MAT3006*: Real Analysis
Homework 5

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Page 63, Problem 15. Let f be a measurable function on E that is finite a.e. on E and $m(E) < \infty$. For each $\epsilon > 0$, show that there is a measurable set F contained in E and a sequence $\phi_n(x)$ of simple functions on E such that $\phi_n \to f$ uniformly on F and $m(E \setminus F) < \epsilon$.

Define $E_k = \{x \in E \,|\, |f(x)| \geq k\}$ then $E_k \in \mathcal{M}$, E_k is decreasing and f is bounded outside E_k. Since f is finite a.e. on E, it is not hard (see Extra Problem 3 below for details) to prove $\lim_{k\to\infty} E_k = 0$. Therefore, for each $\epsilon > 0$, there exists K such that $m(E_K) < \epsilon$ and f is bounded on $E \setminus E_K$. Let $F = E \setminus E_K$, then since f is bounded on F, by approximation theorem, there exists a sequence of simple functions ϕ_n on E such that $\phi_n \to f$ uniformly on F.

Page 63, Problem 16. Let I be a closed, bounded interval and E a measurable subset of I. Let $\epsilon > 0$. Show that there is a step function h on I and a measurable subset F of I for which $h = I_E$ on F and $m(I \setminus F) < \epsilon$.

Since $E \in \mathcal{M}$, there exists $U = \bigcup_{k=1}^N C_k$ where C_k's are closed (bounded) intervals and $m(E \triangle U) < \epsilon$. Let $F = I \setminus (E \triangle U)$, then we have

$$F = I \cap (E \triangle U)^c = I \cap ((E \cup U) \cap (E \cap U)^c)^c = [I \setminus (E \cup U)] \cup [I \cap (E \cap U)]$$

Define $h(x)$ on F by $h(x) = 1$ if $x \in I \cap U$ and $h(x) = 0$ if $x \in I \setminus U$. Then for $x \in F$, if $x \in E$, then $x \in I \cap (E \cap U)$ and $h(x) = 1$; if $x \notin E$, then $x \in I \setminus (E \cup U)$, so $h(x) = 0$. Therefore, on F, $h(x) = I_E(x)$. It is trivial that $m(I \setminus F) < \epsilon$. Also, since $U = \bigcup_{k=1}^N C_k$, $I \cap U = \bigcup_{k=1}^N (I \cap C_k)$, and $h(x) = \sum_{k=1}^N I_{I \cap C_k}(x)$, which is indeed a step function.

Page 67, Problem 31. Let f_n be a sequence of measurable functions on E that converges to the real-valued f pointwise on E. Show that $E = \bigcup_{k=1}^\infty E_k$, where for each k, E_k is measurable, and f_n converges uniformly to f on each E_k if $k > 1$, and $m(E_1) = 0$.

First consider when $m(E) < \infty$. By Egorov's theorem, $f_n \to f$ a.u. on E. Thus, for all $k \geq 1$, there exists $F_k \in \mathcal{M}$ and $F_k \subset E$ s.t. $m(F_k) < \frac{1}{2^k}$ and $f_n \to f$ uniformly on $E \setminus F_k$. Let $E_k = E \setminus F_k$ for $k \geq 2$ and $E_1 = E \setminus \bigcup_{k=2}^\infty E_k = \bigcap_{k=2}^\infty F_k$. Consider $m(E_1) \leq m(F_k) > \frac{1}{2^k}$ for all $k \geq 2$, thus let $k \to \infty$, we obtain $m(E_1) = 0$.

