MAT3006*: Real Analysis
Homework 4

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Due date: Mar. 6, 2020

Extra Problem 1. Define $f : [a, b] \mapsto \mathbb{R}$ such that for all $E \subset [a, b]$ and $E \in \mathcal{M}$, we have $f(E) \in \mathcal{M}$. Prove that for all $Z \subset [a, b]$ with $m(Z) = 0$, we have $m(f(Z)) = 0$.

Suppose there exists $Z \subset [a, b]$ with $m(Z) = 0$ and $m(f(Z)) > 0$, then by a remark in lecture, there exists $S \subset f(Z)$ such that $S \notin \mathcal{M}$. Since for all $E \in \mathcal{M}$, $f(E) \in \mathcal{M}$, we conclude that if $f(E) \notin \mathcal{M}$ then $E \notin \mathcal{M}$. Therefore, $f^{-1}(S) \notin \mathcal{M}$. However, $f^{-1}(S) \subset Z$, but Z is null set, and any subset of a null set is measurable with zero measure, so we obtain a contradiction. This shows that, for all $Z \subset [a, b]$ with $m(Z) = 0$, we have $m(f(Z)) = 0$.

Extra Problem 2. Let f be defined on $E \in \mathcal{M}$, and f be finite on E. Prove that the following are equivalent:

(i) f is measurable on E;

(ii) $f^{-1}(G) \in \mathcal{M}$ for all open set $G \subset \mathbb{R}$.

(iii) $f^{-1}(F) \in \mathcal{M}$ for all closed set $F \subset \mathbb{R}$.

(iv) $f^{-1}(B) \in \mathcal{M}$ for all Borel set $B \subset \mathbb{R}$.

We first prove (ii) $\Longrightarrow$ (i), (iii) $\Longrightarrow$ (i), and (iv) $\Longrightarrow$ (i). Recall that f is measurable if $f^{-1}((-\infty, t)) \in \mathcal{M}$ for all $t \in \mathbb{R}$. If we assume (ii), since all $(-\infty, t)$ is open, (i) automatically holds. If we assume (iii), since $(-\infty, t)^c$ is a closed set for all t, $f^{-1}((-\infty, t)^c) \in \mathcal{M}$. By a useful identity, $f^{-1}((-\infty, t)^c) = [f^{-1}((-\infty, t))]^c \in \mathcal{M}$. Thus, $f^{-1}((-\infty, t)) \in \mathcal{M}$, and (i) is established. If we assume (iv), since $(-\infty, t)$ must be Borel set, (i) automatically holds.

Now we prove (i) $\Longrightarrow$ (iv). To prove (i) $\Longrightarrow$ (iv), construct $\mathcal{S} = \{E \in \mathbb{R} \,|\, f^{-1}(E) \in \mathcal{M}\}$. If f is measurable, then $f^{-1}((-\infty, t)) \in \mathcal{M}$. Also, by useful identity in lecture, $f^{-1}((t, \infty)) \in \mathcal{M}$. Since $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$, and any open interval (a, b) can be written as $(-\infty, b) \cap (a, \infty)$, $f^{-1}(I) \in \mathcal{M}$ for all open interval I. Furthermore, for each open set G, $G = \bigcup_{k=1}^{\infty} I_k$, since $f^{-1}(G) = \bigcup_{k=1}^{\infty} f^{-1}(I_k)$, $f^{-1}(G) \in \mathcal{M}$. Actually we have already proved (i) $\Longrightarrow$ (ii).

Now we try to show $\mathcal{S}$ is a σ-algebra. First, $f^{-1}(\varnothing) = \varnothing \in \mathcal{M}$ and $f^{-1}(\mathbb{R}) = E \in \mathcal{M}$. Second, for any $A \subset \mathbb{R}$, if $f^{-1}(A) \in \mathcal{M}$, again since $f^{-1}(A^c) = [f^{-1}(A)]^c$, $f^{-1}(A^c) \in \mathcal{M}$. This actually verifies (ii) $\Longrightarrow$ (iii), hence (i) $\Longrightarrow$ (iii).

