Real Analysis

Lebesgue Intégration

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Rien ne nous rend si grand qu'une grande douleur.

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Chapter 1 Lesbegue Measurable Sets

1.1 Rectangles

Definition 1.1. Closed & Open Rectangles

*A **closed rectangle** R in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ with the form $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$.*

*An **open rectangle** R in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ with the form $R = (a_1, b_1) \times \cdots \times (a_n, b_n)$.*

Here a_k, b_k are real numbers for $k = 1, \ldots, n$. ♣

Remark Give real numbers a_k, b_k for $k = 1, \ldots, n$, we can define more general rectangles in a similar way, i.e., a rectangle R in $\mathbb{R}^n$ has the form of $I_{a_1}^{b_1} \times I_{a_2}^{b_2} \times \cdots \times I_{a_n}^{b_n}$, where I_x^y is any kinds of bounded intervals in $\mathbb{R}$ (open, closed, or half-open half-closed) with two end points $x \leq y$.

Note *In fact we can define even more general rectangles (e.g. $[0,1] \times [0,1]$ rotated by $30°$), but it is meaningless for our study. Therefore, unless specified, the most general case we need to consider whenever we talk about rectangles is the one defined in the above remark.*

Definition 1.2. Volume of Rectangles

*The **volume** of any rectangles $R = I_{a_1}^{b_1} \times \cdots \times I_{a_n}^{b_n}$ in $\mathbb{R}^n$ is $|R| = \prod_{i=1}^{n}(b_i - a_i)$.* ♣

Definition 1.3. Almost Disjoint Union of Rectangles

*A rectangle is the **almost disjoint** union of a collection of rectangles if the interior of the rectangles in this collection are pairwise disjoint. We can also say the rectangles in this collection are **almost disjoint**.* ♣

Exercise 1.1 If a rectangle R is the almost disjoint union of finitely many rectangles $R_1, \ldots, R_M$, prove that $|R| = \sum_{m=1}^{M} |R_m|$.

Proof Let $R = I_1 \times \cdots \times I_J$, where I_j's are intervals (1-dim rectangles) for $j = 1, \ldots, J$.

- Special case: If R_m's form a grid (each cell of the grid is a rectangle R_m) of R, i.e., for each j, there exist almost disjoint intervals $I_{j,1}, \ldots, I_{j,N_j}$ s.t. $I_j = \bigcup_{n=1}^{N_j} I_{j,n}$ and for each R_m, we can find $1 \leq n_j^m \leq N_j$ for $j = 1, \ldots, J$ s.t. $R_m = I_{1,n_1^m} \times \cdots \times I_{J,n_J^m}$. By Definition 1.2, $|R_m| = \prod_{j=1}^{J} |I_{j,n_j^m}|$. This implies $\sum_{m=1}^{M} |R_m| = \sum_{m=1}^{M} \prod_{j=1}^{J} |I_{j,n_j^m}|$. Since all R_m's form a grid of R, $M = \prod_{j=1}^{J} N_j$ and summation over n_j^m is equivalent to the following form

$$\sum_{m=1}^{M} \prod_{j=1}^{J} |I_{j,n_j^m}| = \sum_{n_J=1}^{N_J} \cdots \sum_{n_2=1}^{N_2} \sum_{n_1=1}^{N_1} \prod_{j=1}^{J} |I_{j,n_j}| \tag{1.1}$$

For almost disjoint intervals, $|I_j| = \sum_{n=1}^{N_j} |I_{j,n}|$ for each fixed j. This implies that

$$\sum_{n_J=1}^{N_J} \cdots \sum_{n_2=1}^{N_2} \sum_{n_1=1}^{N_1} \prod_{j=1}^{J} |I_{j,n_j}| = \sum_{n_J=1}^{N_J} \cdots \sum_{n_2=1}^{N_2} \prod_{j=2}^{J} |I_{j,n_j}| \sum_{n_1=1}^{N_1} |I_{1,n_1}| \tag{1.2}$$

$$= |I_1| \sum_{n_J=1}^{N_J} \cdots \sum_{n_2=1}^{N_2} \prod_{j=2}^{J} |I_{j,n_j}| \tag{1.3}$$

Inductively, we can finally obtain $\sum_{n_J=1}^{N_J} \cdots \sum_{n_2=1}^{N_2} \sum_{n_1=1}^{N_1} \prod_{j=1}^{J} |I_{j,n_j}| = \prod_{j=1}^{J} |I_j|$. Therefore, $\sum_{m=1}^{M} |R_m| = \prod_{j=1}^{J} |I_j| = |R|$.

- General case: In general, R_m's themselves may not be able to form a grid of R, but we can partition each R_m into smaller rectangles so that the finer partition forms a grid of R. This can be done by simply extending each side of each R_m until they intersect the edge of R. Then each of the orginal small rectangles R_m is the almost disjoint union of some (may be just one) smaller rectangle(s), denoted as $R_m = \bigcup_{k=1}^{i_m} R_m^k$ for $m = 1, \ldots, M$. Apply special case on R, $|R| = \sum_{m=1}^{M} \sum_{k=1}^{i_m} |R_m^k|$. Also notice that for each fixed m, R_m^k's form a grid of R_m (**why?**). By applying special case to each R_m, we have $|R_m| = \sum_{k=1}^{i_m} |R_m^k|$. This implies the desired result $|R| = \sum_{m=1}^{M} |R_m|$.

$\square$

Exercise 1.2 Let $R, R_1, \ldots, R_m$ be rectangles s.t. $R \subset \bigcup_{k=1}^{m} R_k$, then $|R| \leq \sum_{k=1}^{m} |R_k|$.

Proof Take a large rectangle R' that contains $R, R_1, \ldots, R_m$. Extend all sides of $R, R_1, \ldots, R_m$ until they intersect the edge of R' to obtain smaller rectangles $\tilde{R}_1, \ldots, \tilde{R}_n$. In this way, R' is the almost disjoint union of $\tilde{R}_1, \ldots, \tilde{R}_n$; each of $R, R_1, \ldots, R_m$ is also the almost disjoint union of some of $\tilde{R}_1, \ldots, \tilde{R}_n$. By Exercise 1.1, $|R| = \sum_{\tilde{R}_j \subset R} |\tilde{R}_j|$. Note that each $\tilde{R}_j \subset R$ must be contained in one of $R_1, \ldots, R_m$. Therefore, $\sum_{\tilde{R}_j \subset R} |\tilde{R}_j| \leq \sum_{k=1}^{m} \sum_{\tilde{R} \subset R_k} |\tilde{R}| = \sum_{k=1}^{m} |R_k|$.

$\square$

Note *Actually for each of $R', R, R_1, \ldots, R_m$, we can find a subset of $\tilde{R}_1, \ldots, \tilde{R}_n$ to form a grid of it.*

Corollary 1.1. of Exercise 1.2

Let R be a rectangle, $\{R_k\}_{k=1}^{\infty}$ be almost disjoint rectangles, and $R \supset \bigcup_{k=1}^{\infty} R_k$. Then $|R| \geq \sum_{k=1}^{\infty} |R_k|$.

