MAT3006*: Real Analysis
Homework 9

李肖鹏 (116010114)

Due date: April. 10, 2020

Extra Problem 1. Suppose $f \in L^1(E)$, $E \in \mathcal{M}$. $E = \bigcup_{k=1}^{\infty} E_k$, $E_k \in \mathcal{M}$, pairwise disjoint. Prove that $\int_E f(x)\,dx = \sum_{k=1}^{\infty} \int_{E_k} f(x)\,dx$.

Let $f_k(x) = f(x) I_{E_k}(x)$, then since $E = \bigcup_{k=1}^{\infty} E_k$ and E_k's pairwise disjoint, we have $f(x) = \sum_{k=1}^{\infty} f_k(x)$ and $f_k \in L^1(E)$. Also,

$$\sum_{k=1}^{\infty} \int_E |f_k(x)|\,dx = \sum_{k=1}^{\infty} \int_{E_k} |f(x)|\,dx = \int_E |f(x)|\,dx < \infty$$

where the last equality comes from the nonnegative version of the desired results. Then by general integration term by term property,

$$\int_E f(x)\,dx = \int_E \sum_{k=1}^{\infty} f_k(x)\,dx = \sum_{k=1}^{\infty} \int_E f_k(x)\,dx = \sum_{k=1}^{\infty} \int_{E_k} f(x)\,dx$$

This shows the desired results.

Extra Problem 2. Prove that for all $f \in L^1(E)$, $E \in \mathcal{M}$, there exists a sequence $f_k(x) \in L^1(E)$, s.t. f_k is bounded on E and $f_k \to f$ in $L^1(E)$ as $k \to \infty$.

Define for all $k \in \mathbb{N}^+$,

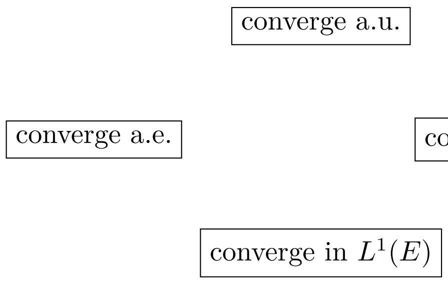
$$f_k(x) = \begin{cases} f(x) & \text{if } |f(x)| \le k \\ k & \text{if } f(x) > k \\ -k & \text{if } f(x) < -k \end{cases}$$

Then it is obvious that $f_k(x)$ is bounded and $|f_k(x)| \le |f(x)|$ for all $x \in E$. Also, $f_k(x) \to f(x)$ pointwise (if convergent to infinity is accepted, otherwise convergent a.e. is also enough). Notice that $|f_k - f| \le 2|f| \in L^1(E)$, by DCT, $\int_E |f_k - f|\,dx \to 0$ as $k \to \infty$.

Extra Problem 3. Prove that for all $f \in L^1(E)$, $E \in \mathcal{M}$, there exists simple functions $f_k(x) \in L^1(E)$ s.t. $f_k \to f$ in $L^1(E)$.

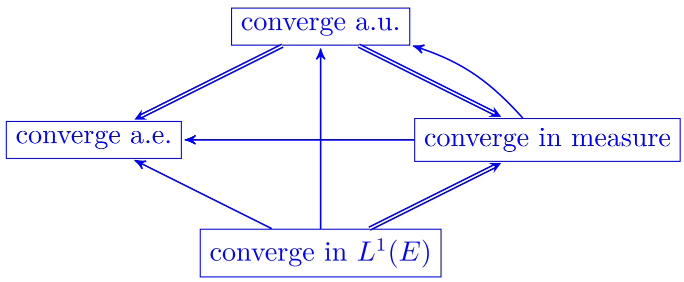
Recall there exists simple measurable function $\phi_k(x)$ s.t. $|\phi_k(x)| < \infty$ and $|\phi_k(x)| \le |f(x)|$ for all $x \in E$ converging pointwise to f on E. Therefore, $\phi_k(x) \in L^1(E)$ and $f - \phi_k(x)$ is well defined ($\infty - \infty$ will not exist). Let $f_k = \phi_k$, since $|f(x) - f_k(x)| \le 2|f(x)| \in L^1(E)$, by DCT, $f_k \to f$ in $L^1(E)$.

