MAT3006*: Real Analysis
Homework 8

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Extra Problem 1. Let $f_k(x)$ be nonnegative and measurable on $[0,1]$ s.t. $f_k(x) \to \infty$ a.e. on $[0,1]$. Prove that $\int_0^1 f_k(x)\,dx \to \infty$.

Denote $a_k = \int_0^1 f_k(x)\,dx$, suppose $a_k \not\to \infty$, then there exists a finite M and a subsequence a_{k_j} of a_k s.t. $a_{k_j} \le M$ for all $j \in \mathbb{N}^+$. By Fatou's lemma,

$$\int_0^1 \varliminf_{j\to\infty} f_{k_j}(x)\,dx \le \varliminf_{j\to\infty} \int_0^1 f_{k_j}(x)\,dx \le \varliminf_{j\to\infty} M = M < \infty$$

If we let $A = \{x \,|\, f_k(x) \to \infty\}$, since f_k is nonnegative, we have

$$\int_0^1 \varliminf_{j\to\infty} f_{k_j}(x)\,dx \ge \int_A \varliminf_{j\to\infty} f_{k_j}(x)\,dx = \int_A \infty\,dx = \infty \cdot m(A) = \infty$$

since $m(A) = 1$. Therefore, we have $\infty < \infty$ which is a contradiction. This impiles $a_k \to \infty$.

Extra Problem 2. Let $f_k(x)$ be nonnegative and measurable on $E \in \mathcal{M}$, $f_k \to f_\infty$ in measure on E. Prove that $\int_E f_\infty(x)\,dx \le \varliminf_{k\to\infty} \int_E f_k(x)\,dx$.

Since $f_k \to f_\infty$ in measure on E, by definition, f_k, f_∞ are a.e. finite on E. Suppose $E_k = \{x \in E \,|\, f_k(x) = \infty\}$ for all k and $E_\infty = \{x \in E \,|\, f_\infty(x) = \infty\}$, then $m(E_k) = 0$ and $m(E_\infty) = 0$. Thus, denote $F = E_\infty \cup \bigcup_{k=1}^\infty E_k$, $m(F) = 0$. Therefore, it suffices to show

$$\int_{E\setminus F} f_\infty(x)\,dx \le \varliminf_{k\to\infty} \int_{E\setminus F} f_k(x)\,dx$$

Let $A = E \setminus F$, then on A, f_k and f_∞ are everywhere finite. Since $f_k \to f_\infty$ in measure, there exists a subsequence $f_{k_j} \to f_\infty$ a.e.. Therefore, $\varliminf_{k\to\infty} f_k(x) \le \lim_{j\to\infty} f_{k_j}(x) \le f_\infty(x)$.

Let $B = \{x \in A \,|\, \varliminf_{k\to\infty} f_k(x) < f_\infty(x)\}$, and suppose $m(B) > 0$. Denote $B_k = \{x \in A \,|\, \varliminf_{k\to\infty} f_k(x) \le f_\infty(x) - \frac{1}{k}\}$, so B_k increases to B and thus $m(B_k) \to m(B)$. Take $\delta = \frac{m(B)}{2}$, there exists N s.t. $m(B_N) \ge \delta$. For each fixed $x \in B_N$, since $\varliminf_{k\to\infty} f_k(x) = \sup_{k\ge1} \inf_{n\ge k} f_n(x)$, we have $\inf_{n\ge k} f_n(x) \le f_\infty(x) - \frac{1}{N} < f_\infty(x) - \frac{1}{2N}$ for all k. This implies for all k, there exists $n \ge k$ s.t. $f_n(x) < f_\infty(x) - \frac{1}{2N}$. Therefore, we can pick a subsequence k_n s.t. k_n is strictly increasing and $f_{k_n}(x) < f_\infty(x) - \frac{1}{2N}$. Therefore, $x \in \{x \in A \,|\, |f_k(x) - f_\infty(x)| > \frac{1}{2N}\}$ for all $k = k_n$. Therefore, $m(\{x \in A \,|\, |f_{k_n}(x) - f_\infty(x)| > \frac{1}{2N}\}) \ge \delta$ for all $n \ge 1$. This contradicts that $f_k \to f_\infty$ in measure.