Proof For each fixed n, $R \supset \bigcup_{k=1}^{n} R_k$, so similar to the proof of Exercise 1.2, extend all sides of R_k for $k = 1, \ldots n$ until they intersect the edge of R to obtain almost disjoint rectangles $\tilde{R}_1^n, \ldots, \tilde{R}_{M_n}^n$ s.t. $R = \bigcup_{l=1}^{M_n} \tilde{R}_l^n$. By Exercise 1.1, $|R| = \sum_{l=1}^{M_n} |\tilde{R}_l^n|$. Note that each R_k is the almost disjoint union of some $\tilde{R}_l^n$, so we have $\sum_{k=1}^{n} |R_k| \leq \sum_{k=1}^{n} \sum_{\tilde{R}_l^n \subset R_k} |\tilde{R}_l^n| \leq \sum_{l=1}^{M_n} |\tilde{R}_l^n|$. This implies $|R| \geq \sum_{k=1}^{n} |R_k|$ for all $n \in \mathbb{N}^+$. Take $n \to \infty$, we have $|R| \geq \sum_{k=1}^{\infty} |R_k|$. $\square$

Problem 1.1 Prove that every open set O of $\mathbb{R}$ is the countable union of disjoint open intervals.

Exercise 1.3 Any open set G in $\mathbb{R}^n$ can be decomposed into almost disjoint countable union of closed cubes (closed rectangles with equal-length edges).

Proof Divide $\mathbb{R}^n$ into cubes $[k_1, k_1+1] \times \cdots \times [k_n, k_n+1]$ ($k_1, \ldots, k_n$ are integers). Denote P_1 to be the collection of all these cubes. Now divide each of the cubes in P_1 into 2^n closed subcubes s.t. all subcubes are almost disjoint, and denote the collection of all such subcubes as P_2. Keep doing such kind of subdivision, and we will obtain P_k for all $k \in \mathbb{N}^+$. Note that all cubes in P_k are almost disjoint, any cubes in P_k is the union of 2^n cubes in P_{k+1}, and P_k is countable. Let H_1 be the set of all cubes in P_1 and contained in G; H_k to be the set of all cubes in P_k but not in any cubes in $H_1, \ldots, H_{k-1}$ and contained in G for any $k \geq 2$.

Claim: $G = \bigcup_{k=1}^{\infty} \bigcup_{c \in H_k} c$ where c represents cube. Since each $c \in G$, it is easy to see $\bigcup_{k=1}^{\infty} \bigcup_{c \in H_k} c \subset G$. Fix arbitrary $x \in G$, denote $x = (x_1, \ldots, x_n)$. Then for each fixed $k \geq 1$, there exists integer $a_{l,k}$ for $l = 1, \ldots, n$ s.t. $\frac{a_{l,k}}{2^k} \leq x_l \leq \frac{a_{l,k}+1}{2^k}$. Let

$$c_k = \left[\frac{a_{1,k}}{2^k}, \frac{a_{1,k}+1}{2^k}\right] \times \cdots \times \left[\frac{a_{n,k}}{2^k}, \frac{a_{n,k}+1}{2^k}\right]$$

Then $x \in c_k \in P_k$ for all $k \geq 1$. Since x is an interior point of G, there exists large enough K s.t. $c_K \subset G$. If c_K is not in $H_1, \ldots, H_{K-1}$, then since it is in P_K and contained in G, it must be in H_K, This shows $c_K \in \bigcup_{k=1}^{K} \bigcup_{c \in H_k} c$, so $x \in \bigcup_{k=1}^{\infty} \bigcup_{c \in H_k} c$ and the claim is proved. Note that each H_k is countable because $H_k \subset P_k$ and P_k is countable, so $\bigcup_{k=1}^{K} \bigcup_{c \in H_k} c$ is a countable union. Also notice that cubes in different H_k's are almost disjoint, and since all cubes in P_k are almost disjoint, cubes in H_k are also almost disjoint. □

1.2 Cantor Set

Definition 1.4. Cantor Set

*Let $F_0 = [0, 1]$. Divide F_0 into 3 equal-length subintervals and remove the center interval $(\frac{1}{3}, \frac{2}{3})$. Let $F_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ be the remaining set. Divide each interval in F_1 into 3 equal-length subintervals and remove the center intervals $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. Let $F_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ be the remaining set. Repeat the removing process to obtain $F_3, \ldots, F_n, \ldots$, and the **Cantor set** is defined to be $C = \bigcap_{n=1}^{\infty} F_n$.* ♣

Property *The Cantor set C*

1. *is a closed set*
2. *contains all end points of the subintervals*
3. *is nowhere dense in $\mathbb{R}$*
4. *is a perfect set*

Proof

1. Since F_k is finite union of closed sets, F_k is closed for all $k \geq 1$. Since the intersection of closed sets is always closed, C is closed.
2. Trivial.

3. For every $x \in C$, we want to show that for all $\delta > 0$, $(x-\delta, x+\delta) \not\subset C$. Since $x \in C$, $x \in F_n$ for all n, and thus x is in one of the closed subinterval(s) I_n of length $\frac{1}{3^n}$. Take n large s.t. $I_n \subset (x-\delta, x+\delta)$. When we construct F_{n+1}, center part of I_n needs to be removed from I_n, so $(x-\delta, x+\delta) \not\subset F_{n+1}$ and this shows $(x-\delta, x+\delta) \not\subset C$.
4. Denote C' as the set of all limit points of C. Since C is closed, $C' \subset C$, so we only need to prove $C \subset C'$. For each $x \in C$, $x \in F_n$ for all n, so x is in some closed subinterval I_n of length $\frac{1}{3^n}$. Let x_n be an end point of I_n, then as $n \to \infty$, $x_n \to x$. Since $x_n \in C$ for all n, x is a limit point of C and $x \in C'$. Since x is arbitrary in C, $C \subset C'$.

□

Now we want to prove a very famous proposition about Cantor set. We will first state this proposition, then prove two useful facts in the exercises following with it, and at last, we will provide a proof of the proposition.

Proposition 1.1. Cardinality of Cantor Set

The Cantor set C is equivalent to $[0,1]$ in cardinality.

Exercise 1.4 Let $D = \{\sum_{k=1}^{\infty} \frac{a_k}{3^k} \mid a_k \in \{0,2\}, \forall\, k \in \mathbb{N}^+\}$. Prove that $C = D$.

Proof Recall $C = \bigcap_{k=1}^{\infty} F_k$ where F_k is defined in Definition 1.4. First we use induction to prove for all $k \geq 1$, if a is the left end point of one subinterval constituting F_k, then a can be written as $\sum_{n=1}^{\infty} a_n 3^{-n}$, where $a_n \in \{0,2\}$ for $1 \leq n \leq k$ and $a_n = 0$ for $n > k$. If $k = 1$, then $a = 0$ or $a = \frac{2}{3}$. If $a = 0$, then just let $a_n = 0$ for all $n \geq 1$; if $a = \frac{2}{3}$, let $a_1 = 2$ and $a_n = 0$ for $n \geq 2$, so our claim is true for $k = 1$. Now we assume our claim is true for some k and we want to prove it is also true for $k+1$. Suppose a is the left end point of one subinterval ($[a,b]$) constituting F_{k+1}, if it is also the left end point of one subinterval constituting F_k, then by induction hypothesis we have already proved our claim for $k+1$. If a is not the left end point of one subinterval in F_k, then there exists $[c,b]$ in F_k s.t. $[a,b] \subset [c,b]$. By construction $a = c + 2/3^{k+1}$, and combined with induction hypothesis on c, $a = \sum_{n=1}^{k} a_n 3^{-n} + 2/3^{k+1}$. This shows $a = \sum_{n=1}^{\infty} a_n 3^{-n}$ s.t. $a_n \in \{0,2\}$ for $1 \leq n \leq k+1$ and $a_n = 0$ for $n > k+1$, and this finishes our induction.