Extra Problem 4. Use "$\Longrightarrow$" to denote "implies" and "$\longrightarrow$" to denote "after passing to a subsequence implies", complete the following diagram

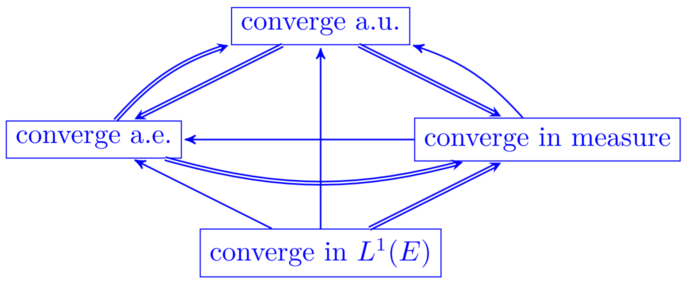


in general case, special case when $m(E) < \infty$, and special case when $|f_k| \leq g \in L^1(E)$ respectively.

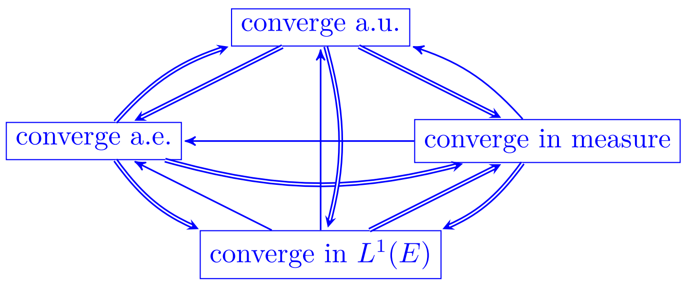
In general case, the diagram is



In $m(E) < \infty$ case, the diagram is



In dominated case, the diagram is



Extra Problem 5. Suppose $f \in L^1(E)$. Prove that for all $\epsilon > 0$, there exists $\delta > 0$ s.t. for all $e \subset E$, $e \in \mathcal{M}$, with $m(e) < \delta$, we have $\int_e |f(x)|\, dx < \epsilon$.

By Extra Problem 2, for all $\epsilon > 0$, there exists a bounded $L^1(E)$-integrable function g s.t. $\int_E |f - g| \, dx < \epsilon/2$ and $|g(x)| \leq M$ for all $x \in E$. Take $\delta = \frac{\epsilon}{2M}$, we have

$$\int_e |f| \, dx \leq \int_e |f - g| \, dx + \int_e |g| \, dx < \frac{\epsilon}{2} + M\frac{\epsilon}{2M} = \epsilon$$

Therefore, we prove the absolute continuity of integrals.

Extra Problem 6. Let $f_k \in L^1(E)$ be s.t. $f_k \to f_\infty$ a.e. on E. Suppose $m(E) < \infty$. Prove that $f_\infty \in L^1(E)$ and $f_k \to f_\infty$ in $L^1(E)$ if and only if for all $\epsilon > 0$, there exists $\delta > 0$ s.t. $\int_e |f_k(x)| \, dx < \epsilon$ for all $k \geq 1$ whenever $e \subset E$, $e \in \mathcal{M}$ and $m(E) < \delta$.