Then consider $m(E) = \infty$. Let $J_k = E \cap B_k(0)$ and $E = \bigcup_{k=1}^\infty J_k$. Since J_k is bounded, $m(J_k) < \infty$, so for fixed $k \geq 1$, there exists E_i^k s.t. $J_k = \bigcup_{i=1}^\infty E_i^k$ and E_i^k are measurable for all

$i \geq 1$. Also, $m(E_1^k) = 0$ and $f_n \to f$ uniformly on E_i^k for $i \geq 2$. Let $E_1 = \bigcup_{k=1}^{\infty} E_1^k$, then it is obvious that $m(E_1) = 0$. Thus, $E = E_1 \cup \bigcup_{k=1}^{\infty} \bigcup_{i=2}^{\infty} E_i^k$ and after renumbering these countably many sets except E_1, we can obtain the desired result.

Extra Problem 1. Let $f_k(x)$ be measurable on $E \in \mathcal{M}$, where $m(E) < \infty$. Suppose $f_k(x) \to \infty$ a.e. on E as $k \to \infty$, then $f_k \to \infty$ a.u. on E.

Let $g_k(x) = \arctan(f_k(x))$, then it is trivial that $g_k(x)$'s are measurable on E and $g_k(x) \to \frac{\pi}{2}$ a.e. on E. Since $\frac{\pi}{2}$ is a finite number, by Egorov's theorem, $g_k(x) \to \frac{\pi}{2}$ a.u., which means for each $\delta > 0$, there exists E_δ such that $m(E_\delta) < \delta$ and $g_k(x) \to \frac{\pi}{2}$ uniformly on $E \setminus E_\delta$. By definition, $\forall \epsilon > 0$, there exists $N(\epsilon)$ such that for all $k \geq N(\epsilon)$, $|g_k(x) - \frac{\pi}{2}| < \epsilon$ for all $x \in E \setminus E_\delta$. Since $\tan(x)$ is a continuous function on $(-\pi/2, \pi/2)$ and $\tan(x) \to \infty$ as $x \to \pi/2$, for all $M > 0$, there exists $\delta(M)$, such that $\tan(x) > M$ for all $|x - \pi/2| < \delta(M)$. Take $\epsilon = \delta(K)$ above, then for all $K > 0$, there exists $N(\delta(K))$ such that for $k \geq N(\delta(K))$, $|g_k(x) - \frac{\pi}{2}| < \delta(K)$, so $\tan(g_k(x)) > K$ for all $x \in E \setminus E_\delta$. But $\tan(g_k(x))$ is nothing but $f_k(x)$, so this shows $f_k(x) \to \infty$ uniformly on $E \setminus E_\delta$.

Extra Problem 2. Let $E \in \mathcal{M}$, $f_k \to f$ in measure and $g_k \to g$ in measure one E as $k \to \infty$. Prove that $f_k + g_k \to f + g$ in measure on E as $k \to \infty$.

Since $|f_k + g_k - (f+g)| \leq |f_k - f| + |g_k - g|$, if $|f_k + g_k - (f+g)| \geq \delta$, then either $|f_k - f|$ or $|g_k - g|$ must be no less than $\delta/2$. Therefore we can obtain

$$\{x \,|\, |f_n(x) + g_n(x) - (f(x) + g(x))| \geq \delta\} \subset \{x \,|\, |f_n(x) - f(x)| \geq \delta/2\} \cup \{x \,|\, |g_n(x) - g(x)| \geq \delta/2\}$$

Take measure on both sides, and by using subadditivity of Lebesgue measure, we have

$$m\left(\{x \,|\, |f_n(x) + g_n(x) - (f(x) + g(x))| \geq \delta\}\right) \to 0$$

because as $n \to \infty$,

$$m\left(\{x \,|\, |f_n(x) - f(x)| \geq \delta/2\}\right) + m\left(\{x \,|\, |g_n(x) - g(x)| \geq \delta/2\}\right) \to 0$$

Extra Problem 3. Let f_n be measurable on $[0,1]$ with $|f_n(x)| < \infty$ for a.e. $x \in E$. Show that there exists sequence of positive numbers c_n such that $\frac{f_n(x)}{c_n} \to 0$ a.e. on E as $n \to \infty$.