Note that for each fixed $k \geq 1$, the number of left end point of subinterval constituting F_k is exactly 2^k. However, the number of cases that $a_n \in \{0,2\}$ for $1 \leq n \leq k$ and $a_n = 0$ for $n > k$ is also 2^k. This shows if $a = \sum_{n=1}^{\infty} a_n 3^{-n}$ where $a_n \in \{0,2\}$ for $1 \leq n \leq k$ and $a_n = 0$ for $n > k$ must be a left end point of one subinterval constituting F_k. Also, by construction, each subinterval in F_k is of length $1/3^k$, so if a is the left end point of some subinterval, then $b = a + 1/3^k$ is the right end point of that subinterval. Since $1/3^k$ can be written as $\sum_{n=k+1}^{\infty} 2/3^n$, $b = \sum_{n=1}^{\infty} b_n 3^{-n}$ s.t. $b_n = a_n$ for $1 \leq n \leq k$ and $b_n = 2$ for $n > k$ if $a = \sum_{n=1}^{\infty} a_n 3^{-n}$. This implies that if $x = \sum_{n=1}^{\infty} x_n 3^{-n}$ and $y = \sum_{n=1}^{\infty} y_n 3^{-n}$ are in the same subinterval of F_k, $x_n = y_n$ for $1 \leq n \leq k$.

Now if $x \in D$, then $x = \sum_{n=1}^{\infty} a_n 3^{-n}$, and it must lie in $[s_k, t_k]$ where $s_k = \sum_{n=1}^{k} a_n 3^{-n}$ and $t_k = \sum_{n=1}^{k} a_n 3^{-n} + \sum_{n=k+1}^{\infty} 2/3^n$. Since we have shown that such s_k must be a left end point of some subinterval in F_k, $x \in F_k$. Here k is arbitrary, so $x \in C$ and $D \subset C$. Conversely, pick $x \in C$, then for each k, there is an subinterval $[x_k, y_k]$ in F_k containing x, so $x_k \to x$. Also note that $[x_{k+1}, y_{k+1}] \subset [x_k, y_k]$ for all k. Since x_k is the left end point, it has the form of $\sum_{n=1}^{k} a_n 3^{-n}$ where $a_n \in \{0, 2\}$, thus we have $x = \lim_{k\to\infty} \sum_{n=1}^{k} a_n 3^{-n} = \sum_{n=1}^{\infty} a_n 3^{-n}$ where $a_n \in \{0, 2\}$. This shows $x \in D$ and $C \subset D$, so $C = D$. □

Exercise 1.5 Prove that for all $x \in [0, 1]$, there exists $a_n \in \{0, 1\}$ for all $n \geq 1$ s.t. $x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$.

Proof For $x = 0$, we let $a_n = 0$ for all $n \geq 1$; for $x = 1$, we let $a_n = 1$ for all $n \geq 1$, then it is easy to see $x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$. Now let $E_1 = (0, 1)$, and divide $(0, 1)$ into two subintervals with equal length $1/2$. If $x \in (0, \frac{1}{2})$, set $a_1 = 0$ and a_n for $n \geq 2$ is to be determined; if $x = \frac{1}{2}$, set $a_1 = 1$ and set $a_n = 0$ for all $n \geq 2$; if $x \in (\frac{1}{2}, 1)$, set $a_1 = 1$ and a_n for $n \geq 2$ is to be determined. It is easy to see $\frac{a_1}{2} \leq x \leq \frac{a_1+1}{2}$ for all $x \in E_1$. Denote $E_2 = (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, and keep on doing the same procedure, i.e., for each subinterval in $E_k = (0, 1) \setminus \{\frac{m}{2^k}\}_{m=1}^{2^k-1}$, we divide it into two subintervals with equal length $1/2^k$ and if x is in the left subinterval (open), we let $a_k = 0$ and a_n for $n \geq k + 1$ to be determined; if x is the middle point, we let $a_k = 1$ and $a_n = 0$ for $n \geq k + 1$; if x is in the right subinterval (open), then let $a_k = 1$ and a_n for $n \geq k + 1$ to be determined. By this procedure, if $x = \frac{m}{2^k}$ for some k and $1 \leq m \leq 2^k - 1$, then $x = \sum_{n=1}^{k} \frac{a_n}{2^n}$; if x is not of such form, then after k steps, we can determine the value of $a_1, \ldots, a_k$ but a_n for $n \geq k + 1$ cannot be determined. The most important observation is that $\sum_{n=1}^{k} \frac{a_n}{2^n} \leq x \leq \sum_{n=1}^{k} \frac{a_n}{2^n} + \frac{1}{2^k}$. Since the LHS and RHS converges to the same value as $k \to \infty$, they both converge to x, and thus $x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$ where $a_n \in \{0, 1\}$. □

Note *Notice that different from Exercise 1.4, the a_n we find in expression $x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$ may not be unique. Think of number in the form of $\frac{m}{2^k}$ for some k and $m = 1, \ldots, 2^k - 1$.*

After all of the above tedious preparations, we are finally ready to prove Proposition 1.1 by using the above two exercises.

Proof [Proposition 1.1] Since it is trivial that Cantor set C is a subset of $[0, 1]$, the cardinality of C is less than or equal to $[0, 1]$, so we if we can construct a surjective map from C to $[0, 1]$, it is enough to show the cardinality of C is larger than or equal to $[0, 1]$, and thus we proved that they have the same cardinality. By Exercise 1.4, it is equivalent to construct a surjective map from D (defined in Exercise 1.4) to $[0, 1]$. Consider the mapping $f : D \mapsto [0, 1]$,

$$f\left(\sum_{n=1}^{\infty} \frac{a_n}{3^n}\right) = \sum_{n=1}^{\infty} \frac{a_n/2}{2^n}, \quad a_n \in \{0, 2\} \tag{1.4}$$

It is surjective because by Exercise 1.5, each number in $[0, 1]$ can be expressed as $\sum_{n=1}^{\infty} \frac{b_n}{2^n}$ where $b_n \in \{0, 1\}$, and by letting $a_n = 2b_n \in \{0, 2\}$ for all $n \geq 1$, we can find the preimage

$\sum_{n=1}^{\infty} \frac{a_n}{3^n}$, which is a number in C. The only thing we need to do is to prove f is well-defined. Suppose $\sum_{n=1}^{\infty} \frac{a_n}{3^n} = \sum_{n=1}^{\infty} \frac{c_n}{3^n}$, where $a_n, c_n \in \{0, 2\}$ for all $n \geq 1$. If $a_1 \neq c_1$, WLOG, let $a_1 = 0$ and $c_1 = 2$, then

$$\frac{1}{3} = \sum_{n=2}^{\infty} \frac{2}{3^n} \geq \sum_{n=1}^{\infty} \frac{a_n}{3^n} = \sum_{n=1}^{\infty} \frac{c_n}{3^n} \geq \frac{2}{3} \tag{1.5}$$

which is obviously impossible, so $a_1 = c_1$. Inductively, we can show $a_n = c_n$ for all $n \geq 1$, so f is well-defined. □