For "only if" part, since $f_k \to f_\infty$ in $L^1(E)$, there exists k_0 s.t. if $k \geq k_0 + 1$, $\int_E |f_k - f_\infty| \, dx < \epsilon/2$. Also, in this case $f_\infty \in L^1(E)$, so by Extra Problem 5, there exists δ_k ($k \in \mathbb{N}^+ \cup \{\infty\}$) s.t. for all $e \subset E$, $e \in \mathcal{M}$ with $m(e) < \delta$, we have $\int_e |f_k| \, dx < \epsilon/2$. Take $\delta = \min\{\delta_1, \ldots, \delta_{k_0}, \delta_\infty\}$, then for all $k \geq k_0 + 1$,

$$\int_e |f_k(x)| \, dx \leq \int_e |f_k(x) - f_\infty(x)| \, dx + \int_e |f_\infty(x)| \, dx < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This implies for all $\epsilon > 0$, there exists $\delta > 0$ s.t. $\int_e |f_\infty(x)| \, dx > \epsilon$ for all $k \geq 1$ whenever $e \subset E$, $e \in \mathcal{M}$ and $m(E) < \delta$.

For "if" part, note that $m(E) < \infty$, then uniform convergent implies $L^1(E)$ convergence. By Egorov's Theorem, for all $\delta_1 > 0$, there exists $E_{\delta_1} \in \mathcal{M}$, $E_{\delta_1} \subset E$ s.t. $m(E_{\delta_1}) < \delta_1$ and $f_k \to f_\infty$ uniformly on $E \setminus E_{\delta_1}$. Therefore, $\int_{E \setminus E_{\delta_1}} |f_k - f_\infty| \, dx < \epsilon$ if $k \geq k_0$. By assumption, there exists $\delta > 0$ s.t. $\int_e |f_k(x)| \, dx < \epsilon$ as long as $e \subset E$, $e \in \mathcal{M}$ and $m(e) < \delta$. Therefore, take $e = E_{\delta_1}$ and $\delta = \delta_1$,

$$\int_E |f_k - f_\infty| \, dx = \int_{E \setminus E_\delta} |f_k - f_\infty| \, dx + \int_{E_\delta} |f_k| \, dx + \int_{E_\delta} |f_\infty| \, dx$$

By Fatou's lemma,

$$\int_e |f_\infty| \, dx = \int_e \varliminf_{k \to \infty} |f_k| \, dx \leq \varliminf_{k \to \infty} \int_e |f_k| \, dx < \epsilon$$

Therefore, $\int_E |f_k - f_\infty| \, dx < 3\epsilon$, which means $f_k \to f_\infty$ in $L^1(E)$. Since $f_\infty = (f_\infty - f_k) + f_k$, it is easy to see $f_\infty \in L^1(E)$.

Extra Problem 7. Recall that one type of improper integral $\int_a^b f(x) \, dx$ can be regarded as $\lim_{c \to a^+} \int_c^b f(x) \, dx$ where $\int_c^b f(x) \, dx$ is a Riemann integral. If such a limit exists as a finite number, then we say the improper integral $\int_a^b f(x) \, dx$ is convergent. Also, the other type of improper integral $\int_{-\infty}^{\infty} f(x) \, dx = \lim_{a \to -\infty, b \to \infty} \int_a^b f(x) \, dx$ converges if $\int_a^b f(x) \, dx$ is Riemann integral and such limit exists as a finite number.

(i) Suppose the improper integral $\int_a^b f(x) \, dx$ is absolutely convergent. Prove that $f \in L^1([a, b])$ and $\int_a^b f(x) \, dm = \int_a^b f(x) \, dx$.

Let $f_k(x) = f(x) I_{[a + \frac{1}{k}, b]}(x)$ for all $k \geq 1$. Since $f_k(x)$ is Riemann integrable, it must be measurable on $[a, b]$ (can be regarded as the limit of step function, and step function is measurable). Since $|f_k|$ increases to $|f|$ pointwise, by MCT,

$$\int_a^b |f(x)| \, dm = \lim_{k \to \infty} \int_a^b |f_k(x)| \, dm = \lim_{k \to \infty} \int_{a + \frac{1}{k}}^b |f(x)| \, dm = \lim_{k \to \infty} \int_{a + \frac{1}{k}}^b |f(x)| \, dx < \infty$$

Therefore, $f \in L^1([a,b])$, and

$$\int_a^b f(x)\ dx = \lim_{k\to\infty} \int_{a+\frac{1}{k}}^b f(x)\ dx = \lim_{k\to\infty} \int_a^b f_k(x)\ dm = \int_a^b f(x)\ dm$$

where the last equality comes from DCT for f_k with dominating function $f(x)$.