Therefore, $m(B) = 0$, and $\varliminf_{k\to\infty} f_k(x) = f_\infty(x)$ a.e. on A. By Fatou's lemma,

$$\int_A f_\infty(x)\,dx = \int_A \varliminf_{k\to\infty} f_k(x)\,dx \le \varliminf_{k\to\infty} \int_A f_k(x)\,dx$$

Extra Problem 3. Let $E_k \subset [0,1]$, $E_k \in \mathcal{M}$, for all $k \geq 1$ s.t. $m(E_k) \geq \delta > 0$ where δ is a constant. Assume for a sequence a_k we have $\sum_{k=1}^{\infty} |a_k| I_{E_k}(x) < \infty$ a.e. on $[0,1]$. Prove that $\sum_{k=1}^{\infty} |a_k| < \infty$.

Let $f(x) = \sum_{k=1}^{\infty} |a_k| I_{E_k}(x)$, and $B_n = \{x \in [0,1] \,|\, f(x) < n\}$. Since f is finite a.e. on $[0,1]$, $m(B_n) \to 1$ increasingly. There exists N s.t. $B_N \geq 1 - \frac{\delta}{2}$. Also,

$$m(E_k \cap B_N) = m(E_k) + m(B_N) - m(E_k \cup B_N) \geq \delta + 1 - \frac{\delta}{2} - 1 = \frac{\delta}{2}$$

for all $k \geq 1$. Notice that $\int_{B_N} f(x)\, dx \leq N m(B_N) \leq N$, and

$$\int_{B_N} f(x)\, dx = \int_0^1 \sum_{k=1}^{\infty} |a_k| I_{E_k \cap B_N}(x)\, dx = \sum_{k=1}^{\infty} |a_k| m(E_k \cap B_N) \geq \frac{\delta}{2} \sum_{k=1}^{\infty} |a_k|$$

Therefore, $\frac{\delta}{2} \sum_{k=1}^{\infty} |a_k| \leq N$, which implies that $\sum_{k=1}^{\infty} |a_k| < \infty$.

Extra Problem 4. Let $f_k(x)$ be measurable on $E \in \mathcal{M}$ s.t. $|f_k| \leq F$ a.e. on E, where $F \in L^1(E)$ and $f_k \to f_\infty$ in measure on E. Prove that $\int_E |f_k - f_\infty|\, dx \to 0$ as $k \to \infty$. In particular, $\int_E f_k(x)\, dx \to \int_E f_\infty(x)\, dx$ as $k \to \infty$.

Let $A_k = \{x \,|\, F(x) > \frac{1}{k}\}$ and $A = \bigcup_{k=1}^{\infty} A_k = \{x \,|\, F(x) \neq 0\}$. Then A_k increases to A and $E \setminus A = \{x \,|\, F(x) = 0\}$ and for $x \in E \setminus A$, $f_k(x) = 0$, $f_\infty(x) = 0$ for all $k \geq 1$. Since $F \in L^1(E)$, $m(A_k) < \infty$ for all $k \geq 1$. Suppose $\int_E |f_k - f_\infty|\, dx \not\to 0$, then there exists a subsequence f_{k_j} s.t.

$$\epsilon \leq \int_E |f_{k_j} - f_\infty|\, dx \leq 2 \int_{E \setminus A_k} F\, dx + \int_{A_k} |f_k - f_\infty|\, dx$$

for some fixed $\epsilon > 0$. Since $A_k \to A$, F is nonnegative, by MCT,

$$\int_{E \setminus A_k} F\, dx \to \int_{E \setminus A} F\, dx = 0$$

Therefore, there exists N s.t. for all $k \geq N$, $\int_{E \setminus A} F\, dx < \frac{\epsilon}{4}$. This implies $\int_{A_k} |f_{k_j} - f_\infty|\, dx > \frac{\epsilon}{2}$. Since $f_k \to f_\infty$ in measure, there exists a further subsequence $f_{k_{j_m}} \to f_\infty$ a.e. on E. This implies that $|f_\infty| \leq F$ a.e. and $f_\infty \in L^1(E)$. Therefore, by DCT, $\int_E |f_{k_{j_m}} - f_\infty|\, dx \to 0$ which contradicts that $\int_{A_k} |f_{k_j} - f_\infty|\, dx > \frac{\epsilon}{2}$. Therefore, $\int_E |f_k - f_\infty|\, dx \to 0$ and by similar argument in the proof of DCT, we have $\int_E f_k(x)\, dx \to \int_E f_\infty(x)\, dx$ as $k \to \infty$.

Extra Problem 5. Let $f_k(x)$ be measurable and nonnegative on $E \in \mathcal{M}$, where $m(E) < \infty$. Prove that $f_k \to 0$ in measure on E iff $\int_E \frac{f_k(x)}{1+f_k(x)}\, dx \to 0$.

