MAT3006*: Real Analysis
Homework 7

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Extra Problem 1. Let $f(x)$ be measurable and nonnegative on $E \in \mathcal{M}$. Suppose $\int_E f(x)\,dx = 0$. Prove that $f = 0$ a.e. on E.

Let $A = \{x \in E \mid f(x) > 0\} = \bigcup_{k=1}^{\infty} \{x \in E \mid f(x) \geq \frac{1}{k}\}$. Then since f is measurable, A is measurable and it suffices to show that $m(A) = 0$. If $m(A) > 0$, then there exists $k_0 \geq 1$ s.t. $m(\{x \in E \mid f(x) \geq \frac{1}{k_0}\}) > 0$. Denote $E_{k_0} = \{x \in E \mid f(x) \geq \frac{1}{k_0}\}$, then

$$0 = \int_E f(x)\,dx \geq \int_{E_{k_0}} f(x)\,dx \geq \int_{E_{k_0}} \frac{1}{k_0}\,dx = \frac{m(E_{k_0})}{k_0} > 0$$

which is a contradiction. Thus, $m(A) = 0$ and $f = 0$ a.e. on E.

Extra Problem 2. Let $f(x)$ be nonnegative, measurable, and positive a.e. on $E \in \mathcal{M}$, satisfying $\int_E f(x)\,dx = 0$. Prove that $m(E) = 0$.

Let $E_k = \{x \in E \mid f(x) \geq \frac{1}{k}\}$ for $k \in \mathbb{N}^+$, then since $f(x) > 0$ on E, E_k is measurable and E_k increasing to E. Thus, $m(E_k) \to m(E)$ as $k \to \infty$. Suppose $m(E) = c > 0$, then by definition of limit, there exists $k_0 \geq 1$ s.t. $m(E_{k_0}) \geq \frac{c}{2} > 0$. With nonegativity of f, this implies

$$0 = \int_E f(x)\,dx \geq \int_{E_{k_0}} f(x)\,dx \geq \int_{E_{k_0}} \frac{1}{k_0}\,dx = \frac{m(E_{k_0})}{k_0} \geq \frac{c}{2k_0} > 0$$

This is a contradiction, so $m(E) = 0$.

Extra Problem 3. Let $f(x)$ be nonnegative, measurable on $E \in \mathcal{M}$ s.t. $\int_E f(x)\,dx < \infty$. Let $E_k = \{x \in E \mid f(x) \geq k\}$, $k \geq 1$. Prove that $\sum_{k=1}^{\infty} m(E_k) < \infty$.

Let $E_k = \bigcup_{i=k}^{\infty} F_i$, where $F_i = \{x \in E \mid i \leq f(x) < i+1\}$. Then,

$$\sum_{k=1}^{\infty} m(E_k) = \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} m(F_i) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} m(F_i) = \sum_{i=1}^{\infty} i m(F_i)$$

Consider the integration term by term rule, we have

$$\infty > \int_E f(x)\,dx \geq \sum_{i=1}^{\infty} \int_{F_i} f(x)\,dx \geq \sum_{i=1}^{\infty} \int_{F_i} i\,dx = \sum_{i=1}^{\infty} i m(F_i) \geq 0$$

Therefore, $\sum_{k=1}^{\infty} m(E_k) < \infty$.

Extra Problem 4. Let $f(x)$ be nonnegative, measurable on $E \in \mathcal{M}$, where $m(E) < \infty$. Prove $\int_E f(x)\,dx < \infty$ if and only if $\sum_{k=0}^{\infty} 2^k m(E_{2^k}) < \infty$, where $E_k = \{x \in E \,|\, f(x) \geq k\}$ for all $k \geq 0$.

Notice that $m(E_0) = m(E)$ is a finite number. Also, by Cauchy's condensation test, the series $\sum_{k=0}^{\infty} 2^k m(E_{2^k})$ converges if and only if $\sum_{k=1}^{\infty} m(E_k)$ converges. Thus, we only need to show $\int_E f(x)\,dx < \infty$ if and only if $\sum_{k=1}^{\infty} m(E_k) < \infty$. For the "only if" part, we have already shown it in Extra Problem 3, so it suffices to show the "if" part. Define the same F_i as in Extra Problem 3, but for $k \geq 0$. Then, it suffices to show if $\sum_{i=1}^{\infty} im(F_i) < \infty$, we have $\int_E f(x)\,dx < \infty$. Note that

$$\int_E f(x)\,dx = \sum_{i=0}^{\infty} \int_{F_i} f(x)\,dx \leq m(F_0) + \sum_{i=1}^{\infty} (i+1)m(F_i)$$

Since $m(E) = \sum_{i=0}^{\infty} m(F_i) < \infty$, we have

$$\int_E f(x)\,dx \leq m(F_0) + \sum_{i=1}^{\infty} (i+1)m(F_i) = \sum_{i=0}^{\infty} m(F_i) + \sum_{i=1}^{\infty} im(F_i) < \infty$$

Therefore, we obtain the desired result.

Extra Problem 5. Let $f(x)$ be measurable on $[0,1]$ s.t. $f(x) > 0$, for all $x \in [0,1]$. Prove that for all $q \in (0,1)$, there exists $\delta > 0$ s.t. $\int_E f(x)\,dx > \delta$, whenever $E \subset [0,1]$, $E \in \mathcal{M}$ and $m(E) \geq q$.

Let $E_k = \{x \in [0,1] \,|\, f(x) \geq \frac{1}{k}\}$. Then E_k increases to $[0,1]$, so $m(E_k) \to 1$ as $k \to \infty$. Take k_0 large s.t. $m(E_{k_0}) > 1 - \frac{q}{2}$. Consider

$$m(E \cap E_{k_0}) = m(E) + m(E_{k_0}) - m(E \cup E_{k_0}) > q + 1 - \frac{q}{2} - 1 = \frac{q}{2}$$

Then we have

$$\int_E f(x)\,dx \geq \int_{E \cap E_{k_0}} f(x)\,dx \geq \int_{E \cap E_{k_0}} \frac{1}{k_0}\,dx = \frac{m(E \cap E_0)}{k_0} > \frac{q}{2k_0} > 0$$

This impiles that for all $q \in (0,1)$, we can take $\delta = \frac{q}{2k_0}$, and the desired result holds.