(ii) Suppose $\int_a^b f(x)\ dx$ is an improper integral and $f \in L^1([a,b])$. Prove that $\int_a^b f(x)\ dx$ is absolutely convergent.

Denote $g(c) = \int_c^b |f(x)|\ dx$ where $c \in (a,b]$, and by definition, $g(c) \to g(a)$ as $c \to a+$. Notice that $\int_c^b |f(x)|\ dx$ is nonnegative and increasing in c, the limit always exists (may be infinity). Then $g(a+\frac{1}{n}) \to g(a)$ as $n \to \infty$. Since when $c < a$, $g(c)$ is a Riemann integral,

$$g\left(a+\frac{1}{n}\right) = \int_{a+\frac{1}{n}}^b |f(x)|\ dm = \int_a^b |f_n(x)|\ dm$$

where $f_n(x)$ is defined similar to part (i). Since $|f_n| \le |f|$, $f_n \to f$ pointwise and $|f| \in L^1$, by DCT, $g(a+\frac{1}{n}) \to \int_a^b |f(x)|\ dm$. This shows that $g(a) = \int_a^b |f(x)|\ dm < \infty$. Therefore, $\int_a^b f(x)\ dx$ is absolutely convergent.

(iii) Prove the same result for improper integral $\int_{-\infty}^{\infty} f(x)\ dx$ as in (i) and (ii).

First, suppose $\int_{-\infty}^{\infty} f(x)\ dx$ converges absolutely, then $\int_{-\infty}^{\infty} |f(x)|\ dx < \infty$. This implies $\int_{-\infty}^0 |f(x)|\ dx < \infty$ and $\int_0^{\infty} |f(x)|\ dx < \infty$. Let $f_k^+(x) = f(x)I_{[0,k]}$, then $f_k^+ \to f$ on $[0,\infty)$ pointwise and $|f_k^+|$ is increasing. Thus, by MCT,

$$\int_0^{\infty} |f(x)|\ dm = \lim_{k\to\infty} \int_0^{\infty} |f_k^+|\ dm = \lim_{k\to\infty} \int_0^k |f|\ dm$$

Since $|f|$ is bounded on $[0,k]$, so the Lebesgue integral is equal to Riemann integral, i.e,

$$\lim_{k\to\infty} \int_0^k |f|\ dm = \lim_{k\to\infty} \int_0^k |f|\ dx = \lim_{b\to\infty} \int_0^b |f(x)|\ dx$$

where the last equality holds because the limit exists (monotone bounded). Therefore, we proved that $\int_0^{\infty} |f(x)|\ dm = \int_0^{\infty} |f(x)|\ dx$. Similarly, we can prove $\int_{-\infty}^0 |f|\ dm = \int_{-\infty}^0 |f|\ dx$. Therefore, $\int_{-\infty}^{\infty} |f|\ dm = \int_{-\infty}^{\infty} |f|\ dx < \infty$, so $f \in L^1(\mathbb{R})$.

Then we can let $f_k(x) = f(x)I_{[-k,k]}(x)$. It is obvious that $f_k \to f$ pointwise on $\mathbb{R}$. Also, $|f_k| \le |f| \in L^1(\mathbb{R})$, so by DCT,

$$\int_{-\infty}^{\infty} f(x)\ dm = \int_{-\infty}^{\infty} \lim_{k\to\infty} f_k(x)\ dm = \lim_{k\to\infty} \int_{-\infty}^{\infty} f_k(x)\ dm = \lim_{k\to\infty} \int_{-k}^k f(x)\ dm$$

Since $f(x)$ is bounded on $[-k,k]$, so Lebesgue integral is equal to Riemann integral, i.e.,

$$\lim_{k\to\infty} \int_{-k}^k f(x)\ dm = \lim_{k\to\infty} \int_{-k}^k f(x)\ dx = \int_{-\infty}^{\infty} f(x)\ dx$$

where the last equality holds because the limit of $\int_a^b f(x)\ dx$ exists as a finite numeber as $a \to -\infty$ and $b \to \infty$.