For "if" part, let $g_k(x) = \frac{f_k(x)}{1+f_k(x)}$, then $g_k(x) \in [0,1]$ for all x, k. $\int_E g_k(x)\, dx \to 0$ means $g_k \to 0$ in L^1, thus $g_k \to 0$ in measure on E. By definition, for any $\sigma > 0$, $m(\{x \in E \,|\, g_k(x) > \frac{\sigma}{1+\sigma}\}) \to 0$ as $k \to \infty$. Since $h(x) = \frac{x}{1+x}$ is continuous and strictly increasing on $[0,1]$, so $g_k(x) > \frac{\sigma}{1+\sigma}$ is equivalent to $f_k(x) > \sigma$. Thus, $m(\{x \in E \,|\, f_k(x) > \sigma\}) \to 0$, i.e., $f_k \to 0$ in measure on E.

For "only if" part, suppose $f_k \to 0$ in measure but $\int_E g_k\, dx \not\to 0$, then there exists $\epsilon > 0$, for all N, there exists $k > N$ s.t. $\int_E g_k\, dx \geq \epsilon$. Since $m(E) < \infty$, denote $M = m(E)$ and WLOGG, assume $\epsilon < 2M$. Consider $B_k = \{x \in E \,|\, g_k \geq \frac{\epsilon}{2M}\}$. Claim that $m(B_k) \geq \frac{\epsilon}{2}$. Suppose not,

$$\epsilon \leq \int_E g_k \, dx = \int_{B_k} g_k \, dx + \int_{E \setminus B_k} g_k \, dx \leq m(B_k) + \frac{\epsilon}{2M} m(E \setminus B_k) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which is a contradiction, so $m(B_k) \geq \frac{\epsilon}{2}$. Thus, $m(\{x \in E \,|\, f_k(x) \geq \frac{\epsilon/(2M)}{1+\epsilon/(2M)}\}) \geq \frac{\epsilon}{2}$ for all $k > N$, but this contradicts that $f_k \to 0$ in measure on E, so $\int_E g_k \, dx \to 0$.

Extra Problem 6. Let $f_k(x)$ be nonnegative measurable on $E \in \mathcal{M}$. Let $f \in L^1(E)$ s.t. $f_k \to f$ in measure on E and $\int_E f_k(x) \, dx \to \int_E f(x) \, dx$. Prove that $\int_E |f_k(x) - f(x)| \, dx \to 0$.

Let $g_k = f_k + f - |f_k - f|$, since $f_k \to f$ in measure, $f_k - f \to 0$ in measure, and thus $|f_k - f| \to 0$ in measure. By linearity, $g_k = f_k + f - |f_k - f| \to f + f - 0 = 2f$ in measure. Apply Extra Problem 2 on g_k,

$$\int_E 2f; dx \leq \varliminf_{k\to\infty} \int_E g_k \, dx = \varliminf_{k\to\infty} \int_E [f_k + f - |f_k - f|] \, dx$$

Since $f \in L^1(E)$ and $\int_E f_k \, dx \to \int_E f \, dx$, there exists N s.t. for all $k \geq N$, $f_k \in L^1(E)$. Thus, if we only consider $k \geq N$, by linearity of integral,

$$\varliminf_{k\to\infty} \int_E [f_k + f - |f_k - f|] \, dx = 2\int_E f \, dx - \varlimsup_{k\to\infty} \int_E |f_k - f| \, dx$$

Therefore, $\varlimsup_{k\to\infty} \int_E |f_k - f| \, dx \leq 0$, which implies $\lim_{k\to\infty} \int_E |f_k - f| \, dx = 0$.

Extra Problem 7. Let $c \in \mathbb{R} \setminus \{0\}$ and $a \in \mathbb{R}$. Suppose $f \in L^1(\mathbb{R})$. Prove that $f(cx + a) \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} f(cx + a) \, dx = \frac{1}{|c|} \int_{\mathbb{R}} f(y) \, dy$.

The key is to show for any $E \subset \mathbb{R}$, $E \in \mathcal{M}$, we have $cE \in \mathcal{M}$ and $m(cE) = |c| m(E)$ for all real $c \neq 0$. Let $\{R_k\}_{k=1}^{\infty}$ be L-covering of E, then $\{cR_k\}_{k=1}^{\infty}$ is a L-covering of cE. Also, $m^*(cE) \leq \sum_{k=1}^{\infty} |cR_k| = |c| \sum_{k=1}^{\infty} |R_k|$, so by taking infinimum over all L-covering of E, we have $m^*(cE) \leq |c| m^*(E)$. Since $m^*(E) = m^*(\frac{cE}{c}) \leq \frac{1}{|c|} m^*(cE)$, we obtain $m^*(cE) = |c| m^*(E)$. Note that $f(x) = cx$ is a Lipschitz continuous function, so it maps any measurable set to measurable set. Since $cE = f(E)$, cE is measurable. This shows $m(cE) = cm(E)$.