Third, if $f^{-1}(E_k) \in \mathcal{M}$ for all $i \in \mathbb{N}^+$, then $f^{-1}(\cup_{k=1}^{\infty} E_k) = \bigcup_{k=1} f^{-1}(E_k) \in \mathcal{M}$. Therefore, $\mathcal{S}$

is a σ-algebra contains all open sets, but since $\mathcal{B}$ is the smallest σ-algebra containing all open sets, we have $\mathcal{B} \subset \mathcal{S}$. This finishes the proof of (i) $\Longrightarrow$ (iv).

Extra Problem 3. Prove that monotone increasing function defined on $[a, b]$ is measurable.

If f is increasing function, then the set $\{x \in [a, b] \,|\, f(x) > t\}$ is an interval for all t, and hence measurable. By definition f is measurable.

Extra Problem 4. Let f be defined on $[a, b]$. Suppose for all $[\alpha, \beta] \subset (a, b)$, f is measurable on $[\alpha, \beta]$. Prove f is measurable on $[a, b]$.

Notice that $[a + \frac{1}{n}, b - \frac{1}{n}] \subset (a, b)$ for all $n \geq 1$. By hypothesis, f is measurable on every $[a + \frac{1}{n}, b - \frac{1}{n}]$, thus $\{x \in [a + \frac{1}{n}, b - \frac{1}{n}] \,|\, f(x) > t\} \in \mathcal{M}$. Also, $(a, b) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b - \frac{1}{n}]$, so $\{x \in (a, b) \,|\, f(x) > t\} = \bigcup_{n=1}^{\infty} \{x \in [a + \frac{1}{n}, b - \frac{1}{n}] \,|\, f(x) > t\} \in \mathcal{M}$. Therefore, f is measurable on (a, b). Since $\{a, b\}$ has measure zero, and f is measurable on any zero measure set, so f is also measurable on $[a, b]$.

Extra Problem 5. Let f be differentiable on $[a, b]$. Prove that $f'(x)$ is also measurable on $[a, b]$.

Let $g_n(x) = \frac{f(x+\frac{1}{n})-f(x)}{\frac{1}{n}}$, then $g_n(x)$ is continuous on $[a, b - \frac{1}{n}]$, hence measurable. For each $[\alpha, \beta] \subset (a, b)$, there exists large N_β such that $[\alpha, \beta] \subset [a, b - \frac{1}{n}]$, hence g_n is measurable on $[\alpha, \beta]$ for all $n \geq N_\beta$. Since $f'(x) = \lim_{n\to\infty} g_n(x)$, $f'(x)$ is measurable on $[\alpha, \beta]$. By Extra Problem 4, $f'(x)$ is also measurable on $[a, b]$.

Extra Problem 6. Define $f : \mathbb{R}^2 \mapsto \mathbb{R}$ such that $f(x, y)$ is a measurable function of $x \in \mathbb{R}$ for each fixed y. Also, for each fixed x, f is a continuous function of $y \in \mathbb{R}$. Define $F(x) = \max_{y\in[0,1]} f(x, y)$. Prove that $F(x)$ is measurable on $\mathbb{R}$.

Consider the equality $\{x \in \mathbb{R} \,|\, F(x) > t\} = \bigcup_{k=1}^{\infty} \{x \in \mathbb{R} \,|\, f(x, r_k) > t\}$ where $\{r_k\}_{k=1}^{\infty} = \mathbb{Q} \cap [0, 1]$. If so, then since $f(x, r_k)$ is measurable for each fixed r_k, $\{x \in \mathbb{R} \,|\, f(x, r_k) > t\} \in \mathcal{M}$, so we can easily see $\{x \in \mathbb{R} \,|\, F(x) > t\} \in \mathcal{M}$, and hence F is measurable on $\mathbb{R}$. To check why the equality holds above, we need to use $f(x, y)$ is continuous with respect y. RHS $\subset$ LHS is trivial because for all $x \in$ RHS, there exists k such that $f(x, r_k) > t$, then $F(x) = \max_{y\in[0,1]} f(x, y) \geq f(x, r_k) > t$, so $x \in$ LHS. For $x \in$ LHS, $F(x) = \max_{y\in[0,1]} f(x, y) > t$. Since $f(x, y)$ is continuous with respect to y on compact set $[0, 1]$, there exists $y_0 \in [0, 1]$ such that $f(x, y_0) = F(x)$. If $y_0 \in \mathbb{Q}$, then $x \in$ RHS and we are done. If $y_0 \notin \mathbb{Q}$, then there exists a sequence $a_n \to y_0$ and $a_n \subset \mathbb{Q} \cap [0, 1]$ such that $f(x, a_n) \to f(x, y_0)$. Take N large enough such that $f(x, y_0) - f(x, a_n) \leq \frac{F(x)-t}{2}$, then $f(x, a_n) \geq \frac{F(x)+t}{2} > t$. This shows $x \in$ RHS. Thus, we establish that LHS $=$ RHS.