Problem Set 1.2

1. Let p be a natural number greater than 1, and x a real umber, $0 < x < 1$. Show that there is a sequence $\{a_n\}$ of integers with $0 \leq a_n < p$ for each n such that $x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$ and that this sequence is unique except when x is of the form q/p^n, in which case there are exactly two such sequences. Show that, conversely, if $\{a_n\}$ is any sequence of integers with $0 \leq a_n < p$, the series $\sum_{n=1}^{\infty} \frac{a_n}{p^n}$ converges to a real number x with $0 \leq x \leq 1$.
2. Let A and B be sets. Suppose there exists injective mappings $f : A \mapsto B$ and $g : B \mapsto A$. Prove that $A \sim B$.
3. Let G_k $(k \in \mathbb{N}^+)$ be open and dense in $\mathbb{R}$. Prove that $\bigcap_{k=1}^{\infty} G_k$ is uncountable.
4. Prove that $\frac{1}{4}$ is in Cantor set C.
5. Let $3 \leq p < \infty$. The Cantor-like set is constructed as follows: On the interval $[0, 1]$, first pick the middle point $1/2$ and remove the $1/p$ neighborhood of it. Denote the remaining part of $[0, 1]$ by F_1. Now in the second stage, from each subterval in F_1, remove the $1/p^2$ neighborhood of its middle point. Denote the remaining part as F_2. Repeat this process we get F_n, which consists of 2^n closed subintervals of equal length. Define $C_p = \bigcap_{n=1}^{\infty} F_n$. Prove that
   (a). C_p is nowhere dense;
   (b). C_p is a perfect set;
   (c). the total length of all open inverals removed is equal to $\frac{1}{p-2}$.
6. Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of sets. Define

$$\overline{\lim_{n\to\infty}}\, E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n, \qquad \underline{\lim_{n\to\infty}}\, E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$$

   (a). Prove $\overline{\lim}_{n\to\infty} E_n$ is equal to the set of points who belong to infinitely many E_n's, and

$$\underline{\lim_{n\to\infty}}\, E_n = \{x \,|\, \exists \text{ integer } n_x \geq 1, \text{ s.t. } x \in E_n \text{ whenever } n \geq n_x\}$$

   (b). Suppose $E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots$, find $\underline{\lim}_{n\to\infty} E_n$ and $\overline{\lim}_{n\to\infty} E_n$.
   (c). Suppose $E_n \cap E_m = \varnothing$, if $n \neq m$. Find $\underline{\lim}_{n\to\infty} E_n$ and $\overline{\lim}_{n\to\infty} E_n$.
   (d). Let all $E_n \subset \mathbb{R}^N$. Prove that

$$\left(\overline{\lim_{n\to\infty}}\, E_n\right)^c = \underline{\lim_{n\to\infty}}\, (E_n)^c, \qquad \left(\underline{\lim_{n\to\infty}}\, E_n\right)^c = \overline{\lim_{n\to\infty}}\, (E_n)^c$$

(e). Let $f(x)$, $\{f_n(x)\}_{n=1}^{\infty}$ be defined on a set $E \subset \mathbb{R}^N$. Prove that

$$Z \triangleq \{x \in E \mid f_n(x) \not\to f(x) \text{ as } n \to \infty\} = \bigcup_{l=1}^{\infty} \left(\varlimsup_{k\to\infty} E_l^k \right)$$

where $E_l^k = \left\{x \in E \mid |f_k(x) - f(x)| \geq \frac{1}{l}\right\}$.

7. Let E be a bounded closed subset of $\mathbb{R}^n$. Suppose $\{f_k\}_{k=1}^{\infty}$ are continuous on E and $f_k \to f$ uniformly for some f as $k \to \infty$. Prove that

$$f(E) = \bigcap_{j=1}^{\infty} \left(\overline{\bigcup_{k=j}^{\infty} f_k(E)} \right)$$

1.3 Outer Measure

Definition 1.5. Lebesgue Covering

*Let $E \subset \mathbb{R}^n$, a sequence of open rectangles $\{R_k\}_{k=1}^{\infty}$ is called a **Lebesgue covering** (L-covering) of E if $E \subset \bigcup_{k=1}^{\infty} R_k$.* ♣

Definition 1.6. Outer Measure

*For all $E \subset \mathbb{R}^n$, define **outer measure** of E by*

$$m^*(E) = \inf \left\{ \sum_{k=1}^{\infty} |R_k| \;\middle|\; \{R_k\}_{k=1}^{\infty} \textit{ is a Lebesgue covering of } E \right\}$$

♣

Example 1.1 Let $x_0 \in \mathbb{R}^n$, $E = \{x_0\}$, then one can check by definition that $m^*(E) = 0$.

Next we will see two seemingly intuitive remarks, while they are not easy to prove and will be very handy in the future study.

Remark If we require R_k's to be closed rectangles in the definition of L-covering, then $m^*(E)$ defined in Definition 1.6 does not change.

Proof For simplicity, we denote $m_o^*(E)$ to be the outer measure defined in Definition 1.6, and $m_c^*(E)$ to be outer measure newly defined in this Remark. For any open L-covering $\{R_k\}_{k=1}^{\infty}$ of E, there exists closed L-covering $\{\bar{R}_k\}_{k=1}^{\infty}$ of E and $\sum_{k=1}^{\infty} |R_k| = \sum_{k=1}^{\infty} |\bar{R}_k|$, then

$$\left\{ \sum_{k=1}^{\infty} |R_k| \;\middle|\; \{R_k\}_{k=1}^{\infty} \text{ open } L\text{-covering of } E \right\} \subset \left\{ \sum_{k=1}^{\infty} |R_k| \;\middle|\; \{R_k\}_{k=1}^{\infty} \text{ closed } L\text{-covering of } E \right\}$$

so by property of infimum, $m_o^*(E) \geq m_c^*(E)$.

Also, for all $\epsilon > 0$, there exists closed L-covering of E, $\{F_k\}_{k=1}^{\infty}$ s.t. $m_c^*(E) + \epsilon \geq \sum_{k=1}^{\infty} |F_k|$. Expand each side of each F_k by a factor $1 + \epsilon$ to obtain a larger rectangle $\tilde{R}_k$ s.t. the interior of $\tilde{R}_k$, $\tilde{R}_k^\circ$, contains F_k. Furthermore, $|\tilde{R}_k^\circ| = (1+\epsilon)^n |F_k|$, thus,

$$m_c^*(E) + \epsilon \geq \sum_{k=1}^{\infty} \frac{|\tilde{R}_k^\circ|}{(1+\epsilon)^n} \geq \frac{m_o^*(E)}{(1+\epsilon)^n}$$

Take $\epsilon \to 0$, we have $m_c^*(E) \geq m_o^*(E)$, and so $m_o^*(E) = m_c^*(E)$. □

Remark If we require R_k's to be closed cubes in the definition of L-covering, then $m^*(E)$ defined in Definition 1.6 does not change.

Proof For simplicity, we denote $m_{re}^*(E)$ to be the outer measure defined in Definition 1.6, and $m_{cu}^*(E)$ to be outer measure newly defined in this Remark. Since cube is a special type of rectangle, it is obvious that $m_{cu}^*(E) \geq m_{re}^*(E)$.

