MAT3006*: Real Analysis
Homework 6

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Extra Problem 1. Let f_∞, f_n, $n \in \mathbb{N}^+$ be measurable and finite a.e. on $E \in \mathcal{M}$, and suppose $m(E) < \infty$. Prove that if any subsequence f_{n_k} of f_n contains a subsequence $f_{n_{k_i}}$ which converges to f_∞ a.e. on E as $i \to \infty$, then $f_n \to f_\infty$ in measure on E as $n \to \infty$.

Suppose f_n does not converge to f in measure. Then we know that there exists $\epsilon > 0$ and $\sigma > 0$ and a subsequence f_{n_k} such that $m(\{x \in E \,|\, |f_{n_k} - f| > \sigma\}) > \epsilon$ for all k. However, by assumption this f_{n_k} also has a further subsequence $f_{n_{k_i}}$ such that $f_{n_{k_i}} \to f$ as $i \to \infty$. Therefore, $f_{n_{k_i}} \to f$ in measure, i.e., for all $\sigma > 0$, $m(\{x \in E \,|\, |f_{n_{k_i}} - f| > \sigma\}) \to 0$ as $i \to \infty$, which contradicts the fact that $m(\{x \in E \,|\, |f_{n_k} - f| > \sigma\}) > \epsilon$ for all k. Therefore, $f_n \to f_\infty$ in measure on E.

Extra Problem 2. Let $E \in \mathcal{M}$ and $m(E) < \infty$. Suppose $f_n \to f_\infty$ and $g_n \to g_\infty$ both in measure on E. Prove that $f_n g_n \to f_\infty g_\infty$ in measure as $n \to \infty$.

Consider an arbitrary subsequence of $f_n g_n$, denoted as $f_{n,k} g_{n,k}$. Since $f_{n,k} \to f$ in measure, there exists a subsequence $f_{n,k,i} \to f$ a.e., and since $g_{n,k,i} \to g$ in measure, there exists a subsequence $g_{n,k,i,j} \to g$ a.e. on E. Therefore, we obtain $f_{n,k,i,j} \to f$ a.e. and $g_{n,k,i,j} \to g$ a.e., so $f_{n,k,i,j} g_{n,k,i,j} \to fg$ a.e. and hence $f_{n,k,i,j} g_{n,k,i,j} \to fg$ in measure. Since $f_{n,k,i,j} g_{n,k,i,j}$ is also a subsequence of $f_{n,k}$ and $g_{n,k}$, this implies for each subsequence of $f_n g_n$, there exists a further subsequence $f_{n,k,i,j} g_{n,k,i,j}$ that converges to fg a.e.. By Extra Problem 1, $f_n g_n \to fg$ in measure.

Extra Problem 3. Suppose $f_n \to f_\infty$ in measure on $E \in \mathcal{M}$; g is uniformly continuous on $\mathbb{R}$. Prove that $g \circ f_n \to g \circ f$ in measure as $n \to \infty$.

If g is uniformly continuous, then for any $\epsilon > 0$ s.t. $|g(x) - g(y)| \geq \epsilon$, then there exists δ_ϵ s.t. $|x - y| \geq \delta_\epsilon$ for all $x, y \in E$. Therefore, for all $\sigma > 0$, if $|g(f_n(x)) - g(f_\infty(x))| \geq \epsilon$, then there exists δ_σ such that $|f_n(x) - f_\infty(x)| \geq \delta_\sigma$. This implies

$$m(\{x \in E \,|\, |g(f_n(x)) - g(f_\infty(x)) \geq \epsilon\}) \leq m(\{x \in E \,|\, |f_n(x) - f_\infty(x)| \geq \delta_\sigma\})$$

Let $n \to \infty$, since $f_n \to f_\infty$ in measure, the RHS above tends to zero, which means LHS also tends to zero. This implies $g(f_n(x)) \to g(f_\infty(x))$ in measure since ϵ can be arbitrary.

Extra Problem 4. Let $f_{n,i} \to f_n$ in measure as $i \to \infty$ on $E \in \mathcal{M}$. Also, $f_n \to f_\infty$ in measure as $n \to \infty$. Prove that there exists subsequence $f_{n_m, i_m} \to f_\infty$ a.u. as $m \to \infty$.

Since $f_n \to f_\infty$ in measure, there exists a subsequence f_{n_m} of f_n s.t. $f_{n_m} \to f_\infty$ a.u.. Consider $f_{n_1,i}$, since $f_{n_1,i} \to f_{n_1}$ in measure, there exists a subsequence $f_{n_1,i_j^{(1)}}$ that converges to f_{n_1} a.u. on E. Then consider $f_{n_2,i_j^{(1)}}$, since it is a subsequence of $f_{n_2,i}$, $f_{n_2,i_j^{(1)}} \to f_{n_2}$ in measure, then there exists a subsequence of $f_{n_2,i_j^{(1)}}$, denoted as $f_{n_2,i_j^{(2)}}$ s.t. $f_{n_2,i_j^{(2)}} \to f_{n_2}$ a.u.. Continue this process, we can find $f_{n_m,i_j^{(m)}} \to f_{n_m}$ a.u. and $\{i_j^{(m+1)}\}_{j=1}^\infty \subset \{i_j^{(m)}\}_{j=1}^\infty$ for all $m \geq 1$. Take the diagonal sequence $\{i_j^{(j)}\}_{j=1}^\infty$, then $f_{n_m,i_j^{(j)}} \to f_{n_m}$ a.u. on E for each fixed $m \geq 1$.