Consider any indicator function $f = I_E(x)$ for any measurable set $E \subset \mathbb{R}$. Since $f \in L^1$, $\int_{\mathbb{R}} I_E(x) \, dx = m(E) < \infty$. By translation invariance proved in lecture and the fact we proved above, the set $\frac{E-a}{c}$ is measurable and $m(\frac{E-a}{c}) = \frac{1}{|c|} m(E)$. Since $f(cx + a) = I_E(cx + a) = I_{\frac{E-a}{c}}(x)$,

$$\int_{\mathbb{R}} f(cx + a) \, dx = m\left(\frac{E - a}{c}\right) = \frac{1}{|c|} m(E) = \frac{1}{|c|} \int_{\mathbb{R}} f(y) \, dy$$

Then consider any nonneagtive simple measurable function with the form $f(x) = \sum_{k=1}^{n} a_k I_{E_k}(x)$ where E_k's are measurable with and $a_k > 0$'s are real number. If $f \in L^1(\mathbb{R})$, then we can always define E_k's s.t. $m(E_k) < \infty$ for all k. Then for each k, $I_{E_k}(cx + a) \in L^1(\mathbb{R})$ and thus, as a finite sum of L^1 function, $f(cx + a) \in L^1$. Also, by I.T.T.,

$$\int_{\mathbb{R}} f(cx + a) \, dx = \int_{\mathbb{R}} \sum_{k=1}^{n} a_k I_{E_k}(cx + a) \, dx = \sum_{k=1}^{n} \frac{a_k}{|c|} \int_{\mathbb{R}} I_{E_k}(y) \, dy = \frac{1}{|c|} \int_{\mathbb{R}} f(y) \, dy$$

Next, for any nonnegative measurable function f, there exists nonnegative simple function $\phi_n(x)$ increasing to $f(x)$. Since $f(x) \in L^1$, $\phi_n(x) \in L^1$, and $\phi_n(cx+a) \in L^1$. Then by MCT,

$$\int_{\mathbb{R}} f(cx+a)\,dx = \lim_{n\to\infty} \int_{\mathbb{R}} \phi_n(cx+a)\,dx = \lim_{n\to\infty} \frac{1}{|c|} \int_{\mathbb{R}} \phi_n(y)\,dy = \frac{1}{|c|} \int_{\mathbb{R}} f(y)\,dy$$

which also shows $f(cx+a) \in L^1$. Finally, for general measurable function f, $f = f^+ - f^-$ where f^+, f^- are both nonnegative. If $f \in L^1$, then f^+, f^- are both in L^1, thus $f^+(cx+a), f^-(cx+a)$ are both in L^1, and so $f(cx+a)$ are in L^1. In addition,

$$\int_{\mathbb{R}} f(cx+a)\,dx = \int_{\mathbb{R}} f^+(cx+a)\,dx - \int_{\mathbb{R}} f^-(cx+a)\,dx = \frac{1}{|c|} \int_{\mathbb{R}} f^+(y)\,dy - \frac{1}{|c|} \int_{\mathbb{R}} f^-(y)\,dy$$

Therefore, we finish the whole proof.

Extra Problem 8. Let $E \subset \mathbb{R}$ and $E \in \mathcal{M}$. Suppose $f \in L^1(E)$, and prove that $\int_{\frac{E-a}{c}} f(cx+a)\,dx = \frac{1}{|c|} \int_E f(y)\,dy$ for all $c \neq 0$, $a \in \mathbb{R}$.

Notice that

$$\int_{\frac{E-a}{c}} f(cx+a)\,dx = \int_{\mathbb{R}} I_{\frac{E-a}{c}}(x) f(cx+a)\,dx = \int_{\mathbb{R}} I_E(cx+a) f(cx+a)\,dx$$

Apply Extra Problem 7, we have

$$\int_{\mathbb{R}} I_E(cx+a) f(cx+a)\,dx = \frac{1}{|c|} \int_{\mathbb{R}} I_E(y) f(y)\,dy = \frac{1}{|c|} \int_E f(y)\,dy$$

Therefore, we proved that $\int_{\frac{E-a}{c}} f(cx+a)\,dx = \frac{1}{|c|} \int_E f(y)\,dy$.