If $m_{re}^*(E) = \infty$, then $m_{cu}^*(E) \geq \infty$, so $m_{cu}^*(E) = m_{re}^*(E) = \infty$. Suppose $m_{re}^*(E) < \infty$, for all $\epsilon > 0$, there exists open rectangular covering $\{R_k\}_{k=1}^\infty$ of E s.t. $m_{re}^*(E) + \epsilon > \sum_{k=1}^\infty |R_k|$. Since R_k is an open set, by Exercise 1.3, $R_k = \bigcup_{i=1}^\infty c_{k,i}$ where $c_{k,i}$'s are almost disjoint closed cubes. $R_k = \bigcup_{i=1}^\infty c_{k,i}$ implies $R_k \supset \bigcup_{i=1}^\infty c_{k,i}$, so by Corollary 1.1, we obtain $|R_k| \geq \sum_{i=1}^\infty |c_{k,i}|$. This implies $m_{re}^*(E) + \epsilon > \sum_{k=1}^\infty \sum_{i=1}^\infty |c_{k,i}| \geq m_{cu}^*(E)$, where the last inequality is because $\{c_{k,i}\}_{k,i=1}^\infty$ forms a L-covering defined by using closed cubes. Take $\epsilon \to \infty$, we obtain $m_{re}^*(E) \geq m_{cu}^*(E)$, so $m_{re}^*(E) = m_{cu}^*(E)$. □

Thanks to the above two remarks, from now on, we don't need to clarify the outer measure or L-covering is defined by open rectangles or closed rectangles or closed cubes. Although by default, we will still follow the open rectangle version, the next two exercises illustrate that sometimes it is convenient to use other versions.

✍ **Exercise 1.6** Prove that $m^*(R) = |R|$ for closed rectangle R.

Proof Obviously, $m^*(R) \leq |R|$ if we treat the outer measure here as the closed rectangle version, because R itself is a closed rectangular covering of itself. Now we treat the outer measure as open rectangle version, then for all $\epsilon > 0$, there exists L-covering (open rectangles) $\{R_k\}_{k=1}^\infty$ of R s.t. $m^*(R) + \epsilon > \sum_{k=1}^\infty |R_k|$. Since R is compact, there exists finite subcover $\{R_{k_i}\}_{i=1}^m$ s.t. $\bigcup_{i=1}^m R_{k_i} \supset R$. By Exercise 1.2, $\sum_{i=1}^m R_{k_i} \geq |R|$. Therefore, $m^*(R) + \epsilon > \sum_{k=1}^\infty |R_k| \geq \sum_{i=1}^m R_{k_i} \geq |R|$. Take $\epsilon \to 0$, we obtain $m^*(R) \geq |R|$, so $m^*(R) = |R|$. □

✍ **Exercise 1.7** Prove that $m^*(R) = |R|$ for open rectangle R.

Proof This time if we regard the outer measure as the open rectangle version, R is an L-covering (open) of itself, so $m^*(R) \leq |R|$. Take small $\delta > 0$ and define $R_\delta = [a_1 + \delta, b_1 - \delta] \times \cdots \times [a_n + \delta, b_n - \delta]$, where $R = (a_1, b_1) \times \cdots \times (a_n, b_n)$. Since R is an L-covering (open) of R_δ, $|R| \geq m^*(R_\delta)$. Since any L-covering of R is also an L covering of R_δ, so $m^*(R) \geq m^*(R_\delta)$. By the Exercise 1.6, $m^*(R_\delta) = |R_\delta|$ since R_δ is closed rectangle. Take $\delta \to 0$ on both sides of $m^*(R) \geq |R_\delta|$, we obtain $m^*(R) \geq |R|$, so $m^*(R) = |R|$. □

Problem 1.2 If $E_1 \subset E_2 \subset \mathbb{R}^n$, prove that $m^*(E_1) \leq m^*(E_2)$. Hence, prove that the outer measure of general rectangle R (defined in the Remark after Definition 1.1) is also equal to the volume of R.

Let's end this section by taking a closer look at some fundamental properties of outer measure. Note that you may have already seen or proved some of them.

Property

1. *$m^*(E) \geq 0$, $\forall\, E \subset \mathbb{R}^n$ and $m^*(\varnothing) = 0$. This is called **nonnegativity** of outer measure.*
2. *If $E_1 \subset E_2 \subset \mathbb{R}^n$, then $m^*(E_1) \leq m^*(E_2)$. This is called **monotonicity** of outer measure.*
3. *If $E_k \subset \mathbb{R}^n$ for $k \in \mathbb{N}^+$, then $m^*(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m^*(E_k)$. This is called **$\sigma$-subadditivity** of outer measure.*
4. *Let $E \subset \mathbb{R}^n$ and $y \in \mathbb{R}^n$, then $m^*(E+y) = m^*(E)$. This is called **translation invariance** of outer measure.*
5. *If $E \subset \mathbb{R}^n$, then $m^*(E) = \inf\{m^*(O) \,|\, O \supset E,\, O \text{ is open}\}$.*
6. *Suppose $E_1, E_2 \subset \mathbb{R}^n$, and there exists disjoint open set G, H s.t. $G \supset E_1$, $H \supset E_2$. Then $m^*(E_1 \cup E_2) = m^*(E_1) + m^*(E_2)$.*
7. *Let $E = \bigcup_{k=1}^{\infty} R_k$ where R_k's are almost disjoint rectangles. Then $m^*(E) = \sum_{k=1}^{\infty} |R_k|$.*

Proof

1. Trivial. Please prove it by yourself.
2. Has been proved in Problem 1.2.
3. For each fixed k, by definition of outer measure of E_k, for every $\epsilon > 0$, there exists an L-covering of E_k, $\{R_{k,l}\}_{l=1}^{\infty}$, s.t. $\bigcup_{l=1}^{\infty} R_{k,l} \supset E_k$ and $\sum_{l=1}^{\infty} |R_{k,l}| \leq m^*(E_k) + \epsilon/2^k$. Notice that $\{R_{k,l}\}_{k,l=1}^{\infty}$ is an L-covering of $\bigcup_{k=1}^{\infty} E_k$, so $m^*(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} |R_{k,l}|$. This implies $m^*(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty}(m^*(E_k) + \epsilon/2^k) = \sum_{k=1}^{\infty} m^*(E_k) + \epsilon$. Take $\epsilon \to 0$, we will obtain $m^*(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m^*(E_k)$.
4. Let $\{R_k\}_{k=1}^{\infty}$ be an L-covering of E, then $\{R_k + y\}_{k=1}^{\infty}$ is also an L-covering of $E + y$. This implies $m^*(E+y) \leq \sum_{k=1}^{\infty} |R_k + y| = \sum_{k=1}^{\infty} |R_k|$. Take infimum over all L-covering $\{R_k\}_{k=1}^{\infty}$ of E, and we obtain $m^*(E+y) \leq m^*(E)$. Now we proved for all $F \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $m^*(F+x) \leq m^*(F)$. Let $F = E + y$ and $x = -y$, then we have $m^*(E) = m^*(E + y + (-y)) \leq m^*(E+y)$. Thus, $m^*(E) = m^*(E+y)$.
5. By monotonicity of outer measure, $m^*(O) \geq m^*(E)$ for any open set $O \supset E$. Take infimum over all open set O, we obtain $m^*(E) \leq \inf\{m^*(O) \,|\, O \supset E,\, O \text{ is open}\}$. By definition of $m^*(E)$, for all $\epsilon > 0$, there exists $\{R_k\}_{k=1}^{\infty}$ s.t. $\bigcup_{k=1}^{\infty} R_k \supset E$ and $m^*(E) + \epsilon \geq \sum_{k=1}^{\infty} |R_k|$. Let $G = \bigcup_{k=1}^{\infty} R_k$, then since R_k's are all open, G is open. This implies $m^*(G) \geq \inf\{m^*(O) \,|\, O \supset E,\, O \text{ is open}\}$. By σ-subadditivity of outer measure, $m^*(G) \leq \sum_{k=1}^{\infty} |R_k|$. Thus, $m^*(E) + \epsilon \geq \inf\{m^*(O) \,|\, O \supset E,\, O \text{ is open}\}$. Take $\epsilon \to 0$, we have $m^*(E) \geq \inf\{m^*(O) \,|\, O \supset E,\, O \text{ is open}\}$ and we are done.
6. By σ-subadditivity of outer measure, $m^*(E_1 \cup E_2) \leq m^*(E_1) + m^*(E_2)$ is trivial. For all