Let $g_{m,j} = f_{n_m,i_j^{(j)}}$ and $g_m = f_{n_m}$, then $g_{m,j} \to g_m$ a.u. as $j \to \infty$ and $g_m \to f_\infty$ as $m \to \infty$. For all $m \geq 1$, take a large j_m and $B_m \subset E$ s.t. $m(B_m) < \frac{1}{2^m}$ and $|g_{m,l} - g_m| < \frac{1}{2^m}$ on $E \setminus B_m$ if $j \geq j_m$. WLOG, we can choose j_m s.t. j_m is strictly increasing to infinity as $m \to \infty$. Then we claim that $g_{m,j_m} \to f_\infty$ a.u. on E. For any $\delta > 0$, take $B_\delta \subset E$ s.t. $B_\delta < \frac{\delta}{100}$ and $g_m \to f_\infty$ uniformly on $E \setminus B_\delta$. For any $\epsilon > 0$, take large $M \geq 1$ s.t. $\sum_{m=M}^\infty \frac{1}{2^m} < \frac{\delta}{100}$, $|g_m - f_\infty| < \frac{\epsilon}{2}$ on $E \setminus B_\delta$ and $m > 1 - \log_2 \epsilon$ for all $m \geq M$. Let $E_\delta = B_\delta \cup (\bigcup_{m=M}^\infty B_m)$, then $m(E_\delta) < \delta$. On $E \setminus E_\delta$, for $m \geq M$, we have $|g_{m,j_m} - f_\infty| < |g_{m,j_m} - g_m| + |g_m - f_\infty| < \epsilon$.

Extra Problem 5. Suppose $f_n \to f_\infty$ in measure on $E \in \mathbb{R}$, $E \in \mathcal{M}$. Assume f_n is M-Lipschitz continuous on E for all $n \geq 1$, prove that $f_n \to f_\infty$ a.e. as $n \to \infty$.

Let $E_0 = \{x \in E \,|\, \exists\, \epsilon_x > 0,\, m(N_{\epsilon_x}(x) \cap E) = 0\}$ where $N_{\epsilon_x}(x)$ is the neighborhood of x with radius ϵ_x. Then $E_0 \subset \bigcup_{x \in E_0} N_{\epsilon_x}(x)$. By Lindelöf covering theorem, there exists $\{x_n\}_{n=1}^\infty \subset E_0$ and $\{\epsilon_n\}_{n=1}^\infty \subset \{\epsilon_x\}_{x \in E_0}$ s.t. $E_0 \subset \bigcup_{n=1}^\infty N_{\epsilon_n}(x_n)$ and $m(E \cap N_{\epsilon_n}(x_n)) = 0$ for all $n \geq 1$. Notice that

$$0 \leq m^*(E_0) \leq m^*\left(\bigcup_{n=1}^\infty (E \cap N_{\epsilon_n}(x_n))\right) \leq \sum_{n=1}^\infty m^*(E \cap N_{\epsilon_n}(x_n)) = 0$$

Therefore, $m(E_0) = 0$. For each fixed $x_0 \in E \setminus E_0$, $x_0 \notin E_0$ implies that for all $\epsilon > 0$, $m(E \cap N_\epsilon(x_0)) = 2c(\epsilon; x_0) > 0$. Since $f_n \to f_\infty$ in measure on E as $n \to \infty$, f_n is Cauchy in measure, and there exists $N(c, \epsilon) \geq 1$ s.t. $m(\{x \in E \,|\, |f_n - f_m| > \epsilon\}) < c$ for all $m, n \geq N(c, \epsilon)$. Define

$$A = \{x \in E \,|\, |f_n - f_m| \leq \epsilon\} \cap (E \cap N_\epsilon(x_0))$$

We claim that $A \neq \varnothing$, because otherwise $E \cap N_\epsilon(x_0) \subset \{x \in E \,|\, |f_n - f_m| > \epsilon\}$. Then,

$$2c = m(E \cap N_\epsilon(x_0)) \leq m(\{x \in E \,|\, |f_n - f_m| > \epsilon\}) < c$$

gives a contradiction. Thus, we can pick $y \in A$, and $|f_n(y) - f_m(y)| \leq \epsilon$ when $n, m \geq N(c, \epsilon)$. Hence,

$$|f_n(x_0) - f_m(x_0)| \leq |f_n(x_0) - f_n(y)| + |f_n(y) - f_m(y)| + |f_m(y) - f_m(x_0)| \leq 2M|x_0 - y| + \epsilon$$

Since $y \in A$, $|y - x_0| < \epsilon$, we obtain $|f_n(x_0) - f_m(x_0)| \leq (2M + 1)\epsilon$. This is enough to show $f_n(x_0) \to f_\infty(x_0)$ and since x_0 is arbitrary in E except a zero measure set E_0, $f_n(x) \to f_\infty(x)$ a.e. on E.