Conversely, suppose $f \in L^1(\mathbb{R})$, we want to show the improper integrals $\int_0^\infty |f(x)|\, dx$ and $\int_{-\infty}^0 |f(x)|\, dx$ exist as a fintie number. Denote $g(b) = \int_0^b |f(x)|\, dx$ for any $b \geq 0$, then $g(b)$ is a nonnegative increasing function, so $\lim_{b\to\infty} g(b)$ exists (may be infinity). If this limit is infinity, then by definition, for all $M > 0$, there exists $B > 0$ s.t. for all $b \geq B$, $g(b) \geq M$. However, for each B, we can always find an integer N s.t. $N > B$, so for all $M > 0$, there exists $N \in \mathbb{N}^+$ s.t. for all integer $k \geq N$, $g(k) \geq M$, so $\lim_{k\to\infty} g(k) = \infty$. Note

$$\infty = \lim_{k\to\infty} g(k) = \lim_{k\to\infty} \int_0^k |f(x)|\, dx = \lim_{k\to\infty} \int_0^k |f(x)|\, dm = \lim_{k\to\infty} \int_0^\infty |f_k(x)|\, dm$$

By MCT on $|f_k(x)|$ restricted on $(0,\infty)$, $\lim_{k\to\infty} \int_0^\infty |f_k(x)|\, dm = \int_0^\infty |f(x)|\, dm = \infty$. This is a contradiction because $\infty > \int_{-\infty}^\infty |f(x)|\, dm \geq \int_0^\infty |f(x)|\, dm$. Therefore, $\lim_{b\to\infty} g(b)$ exists as a finite number. This implies the improper integral $\int_0^\infty |f(x)|\, dx$ exists as a finite number. Similarly, we can show $\int_{-\infty}^0 |f(x)|\, dx$ exists as a finite number. This shows $\int_{-\infty}^\infty |f(x)|\, dx$ exists as a finite number.

Extra Problem 8. Let $\alpha > -1$. Define $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha+1}\, dt$. Prove Lebesgue integral

$$\int_0^\infty \frac{e^{-x}}{1-e^{-x}} x^{\alpha+1}\, dm = \Gamma(\alpha) \sum_{n=1}^\infty \frac{1}{n^{\alpha+2}}$$

Is the improper integral $\int_0^\infty \frac{e^{-x}}{1-e^{-x}}\, x^{\alpha+1}\, dx$ convergent absolutely?

Note that by Taylor expansion and I.T.T. for nonegative measurable function,

$$\int_0^\infty \frac{e^{-x}}{1-e^{-x}} x^{\alpha+1}\, dm = \int_0^\infty \sum_{n=1}^\infty e^{-nx} x^{\alpha+1}\, dm = \sum_{n=1}^\infty \int_0^\infty e^{-nx} x^{\alpha+1}\, dm$$

Note that for all $\alpha > -1$, there exists a $K_\alpha > 0$ s.t. for all $x \geq K_\alpha$, $e^{x/2} > x^{\alpha+1}$. Therefore,

$$\begin{aligned}
\int_0^\infty e^{-nx} x^{\alpha+1}\, dm &= \int_0^{K_\alpha} e^{-nx} x^{\alpha+1}\, dm + \int_{K_\alpha}^\infty e^{-nx} x^{\alpha+1}\, dm \\
&\leq K_\alpha^{\alpha+1} \int_0^{K_\alpha} e^{-nx}\, dm + \int_{K_\alpha}^\infty e^{(-n+1/2)x}\, dm \\
&\leq K_\alpha^{\alpha+2} + \int_0^\infty e^{-x/2}\, dm
\end{aligned}$$

