = m^*(E_k \cap A)$. Let $H_k = \bigcap_{n=k}^{\infty} G_n$, H_k is also a G_δ set for all k, and $E_k \subset H_k \subset G_k$. It is easy to see $m^*(E_k \cap A) = m(H_k \cap A)$. Let $H = \bigcup_{k=1}^{\infty} H_k$, then $E \subset H$. Note that $E_k \cap A$ and $H_k \cap A$ are both increasing, so take limit on both sides of $m^*(E_k \cap A) = m(H_k \cap A)$, we obtain $m^*(E \cap A) = m(H \cap A)$. Notice that $H \in \mathcal{M}$, there exists G_δ set O such that $O \supset H$ and $m(O \setminus H) = 0$. Since $O = (O \setminus H) \cup H$, $O \cap A = [(O \setminus H) \cap A] \cup (H \cap A)$, so we have

$$m(O \cap A) \leq m((O \setminus H) \cap A) + m(H \cap A) \leq m(O \setminus H) + m(H \cap A) = m(H \cap A)$$

Also, by monotonicity, $m(O\cap A)\geq m(H\cap A)$, so $m(O\cap A)=m(H\cap A)=m^*(E\cap A)$. The set O is the desired G_δ set.

Extra Problem 9. Let $E\notin\mathcal{M}$. Prove that there exists $\epsilon>0$ such that whenever $A,B\in\mathcal{M}$, $A\supset E$, $B\supset E^c$, we always have $m(A\cap B)\geq\epsilon$.

Suppose not, then for all $k\geq 1$, there exists $A_k,B_k\in\mathcal{M}$ such that $A_k\supset E$, $B_k\supset E^c$, and $m(A_k\cap B_k)\leq\frac{1}{k}$. Let $A=\bigcap_{k=1}^\infty A_k$ and $B=\bigcap_{k=1}^\infty B_k$, then $m(A\cap B)=0$. Also, $A\supset E$ and $B\supset E^c$, so $B^c\subset E$. Notice that $m(A\setminus B^c)=m(A\cap B)=0$ and $m^*(E\setminus B^c)\leq m(A\setminus B^c)=0$, so $m^*(E\setminus B^c)=0$ and $E\setminus B^c\in\mathcal{M}$. Since $B^c\in\mathcal{M}$, $E=(E\setminus B^c)\cup B^c\in\mathcal{M}$. Contradiction!

Extra Problem 10. Let $E\subset\mathbb{R}$ and $E\in\mathcal{M}$. Suppose there exists open intervals I_k for $k\in\mathbb{N}^+$ such that $m(E\cap I_k)\geq\frac{2}{3}m(I_k)$. Prove that $m\left(E\cap\bigcup_{k=1}^\infty I_k\right)\geq\frac{1}{3}m\left(\bigcup_{k=1}^\infty I_k\right)$.

It sufffices to show that for all $n\geq 1$, $m\left(E\cap\bigcup_{k=1}^n I_k\right)\geq\frac{1}{3}m\left(\bigcup_{k=1}^n I_k\right)$ because if so, then since $E\cap\bigcup_{k=1}^n I_k$ and $\bigcup_{k=1}^n I_k$ are both increasing, we can take limit on both sides and the desire inequality holds. First, for each $k=1,\dots,n$, check if there exists $k'\neq k$, $k'=1,\dots,n$ such that $I_{k'}\subset I_k$. If yes, then delete $I_{k'}$, and finally we will obtain a subcollection of open intervals. After relabeling, we obtain $\{I_k\}_{k=1}^m$ with $m\leq n$ and $\bigcup_{k=1}^n I_k=\bigcup_{k=1}^m I_k$.

Furthermore, consider if there exists distinct index $j,k,l=1,\dots,m$ such that $I_j\cap I_k\cap I_l\neq\varnothing$, then $I_j\cup I_k\cup I_l$ is an open interval (a,b). If we denote $I_j=(a_j,b_j)$, $I_k=(a_k,b_k)$ and $I_l=(a_l,b_l)$, then a_j,a_k,a_l must be distinct and so are b_j,b_k,b_l (if two of them are equal, then one interval is contained in another, which is impossible since we have already delete the smaller one in the previous step). Therefore, a,b come from two different intervals and the third interval is contained in the union of those two intervals, so we can delete the third interval without changing $I_j\cup I_k\cup I_l$. Therefore, we can finally obtain a further subcollection of open intervals. After relabeling, we can obtain $\{I_k\}_{k=1}^p$ with $p\leq m$ and $\bigcup_{k=1}^m I_k=\bigcup_{k=1}^p I_k$.Thus, it suffices to show $m\left(E\cap\bigcup_{k=1}^p I_k\right)\geq\frac{1}{3}m\left(\bigcup_{k=1}^p I_k\right)$.

Notice that for this new subcollection $\{I_k\}_{k=1}^p$, if we denote $I_k=(a_k,b_k)$, then WLOG, we can assume $a_1<a_2<\cdots<a_p$ and $b_1<b_2<\cdots<b_p$. Furthermore, $a_{k+2}\geq b_k$ for all $k=1,\dots,p-2$. Thus, an essential observation is that each I_k can only have nonempty intersection with I_{k-1} and I_k, and it will not intersect with $I_1,\dots,I_{k-2},I_{k+2},\dots,I_p$. Then denote $I_{k,j}=I_k\cap I_j$ for all $k\neq j$ and also denote $J_k=I_k\setminus\left(\bigcup_{j\neq k,\ j=1}^p I_{k,j}\right)$. Then we can see that $\bigcup_{k=1}^p I_k=\left(\bigcup_{k=1}^p J_k\right)\cup\left(\bigcup_{k\neq j,\ k,j=1}^p I_{k,j}\right)$. The most important thing is that all $I_{k,j}$ and J_k are all pairwise disjoint. Therefore, we can conclude

$$\begin{aligned}
2m\left(E\cap\bigcup_{k=1}^p I_k\right)&=2m\left[\left(\bigcup_{k=1}^p J_k\cap E\right)\cup\left(\bigcup_{k\neq j} I_{k,j}\cap E\right)\right]\\
&=2\sum_{k=1}^p m(J_k\cap E)+2\sum_{k\neq j,\ k,j=1}^p m(I_{k,j}\cap E)\\
&\geq[m(J_1\cap E)+m(I_{1,2}\cap E)]+[m(I_{1,2}\cap E)+m(J_2\cap E)+m(I_{2,3}\cap E)]+\cdots\\
&=\sum_{k=1}^p m(I_k\cap E)\geq\frac{2}{3}\sum_{k=1}^p m(I_k)\geq\frac{2}{3}m\left(\bigcup_{k=1}^p I_k\right)
\end{aligned}$$

Therefore, $2m\left(E\cap\bigcup_{k=1}^p I_k\right)\geq\frac{2}{3}m\left(\bigcup_{k=1}^p I_k\right)$ implies that $m\left(E\cap\bigcup_{k=1}^p I_k\right)\geq\frac{1}{3}m\left(\bigcup_{k=1}^p I_k\right)$.