$\epsilon > 0$, there exists L-covering $\{R_k\}_{k=1}^\infty$, R_k closed rectangles s.t. $\bigcup_{k=1}^\infty R_k \supset E_1 \cup E_2$ and $m^*(E_1 \cup E_2) + \epsilon \geq \sum_{k=1}^\infty |R_k|$. Since G, H are open, by Exercise 1.3, $G = \bigcup_{m=1}^\infty I_m$ and $H = \bigcup_{m=1}^\infty J_m$, where I_m's, J_m's are closed and almost disjoint cubes. Since $G \cap H = \varnothing$, $I_m \cap J_{m'} = \varnothing$ for any $m, m' \geq 1$. By definition of rectangles, it is easy to see the intersection of two rectangles is either empty or again a rectangle (maybe a rectangle in lower dimension). Therefore, for each fixed $k \geq 1$, $\{R_k \cap I_m\}_{m=1}^\infty$ and $\{R_k \cap J_m\}_{m=1}^\infty$ are closed almost disjoint rectangles (for any $R_k \cap I_m$ or $R_k \cap J_m$ with zero volume, we can ignore it) contained in R_k. By Corollary 1.1, $|R_k| \geq \sum_{m=1}^\infty |R_k \cap I_m| + \sum_{m=1}^\infty |R_k \cap J_m|$. Sum over k on both sides, $\sum_{k=1}^\infty |R_k| \geq \sum_{k=1}^\infty \sum_{m=1}^\infty |R_k \cap I_m| + \sum_{k=1}^\infty \sum_{m=1}^\infty |R_k \cap J_m|$. Since $\bigcup_{k=1}^\infty R_k \cap G \supset E_1$, $\{I_m \cap R_k\}_{k,m=1}^\infty$ is an L-covering (closed) of E_1, we have $\sum_{k=1}^\infty \sum_{m=1}^\infty |R_k \cap I_m| \geq m^*(E_1)$. Similarly, $\sum_{k=1}^\infty \sum_{m=1}^\infty |R_k \cap J_m| \geq m^*(E_2)$. This shows $\sum_{k=1}^\infty |R_k| \geq m^*(E_1) + m^*(E_2)$. Therefore, $m^*(E_1 \cup E_2) + \epsilon \geq m^*(E_1) + m^*(E_2)$. Take $\epsilon \to 0$, we obtain $m^*(E_1 \cup E_2) \geq m^*(E_1) + m^*(E_2)$.

7. By σ-subadditivity of outer measure, $m^*(E) \leq \sum_{k=1}^\infty |R_k|$. For each fixed $n \geq 1$, $\sum_{k=1}^n |R_k| = \sum_{k=1}^n |R_k^\circ|$ and $m^*(E) \geq m^*(\bigcup_{k=1}^n R_k^\circ)$. Since all R_k's are open disjoint rectangles, we can apply Property 6 inductively ($n-1$ times) on $\bigcup_{k=1}^n R_k^\circ$ to obtain $m^*(\bigcup_{k=1}^n R_k^\circ) = \sum_{k=1}^n |R_k^\circ|$. Thus, $m^*(E) \geq \sum_{k=1}^n |R_k|$ and we are done.

□

Corollary 1.2. of Property 5

For $E \subset \mathbb{R}^n$, there exists $\{O_n\}_{n=1}^\infty$ s.t. O_n is open for $n \geq 1$ and $\bigcap_{n=1}^\infty O_n \supset E$ and $m^*(E) = m^*(\bigcap_{n=1}^\infty O_n)$.

Proof Since $m^*(E) = \inf\{m^*(O) \,|\, O \supset E,\, O \text{ is open}\}$, for $n \geq 1$, there exists O_n s.t. O_n is open, $O_n \supset E$ and $m^*(E) + \frac{1}{n} \geq m^*(O_n) \geq m^*(E)$. Since $m^*(O_n) \geq m^*(\bigcap_{n=1}^\infty O_n) \geq m^*(E)$ (by monotonicity of outer measure), we have $m^*(E) + \frac{1}{n} \geq m^*(\bigcap_{n=1}^\infty O_n) \geq m^*(E)$. Take $n \to \infty$, by squeeze theorem, $m^*(E) = m^*(\bigcap_{n=1}^\infty O_n)$. □

Remark Note that $\bigcap_{n=1}^\infty O_n$ is a G_δ-type set (G_δ stands for *Gebiet Durchschnitt* in German), so we can restate the corollary as: for $E \subset \mathbb{R}$, there exists a G_δ set $G \supset E$ and $m^*(G) = m^*(E)$.

Corollary 1.3. of Property 7

Let G be open in $\mathbb{R}^n$, then we have $G = \bigcup_{k=1}^\infty c_k$, where c_k's are almost disjoint closed cubes and $m^*(G) = \sum_{k=1}^\infty |c_k|$.

Problem Set 1.3

1. Let B be the set of rational numbers in the interval $[0,1]$, and let $\{I_k\}_{k=1}^n$ be a finite collection of open intervals that covers B. Prove that $\sum_{k=1}^n m^*(I_k) \geq 1$.
2. Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.

3. Let A and B be bounded sets for which there is an $\alpha > 0$ such that $|a - b| \geq \alpha$ for all $a \in A, b \in B$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.
4. Let F_k for $k \in \mathbb{N}^+$ be nonempty closed subsets of $\mathbb{R}^n$ s.t. $\text{dist}(x_0, F_k) \to \infty$ as $k \to \infty$ for a fixed point $x_0 \in \mathbb{R}^n$. Prove that $\overline{\bigcup_{k=1}^{\infty} F_k} = \bigcup_{k=1}^{\infty} F_k$.
5. Let $E \subset \mathbb{R}$ and define outer Jordan content of E by
$$J_*(E) = \inf\left\{ \sum_{i=1}^{N} |I_i| \;\middle|\; I_i \text{ intervals}, \bigcup_{i=1}^{N} I_i \supset E \right\}$$
   (a). Prove that $J_*(E) = J_*(\bar{E})$.
   (b). Find a countable set $E \subset [0,1]$ such that $J_*(E) = 1$, and $m^*(E) = 0$.
6. Let $A, B \subset \mathbb{R}^n$ with finite outer measure. Prove $|m^*(A) - m^*(B)| \leq m^*(A \triangle B)$.