For each fixed $n \geq 1$, define $E_n^k = \{x \in [0,1] \,|\, |f_n(x)| \geq k\}$ for all $k \geq 1$. It is obvious that $\lim_{k \to \infty} m(E_n^k) = 0$ because if not, then there exists a subsequence k_j such that $m(E_n^{k_j}) \geq \epsilon > 0$ for all j. Since $k_j \to \infty$ as $j \to \infty$, $f_n(x) = \infty$ on a positive measure set, which contradict $f_n(x)$ is finite a.e. $x \in E$. This implies for each fixed n, we can take k_n large enough such that $m(E_n^{k_n}) < \frac{1}{2^n}$ for all $n \geq 1$. Since $\sum_{n=1}^{\infty} m(E_n^{k_n}) < \infty$, by Borel-Cantelli Lemma, $m(\overline{\lim}_{n \to \infty} E_n^{k_n}) = 0$. Take $A = \overline{\lim}_{n \to \infty} E_n^{k_n}$, if $x \notin A$, then there exists M such that for all $n \geq M$, $x \notin E_n^{k_n}$. This means $|f_n(x)| < k_n$ for large n for each fixed x. Therefore, for a fixed $x \notin A$, take $c_n = nk_n$, when n is large, $\frac{f_n(x)}{c_n} \leq \frac{1}{n}$. This implies $\frac{f_n(x)}{c_n} \to 0$ a.e. on E.

Extra Problem 4. Let f_n be measurable on $\mathbb{R}$ and λ_n be a sequence of positive numbers, satisfying

$$\sum_{n=1}^{\infty} m\left(\{x \in \mathbb{R} \mid |f_n(x)| > \lambda_n\}\right) < \infty$$

Prove that $\limsup_{n\to\infty} \frac{|f_n(x)|}{\lambda_n} \le 1$ a.e. on $\mathbb{R}$.

If we denote $E_n = \{x \in \mathbb{R} \mid |f_n(x)| > \lambda_n\}$, by Borel-Cantelli Lemma, $m(\overline{\lim}_{n\to\infty} E_n) = 0$. Let $A = \overline{\lim}_{n\to\infty} E_n$, if $x \notin A$, then there exists N_x such that $|f_n(x)| \le \lambda_n$ for $n \ge N_x$. Therefore, for this fixed x, $\limsup_{n\to\infty} \frac{|f_n(x)|}{\lambda_n} \le 1$. This has already been enough to conclude $\limsup_{n\to\infty} \frac{|f_n(x)|}{\lambda_n} \le 1$ a.e. on $\mathbb{R}$.

Extra Problem 5. Let $f_k(x)$ be real-valued, measurable on $E \in \mathcal{M}$, with $m(E) < \infty$. Prove that $f_k \to 0$ a.e. on E as $k \to \infty$ if and only if

$$\lim_{j\to\infty} m\left(\left\{x \in E \,\middle|\, \sup_{k\ge j} |f_k(x)| \ge \epsilon\right\}\right) = 0$$

for all $\epsilon > 0$.

For "only if" part, let $E_j^\epsilon = \{x \in E \mid \sup_{k\ge j} |f_k(x)| \ge \epsilon\}$. It is easy to see E_j^ϵ is decreasing. Since $m(E) < \infty$, we have $\lim_{j\to\infty} m(E_j^\epsilon) = m(\cap_{j=1}^\infty E_j^\epsilon) = m(\{x \in E \mid \overline{\lim}_{k\to\infty} |f_k(x)| \ge \epsilon\})$. Since $f_k \to 0$ a.e., $\overline{\lim}_{k\to\infty} |f_k(x)| = 0$ a.e., thus we have $m(\{x \in E \mid \overline{\lim}_{k\to\infty} |f_k(x)| \ge \epsilon\}) = 0$.