Extra Problem 7. Let $E \subset \mathbb{R}^n$. Prove that $E \in \mathcal{M}$ if and only if $I_E(x)$ is measurable on $\mathbb{R}^n$, where $I_E(x)$ is the indicator function of set E.

If $I_E(x)$ is measurable on $\mathbb{R}^n$, then since $\{1\}$ is a closed set, and by Extra Problem 2, $E = f^{-1}(\{1\}) \in \mathcal{M}$. If E is measurable, then consider $I_E^{-1}((t, \infty))$, if $t \leq 0$, then $I_E^{-1}((t, \infty)) = \mathbb{R} \in \mathcal{M}$;

if $t \in (0,1)$, then $I_E^{-1}((t,\infty)) = E \in \mathcal{M}$; if $t \geq 1$, then $I_E^{-1}((t,\infty)) = \varnothing \in \mathcal{M}$. Therefore, $I_E^{-1}((t,\infty)) \in \mathcal{M}$ for all $t \in \mathbb{R}$, so $I_E(x)$ is measurable.

Extra Problem 8. Let f be real-valued and measurable on $E \in \mathcal{M}$ with $m(E) < \infty$. Prove that for all $\epsilon > 0$, there exists bounded measurable function $g(x)$ defined on E such that $m(\{x \in E \,|\, f(x) \neq g(x)\}) < \epsilon$.

For all $k \geq 1$, let $E_k = \{x \in E \,|\, |f(x)| > k\}$, then $E_{k+1} \subset E_k$ for all $k \geq 1$. Also, $\lim_{k\to\infty} E_k = \bigcap_{k=1}^{\infty} E_k = \varnothing$ because f is real-valued. Since f is measurable on $E \in \mathcal{M}$, $|f|$ is measurable, so $E_k \in \mathcal{M}$ for all $k \geq 1$. Furthermore, $m(E_1) \subset m(E) < \infty$, hence we have $\lim_{k\to\infty} m(E_k) = m(\lim_{k\to\infty} E_k) = 0$. Therefore, for all $\epsilon > 0$, there exists $K \geq 1$ such that $m(E_k) < \epsilon$ for all $k \geq K$. Let us take $k = K$, and define

$$g(x) = \begin{cases} f(x) & x \in E \setminus E_K \\ 0 & x \in E_K \end{cases}$$

Then $m(\{x \in E \,|\, f(x) \neq g(x)\}) \leq m(E_K) < \epsilon$.

Extra Problem 9. Construct an example in which f is measurable and g is continuous, but $f \circ g$ is not measurable.

Recall Cantor function $c(x)$, define $\hat{c}(x) = x + f(x)$, then $\hat{c}(x)$ is strictly increasing and continuous on $[0,1]$. Since $\hat{c}^{-1}$ is also continuous, we take $g = \hat{c}^{-1}$. Also, we proved that $m(\hat{c}(C)) = 1$ for Cantor set C, so there exists non-measurable set $S \subset \hat{c}(C)$, and $B = \hat{c}^{-1}(S) \subset C$ is measurable. Take $f = I_B$ to be the indicator function of B, by Extra Problem 7, f is measurable. We claim that $h = f \circ g$ is not measurable. Suppose it is measurable, then since $\{1\}$ is closed, $h^{-1}(\{1\})$ must be measurable. However, $h^{-1}(\{1\}) = g^{-1} \circ f^{-1}(\{1\}) = g^{-1}(B) = \hat{c}(B) = S \notin \mathcal{M}$, which is a contradiction. Therefore, $f \circ g$ is not measurable.