1.4 Lebesgue Measurable Sets

Definition 1.7. Lebesgue Measurable Sets (Inner regularity)

*We say $E \subset \mathbb{R}^n$ is **Lebesgue measurable** if $\forall \epsilon > 0$, there exists open $G \supset E$ s.t. $m^*(G \setminus E) < \epsilon$. Denote the collection of all **Lebesgue measurable** sets as $\mathcal{M}$ and the **Lebesgue measure** of E is $m(E) = m^*(E)$.* ♣

Note *We want to define such a new collection of sets because there exists $E_1, E_2 \subset \mathbb{R}^n$ s.t. $E_1 \cap E_2 = \varnothing$, but $m^*(E_1 \cup E_2) < m^*(E_1) + m^*(E_2)$. We don't like such kind of strange sets, and this phenomenon can only happen when the sets do not satisfy Definition 1.7.*

The following are some basic and fundamental properties of Lebesgue measurable sets. For some of them, we will leave the proof as an exercise in Problem Set 1.4, but you can use these properties freely when you prove other statements.

Property

1. *If $O \subset \mathbb{R}^n$ is open, then $O \in \mathcal{M}$.*
2. *If $E \subset \mathbb{R}^n$ and $m^*(E) = 0$, then $E \in \mathcal{M}$.*
3. *If $E_k \in \mathcal{M}$ for all $k \geq 0$, then $\bigcup_{k=1}^{\infty} E_k \in \mathcal{M}$.*
4. *If $F \subset \mathbb{R}^n$ is closed, then $F \in \mathcal{M}$.*
5. *If $E \in \mathcal{M}$, then $E^c \in \mathcal{M}$.*
6. *If $E_k \in \mathcal{M}$ for all $k \geq 1$, then $\bigcap_{k=1}^{\infty} E_k \in \mathcal{M}$.*
7. *If $E_k \in \mathcal{M}$ for all $k \geq 1$, and E_k's pairwise disjoint, then $m(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m(E_k)$. This is called **$\sigma$-additivity** of Lebesgue measure.*

Proof

1. Trivial.
2. Trivial.
3. Since $E_k \in \mathcal{M}$, for all $\epsilon > 0$, there exists open $G_k \supset E_k$ s.t. $m(G_k \setminus E_k) < \frac{\epsilon}{2^k}$. Since

$\bigcup_{k=1}^{\infty} G_k \setminus \bigcup_{k=1}^{\infty} E_k \subset \bigcup_{k=1}^{\infty}(G_k \setminus E_k)$, by monotonicity and σ-subadditivity of outer measure, $m^*\left(\bigcup_{k=1}^{\infty} G_k \setminus \bigcup_{k=1}^{\infty} E_k\right) \leq m^*\left(\bigcup_{k=1}^{\infty}(G_k \setminus E_k)\right) \leq \sum_{k=1}^{\infty} m^*(G_k \setminus E_k)$. This implies $m^*\left(\bigcup_{k=1}^{\infty} G_k \setminus \bigcup_{k=1}^{\infty} E_k\right) < \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon$. Let $G = \bigcup_{k=1}^{\infty} G_k$, then G is the desired open set to prove $\bigcup_{k=1}^{\infty} E_k$ is Lebesgue measurable.

4. Special case: If F is bounded, then F is compact and $m^*(F) < \infty$. By Property 5 of outer measure, for all $\epsilon > 0$, there exists open $G \supset F$ s.t. $m^*(G) \leq m^*(F) + \epsilon$. Since $G \setminus F$ is open, by Exercise 1.3, $G \setminus F = \bigcup_{k=1}^{\infty} c_k$ where c_k's are almost disjoint closed cubes. Observe that $\bigcup_{n=1}^{k} c_n$ is compact for any fixed $k \geq 1$. Note that two disjoint compact sets in $\mathbb{R}^n$ can be separated by two disjoint open sets (This is a famous fact in basic topology). Therefore, by Property 6 and monotonicity of outer measure,
$$m^*(F) + m^*\left(\bigcup_{n=1}^{k} c_n\right) = m^*\left(F \cup \left(\bigcup_{n=1}^{k} c_n\right)\right) \leq m^*(G)$$
By Property 7 of outer measure, $m^*(\bigcup_{n=1}^{k} c_n) = \sum_{n=1}^{k} |c_n|$. This shows $m^*(G) - m^*(F) \geq \sum_{n=1}^{k} |c_n|$ for all $k \geq 1$. Send $k \to \infty$, $m^*(G) - m^*(F) \geq \sum_{n=1}^{\infty} |c_n|$. Again by Property 7, $m^*(G \setminus F) = \sum_{k=1}^{\infty} |c_k|$. Therefore, $m^*(G \setminus F) \leq \epsilon$ and $F \in \mathcal{M}$.
General case: Note that any closed set F in $\mathbb{R}^n$ can be decomposed as countable union of compact sets ($F = \bigcup_{k=1}^{\infty} F \cap B_k$, where B_k is the closed ball with radius k centered at the orgin). For each compact set $F \cap B_k$, we can use special case to prove it is in $\mathcal{M}$, and then by Property 3, F is in $\mathcal{M}$.
5. Question 1. in Problem Set 1.4.
6. Question 3. in Problem Set 1.4.
7. Question 4. in Problem Set 1.4.

□

The following are two extremely handy corollaries of the above properties. We only display the statement here and leave the proof as exercise in Problem Set 1.4, Question 5. and 6..

Corollary 1.4. of σ-additivity

Suppose $E, F \in \mathcal{M}$, $F \subset E$ with $m(F) < \infty$, then $m(E \setminus F) = m(E) - m(F)$. ♡

Corollary 1.5. Continuity of Lebesgue Measure

Suppose $E_k \in \mathcal{M}$ for all $k \geq 1$,

1. $E_1 \subset E_2 \subset \cdots \subset E_k \subset \cdots$, *then* $m(\lim_{k\to\infty} E_k) = \lim_{k\to\infty} m(E_k)$ *where* $\lim_{k\to\infty} E_k = \bigcup_{k=1}^{\infty} E_k$.
2. $E_1 \supset E_2 \supset \cdots \supset E_k \supset \cdots$ *and there exists* $k_0 \geq 1$ *s.t.* $m(E_{k_0}) < \infty$, *then* $m(\lim_{k\to\infty} E_k) = \lim_{k\to\infty} m(E_k)$ *where* $\lim_{k\to\infty} E_k = \bigcap_{k=1}^{\infty} E_k$. ♡

Next, we introduce some basic concepts that we will use in our later study.

Definition 1.8. σ-algebra

*A collection of sets in $\mathbb{R}^n$ which is closed under countable unions, intersections and complement are called **σ-algebra**.*

Note *By Property 5, 6 and 7 above, we can see $\mathcal{M}$ is a σ-algebra.*

Definition 1.9. Borel σ-algebra

***Borel σ-algebra** $\mathcal{B}$ is the smallest σ-algebra that contains all open sets in $\mathbb{R}^n$. Any sets in $\mathcal{B}$ are called **Borel measurable** sets.*

Note *Later we will study a famous example which indicates that $\mathcal{B}$ is strictly contained in $\mathcal{M}$.*

Recall in the Remark of Corollary 1.2, we have mentioned the so-called G_δ set. Now let's give a formal definition of it and another type of set: F_σ set.