Extra Problem 6. Let f be real-valued and defined on $E \in \mathbb{R}^n$, $E \in \mathcal{M}$, satisfying $\forall\, \delta > 0$, there exists closed $F_\delta \subset E$ s.t. $m(E \setminus F_\delta) < \delta$ and $f\big|_{F_\delta}$ is continuous. Prove f is measureable on E.

By assumption, for all $n \in \mathbb{N}$, there exists closed $F_n \subset E$ s.t. $m(E \setminus F_n) < \frac{1}{n}$ and $f\big|_{F_n}$ is continuous. Note that $E \setminus \bigcup_{n=1}^{\infty} F_n \subset E \setminus F_n$ for all $n \in \mathbb{N}^+$, define $Z = E \setminus \bigcup_{n=1}^{\infty} F_n$, we have $m(Z) \leq m(E \setminus F_n) < \frac{1}{n}$. Hence, $m(Z) = 0$. For all $t \in \mathbb{R}$, since $f\big|_{F_n}$ is continuous, $\{x \in F_n \,|\, f(x) > t\}$ is open hence measurable. Since any subset of a null set is measurable, $\{x \in Z \,|\, f(x) > a\} \in \mathcal{M}$. Therefore, $\{x \in E \,|\, f(x) > t\} = \{x \in Z \,|\, f(x) > a\} \cup \bigcup_{n=1}^{\infty} \{x \in F_n \,|\, f(x) > t\}$ is measurable. This implies f is measurable on E.

Extra Problem 7. Let f be real-valued, measurable on a finite interval $[a, b]$. Prove that there exists sequence h_k s.t. $h_k \to 0$, $f(x + h_k) \to f(x)$ for a.e. $x \in [a, b]$ as $k \to \infty$.

Denote $E = [a, b]$. By Lusin's theorem, for every $k \in \mathbb{N}^+$, there exists closed $F_k \subset E$ s.t. $m(E \setminus F_k) < \frac{b-a}{2^{k+2}}$ and $f\big|_{F_k}$ is continuous. Note that $m(E \setminus F_k) = m(E) - m(F_k) = b - a - m(F_k)$, we can conclude $m(F_k) > \left(1 - \frac{1}{2^{k+2}}\right)(b - a)$. Also notice that E is bounded, so F_k is compact and $f\big|_{F_k}$ is uniformly continuous. Thus, there exists small h_k s.t. for all $x, y \in F_k$, $|x - y| < h_k$, we have $|f(x) - f(y)| < \frac{1}{k}$. WLOG, assume $h_k < \frac{b-a}{2^{k+1}} \to 0$ as $k \to \infty$.

Now we need to estimate $m(F_k \cap (F_k - h_k))$. Consider

$$\begin{aligned} m(F_k \cap (F_k - h_k)) &= m(F_k) + m(F_k - h_k) - m(F_k \cup (F_k - h_k)) \\ &= 2m(F_k) - m(F_k \cup (F_k - h_k)) \\ &> \left(2 - \frac{1}{2^{k+1}}\right)(b - a) - m(E \cup (E - h_k)) \\ &= \left(2 - \frac{1}{2^{k+1}}\right)(b - a) - ((b - a) + h_k) \\ &> \left(1 - \frac{1}{2^k}\right)(b - a) \end{aligned}$$

Therefore, let $E_k = E \setminus (F_k \cap (F_k - h_k))$, we have $m(E_k) \leq b - a - (b - a)\left(1 - \frac{1}{2^k}\right) = \frac{b-a}{2^k}$ for all $k \geq 1$. Therefore, $\sum_{k=1}^{\infty} m(E_k) < \infty$. By Borel-Cantelli lemma, $m(\overline{\lim}_{k\to\infty} E_k) = 0$. For $x \in E \setminus \overline{\lim}_{k\to\infty} E_k = \underline{\lim}_{k\to\infty} (E \setminus E_k) = \underline{\lim}_{k\to\infty} [F_k \cap (F_k - h_k)]$, there exists N_x s.t. for $k \geq N_x$, $x \in E \setminus E_k$. This shows $x \in F_k$ and $x + h_k \in F_k$. By uniform continuity, $|f(x) - f(x + h_k)| < \frac{1}{k}$. Therefore, if $k \to \infty$, $h_k \to 0$ and $f(x + h_k) \to f(x)$. This proves $f(x + h_k) \to f(x)$ for a.e. $x \in [a, b]$.