It is easy to see the Cauchy-Riemann integral $\int_0^\infty e^{-x/2}\, dx$ is absolutely convergent, so by previous result, the Lebesgue integral $\int_0^\infty e^{-x/2}\, dm$ is equal to Cauchy-Riemann integral and hence a finite number. Thus, $\int_0^\infty e^{-nx} x^{\alpha+1}\, dm$ is finite and we can apply change of variable $t = nx$ to Lebesgue integral because of HW8, Extra Problem 7, i.e.,

$$\int_0^\infty e^{-nx} x^{\alpha+1}\, dm = \frac{1}{n^{\alpha+2}} \int_0^\infty e^{-t} t^{\alpha+1}\, dm = \frac{1}{n^{\alpha+2}} \Gamma(\alpha)$$

where the last equality is because $\int_0^\infty e^{-t} t^{\alpha+1}\, dm$ is also finite, and thus it is equal to its Cauchy-Riemann integral $\int_0^\infty e^{-t} t^{\alpha+1}\, dt$. When $\alpha > -1$, $\alpha + 2 > 1$, so the series $\sum_{n=1}^\infty \frac{1}{n^{\alpha+2}}$ converges. Since $\Gamma(\alpha)$ is finite (by what we proved above with $n = 1$), $\int_0^\infty \frac{e^{-x}}{1-e^{-x}} x^{\alpha+1}\, dm$ is finite, again by Extra Problem 7, its corresponding Cauchy Riemann integral converges absolutely.

Extra Problem 9. Let $f(x,y) \in L^1(E_1 \times E_2)$, where $x \in E_1 \subset \mathbb{R}^{n_1}$, $E_1 \in \mathcal{M}$ and $y \in E_2 \subset \mathbb{R}^{n_2}$, $E_2 \in \mathcal{M}$. Prove that $\int_{E_2} f(x,y)\, dy \in L^1(E_1)$ and $\int_{E_1} f(x,y)\, dx \in L^1(E_2)$.

Note that $g(x,y) = |f(x,y)| I_{E_1}(x) I_{E_2}(y)$ is nonnegative and measurable on $\mathbb{R}^{n_1+n_2}$. Therefore, by Fubini's theorem (nonnegative version), we have

$$\int_{\mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_1}} g(x,y)\, dx\, dy = \int_{\mathbb{R}^{n_1+n_2}} g(x,y)\, d(x,y)$$

which is equivalent to (since $f \in L^1(E_1 \times E_2)$),

$$\int_{E_2} \int_{E_1} |f(x,y)|\, dx\, dy = \int_{E_1 \times E_2} |f(x,y)|\, d(x,y) < \infty$$

This impiles that $\int_{E_1} |f(x,y)|\, dx \in L^1(E_2)$. Note that

$$\left| \int_{E_1} f(x,y)\, dx \right| \leq \int_{E_1} |f(x,y)|\, dx < \infty$$

Thus, $\int_{E_1} f(x,y)\, dx \in L^1(E_2)$. Similarly, by the other part of Fubini, we also have

$$\int_{E_1} \int_{E_2} |f(x,y)|\, dy\, dx = \int_{E_1 \times E_2} |f(x,y)|\, d(x,y) < \infty$$

so $\int_{E_2} |f(x,y)|\, dy \in L^1(E_1)$. For the same reason, $\int_{E_2} f(x,y)\, dy \in L^1(E_1)$.

Extra Problem 10. Let $f(x)$ be nonnegative on $E \in \mathcal{M}$, $E \subset \mathbb{R}^n$. Let $A = \{(x,y) \in E \times \mathbb{R} \,|\, 0 \leq y \leq f(x)\}$. Prove that f is measurable on E iff $A \subset \mathbb{R}^{n+1}$ is measurable. Also prove if $f(x)$ is measurable on E, then $\int_E f(x)\, dx = m(A)$.