For "if" part, let $Z = \{x \in E \mid f_k(x) \not\to 0\}$, and $E_l^k = \{x \in E \mid |f_k(x)| \ge \frac{1}{l}\}$. Then by the proof of Egorov's theroem, we know $Z = \bigcup_{l=1}^\infty \lim_{j\to\infty} F_l^j$ where $F_l^j = \bigcup_{k=j}^\infty E_l^k$. If $x \in F_l^j$, then there exists $k_0 \ge j$ such that $x \in E_l^{k_0}$. This shows $|f_{k_0}(x)| \ge \frac{1}{l}$. Then $\sup_{k\ge j} |f_k(x)| \ge |f_{k_0}(x)| \ge \frac{1}{l}$. Take $\epsilon = \frac{1}{l}$ in the hypothesis, $x \in \{x \in E \mid \sup_{k\ge j} |f_k(x)| \ge \frac{1}{l}\}$. Therefore, $F_l^j \subset \{x \in E \mid \sup_{k\ge j} |f_k(x)| \ge \frac{1}{l}\}$. Therefore, by taking measure on both sides and taking limit w.r.t. j, we have $\lim_{j\to\infty} m(F_l^j) = 0$. Since F_l^j is decreasing and with finite measure, $\lim_{j\to\infty} m(F_l^j) = m(\lim_{j\to\infty} F_l^j) = 0$. Therefore, $m(Z) = 0$, i.e., $f_k \to 0$ a.e. on E.

Extra Problem 6. Let $f_{k,i}(x)$, $1 \le k < \infty$, $1 \le i < \infty$, be real-valued and measurable on $[0,1]$, satisfying

(i) For each fixed $k \ge 1$, $f_{k,i} \to f_k$ a.e. on $[0,1]$ as $i \to \infty$ with some f_k real-valued on $[0,1]$.

(ii) $f_k \to g$ a.e. on $[0,1]$ as $k \to \infty$, with some g real-valued on $[0,1]$.

Prove that there exists k_j and i_j such that $f_{k_j,i_j} \to g$ a.e. on $[0,1]$ as $j \to \infty$.

Denote $E = [0,1]$. Consider $\{f_{1,i}\}$, by Egorov theorem, take $\delta = 1/2$, there exists $E_1 \subset E$ s.t. $m(E_1) < 1/2$. Also, by definition of uniform convergence, there exists i_1 s.t. $|f_{1,i_1}(x) - f_1(x)| < \epsilon/2$ for all $x \in E \setminus E_1$. Similarly, in general, we will obtain E_j s.t. $m(E_j) < 1/2^j$ and i_j s.t. $|f_{j,i_j}(x) - f_j(x)| < \epsilon/2$ for all $x \in E \setminus E_j$. WLOG, we can assume i_j is strictly increasing to infinity as $j \to \infty$. Let $A = \overline{\lim}_{j\to\infty} E_j$, since $\sum_{j=1}^\infty m(E_j) = \sum_{j=1}^\infty 1/2^j < \infty$, by Borel-Cantelli Lemma, $m(A) = 0$. Define $E_0 = E \setminus A$, then $E_0 = \underline{\lim}_{j\to\infty}(E \setminus E_j)$. Therefore, for each fixed $x \in E_0$, there exists $j_x \ge 1$ s.t. $x \in E \setminus E_j$ for all $j \ge j_x$ and $|f_{j,i_j} - f_j| < \epsilon/2$. Since $f_j \to g$ a.e., there exists $Z \subset E$ and $m(Z) = 0$ s.t. for each j and each fixed $x \in E \setminus Z$, there exists K s.t. for all $j \ge K$,

$|f_j(x) - g(x)| < \epsilon/2$. This implies for each fixed $x \in E_0$, there exists $M = \max\{j_x, K\}$ s.t. for all $j \geq M$,

$$|f_{j,i_j}(x) - g(x)| \leq |f_{j,i_j} - f_j| + |f_j - g| < \epsilon/2 + \epsilon/2 = \epsilon$$

Thus, $f_{j,i_j} \to g(x)$ a.e. on E.