For "only if" part, let $F(x,y) = f(x)$, then $\{(x,y) \,|\, F(x,y) > t\} = \{x \,|\, f(x) > t\} \times \mathbb{R} \in \mathcal{M}(\mathbb{R}^{n+1})$ because $\{x \,|\, f(x) > t\} \in \mathcal{M}(\mathbb{R}^n)$. Let $G(x,y) = y - f(x)$, then $G(x,y)$ is measurable in $\mathbb{R}^{n+1}$ and $A = \{(x,y) \in E \times \mathbb{R} \,|\, G(x,y) \leq 0\} \cap \{(x,y) \in E \times \mathbb{R} \,|\, y \geq 0\} \in \mathcal{M}(\mathbb{R}^{n+1})$ because the first part is in $\mathcal{M}(\mathbb{R}^{n+1})$ and the second part is closed in $\mathbb{R}^{n+1}$.

For "if" part, since $A \in \mathcal{M}(\mathbb{R}^{n+1})$, by Lemma 2 in lecture, $I_A(x,y)$ is measurable in $\mathbb{R}^{n+1}$. By Fubini's theorem (nonnegative version) $\int_{\mathbb{R}} I_A(x,y)\, dy$ is measurable in $x \in \mathbb{R}^n$. Notice that $f(x) = \int_{\mathbb{R}} I_A(x,y)\, dy$ for each fixed $x \in E$, so we are done.

The last claim is easy to see, since

$$m(A) = \int_{\mathbb{R}^{n+1}} I_A(x,y)\, d(x,y) = \int_{\mathbb{R}^n} \int_{\mathbb{R}} I_A(x,y)\, dy\, dx = \int_E f(x)\, dx$$

Extra Problem 11. Suppose $f(x)$ is measurable on $E \subset \mathbb{R}^n$, $E \in \mathcal{M}$. Fora all $\lambda \geq 0$, let $F(\lambda) = m(\{x \in E \,|\, |f(x)| > \lambda\})$. Prove that if $|f|^p \in L^1(E)$ where $p \geq 1$, then $\int_E |f(x)|^p\, dx = p \int_0^\infty \lambda^{p-1} F(\lambda)\, d\lambda$.

Denote $A = \{(x,\lambda) \in E \times \mathbb{R} \,|\, 0 \leq \lambda < |f(x)|\}$. Then since f is measurable on E, by Extra Problem 10, A is measurable in $\mathbb{R}^{n+1}$. Note that $I_A(x,\lambda) = I_A(x,\lambda) I_E(x) I_{\mathbb{R}_+}(\lambda)$ and by Lemma 2 in lecture it is a measurable function on $\mathbb{R}^{n+1}$. This implies $p\lambda^{p-1}I_A(x,\lambda)I_E(x)I_{\mathbb{R}_+}(\lambda)$ is a nonnegative measurable function on $\mathbb{R}^{n+1}$. By Fubini's theorem (nonnegative),

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} p\lambda^{p-1}I_A(x,\lambda)I_E(x)I_{\mathbb{R}_+}(\lambda)\ d(x\lambda) &= \int_{\mathbb{R}}\int_{\mathbb{R}^n} p\lambda^{p-1}I_A(x,\lambda)I_E(x)I_{\mathbb{R}_+}(\lambda)\ dx\ d\lambda \\ &= p\int_0^\infty \lambda^{p-1}\int_E I_A(x,\lambda)\ dx\ d\lambda = p\int_0^\infty \lambda^{p-1}F(\lambda)\ d\lambda\end{aligned}$$

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} p\lambda^{p-1}I_A(x,\lambda)I_E(x)I_{\mathbb{R}_+}(\lambda)\ d(x\lambda) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}} p\lambda^{p-1}I_A(x,\lambda)I_E(x)I_{\mathbb{R}_+}(\lambda)\ d\lambda\ dx \\ &= \int_E\int_0^\infty p\lambda^{p-1}I_A(x,\lambda)\ d\lambda\ dx \\ &= \int_E\int_0^{|f(x)|^p} 1\ dy\ dx = \int_E |f(x)|^p\ dx \qquad \text{(Take } y=\lambda^p\text{)}\end{aligned}$$

Therefore, $\int_E |f(x)|^p\ dx = p\int_0^\infty \lambda^{p-1}F(\lambda)\ d\lambda$.