Definition 1.10. F_σ & G_δ Set

An $\boldsymbol{F_\sigma}$ set is the countable union of closed sets. A $\boldsymbol{G_\delta}$ set is the countable intersection of open sets.

At the end of this section, we will show our main result which illustrates the relation between Lebesgue measurable set and Borel measurable set.

Theorem 1.1

For all $E \subset \mathbb{R}^n$, the following are equivalent:

1. $E \in \mathcal{M}$
2. *For all $\epsilon > 0$, there exists closed $F \subset E$ s.t. $m^*(E \setminus F) < \epsilon$.*
3. *There exists G_δ set s.t. $G \supset E$ and $m^*(G \setminus E) = 0$.*
4. *There exists F_σ set s.t. $F \subset E$ and $m^*(E \setminus F) = 0$.*
5. *If $m^*(E) < \infty$, for all $\epsilon > 0$, there exists finitely many closed cubes $c_1, \ldots, c_k$ s.t. $U = \bigcup_{i=1}^k c_i$ satisfies $m^*(U \triangle E) < \epsilon$, where the symmetric difference $U \triangle E = (U \setminus E) \cup (E \setminus U)$.*

Proof

- $1 \to 2$: Question 2. in Problem Set 1.4.
- $2 \to 4$: By assumption, for all $k \geq 1$, there exists closed $F_k \subset E$ s.t. $m^*(E \setminus F_k) < \frac{1}{k}$. Take $F = \bigcup_{k=1}^\infty F_k$, then $F \subset E$ and F is F_σ-type. Note that $E \setminus F \subset E \setminus F_k$ for all $k \geq 1$. Therefore, $m^*(E \setminus F) \leq m^*(E \setminus F_k) < \frac{1}{k}$. Take $k \to \infty$, we have $m^*(E \setminus F) = 0$.
- $4 \to 1$: Since $m^*(E \setminus F) = 0$, then by Property 2, $E \setminus F \in \mathcal{M}$. Since F is the countable union of closed set, by Property 3 and 4, $F \in \mathcal{M}$. Since $E = F \cup (E \setminus F)$, by Property 3 again, $E \in \mathcal{M}$.
- $1 \to 3$: For $k \geq 1$, there exists open G_k s.t. $G_k \supset E$ and $m^*(G_k \setminus E) < \frac{1}{k}$. Let

$G = \bigcap_{k=1}^{\infty} G_k$, then G is G_δ-type and $G \supset E$. Note that $G \setminus E \subset G_k \setminus E$, so $m^*(G \setminus E) \leq m^*(G_k \setminus E) < \frac{1}{k}$ for all k. Take $k \to \infty$, we have $m^*(G \setminus E) = 0$.

- $3 \to 1$: Note that $E^c = (G \setminus E) \cup G^c$. Since $m^*(G \setminus E) = 0$, $G \setminus E \in \mathcal{M}$. Also, G is the countable intersection of open sets, so $G \in \mathcal{M}$ by Property 6, $G \in \mathcal{M}$. By Property 5, $G^c \in \mathcal{M}$. Therefore, $E^c \in \mathcal{M}$ and $E \in \mathcal{M}$.
- $1 \to 5$: For all $\epsilon > 0$, there exists open $O \supset E$ s.t. $m^*(O \setminus E) < \frac{\epsilon}{100}$. By Corollary 1.3 and Exercise 1.6, we have $O = \bigcup_{k=1}^{\infty} c_k$, where c_k's are almost disjoint closed cubes and $m^*(O) = \sum_{k=1}^{\infty} m^*(c_k)$. Since we assume $m^*(E) < \infty$, by σ-subadditivity of outer measure, $m^*(O) \leq m^*(O \setminus E) + m^*(E) < \infty$. Thus, the series $\sum_{k=1}^{\infty} m^*(c_k)$ converges and there exists N s.t. $\sum_{k=N+1}^{\infty} m^*(c_k) < \epsilon/100$.
  Claim: $U = \bigcup_{k=1}^{N} c_k$ will satisfy $m^*(U \triangle E) < \epsilon$. Observe that
  $$U \triangle E = (U \setminus E) \cup (E \setminus U) \subset (O \setminus E) \cup (O \setminus U)$$
  Thus, $m^*(U \triangle E) \leq m^*(O \setminus E) + m^*(\bigcup_{k=N+1}^{\infty} c_k) < \frac{\epsilon}{100} + \frac{\epsilon}{100} < \epsilon$.
- $5 \to 1$: By assumption, for all $\epsilon > 0$, there exists $U = \bigcup_{k=1}^{N} c_k$ s.t. $m^*(E \triangle U) < \frac{\epsilon}{100}$. Also, for all $\epsilon > 0$, there exists open $G \supset E$, s.t. $m^*(G) < m^*(E) + \frac{\epsilon}{100}$. Let $A = U \cap G$, since $A \triangle E \subset U \triangle E$, we have $m^*(A \triangle E) < \frac{\epsilon}{100}$. Since $A \triangle E = (A \setminus E) \cup (E \setminus A)$, $m^*(E \setminus A) < \frac{\epsilon}{100}$ and $m^*(A \setminus E) < \frac{\epsilon}{100}$. This shows
  $$m^*(E) \leq m^*(E \setminus A) + m^*(A) < \frac{\epsilon}{100} + m^*(A)$$
  Also, $m^*(G \setminus E) \leq m^*(G \setminus A) + m^*(A \setminus E)$. Since $A \in \mathcal{M}$ and $m(A) < \infty$, by Corollary 1.4, $m^*(G \setminus A) = m^*(G) - m^*(A) < m^*(E) - m^*(A) + \frac{\epsilon}{100}$. Therefore, $m^*(G \setminus E) < m^*(E) - m^*(A) + \frac{\epsilon}{100} + \frac{\epsilon}{100} < \epsilon$ implies $E \in \mathcal{M}$.

$\square$

In Problem Set 1.4, Question 12. we will introduce another definition of Lebesgue measurable sets, which is well-known as Carathéodory property, and you will prove the equivalence of Carathéodory property and Definition 1.7. In case you may need to use this property, we display it here without proving it.

Theorem 1.2. Carathéodory Property

$E \in \mathcal{M}$ if and only if for all $T \subset \mathbb{R}^n$, $m^*(T) = m^*(T \cap E) + m^*(T \cap E^c)$. ♡

Problem Set 1.4

1. Prove that if $E \in \mathcal{M}$, then $E^c \in \mathcal{M}$.
2. If $E \in \mathcal{M}$, prove that for all $\epsilon > 0$, there exists closed subset $F \subset E$ s.t. $m^*(E \setminus F) < \epsilon$.
3. If $E_k \in \mathcal{M}$ for $k = 1, 2, \ldots$, prove that $\bigcap_{k=1}^{\infty} E_k \in \mathcal{M}$.
4. Let $E_k \in \mathcal{M}$ for $k \in \mathbb{N}^+$, and E_k's pairwise disjoint. Prove $m(\cup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m(E_k)$.

5. For all $E, F \in \mathcal{M}$ such that $F \subset E$, prove that $m(E \setminus F) + m(F) = m(E)$. Furthermore, if $m(F) < \infty$, then $m(E \setminus F) = m(E) - m(F)$.
6. Supose $E_k \in \mathcal{M}$ for all $k = 1, 2, \ldots$, prove
   (a). If $E_1 \subset E_2 \subset \cdots \subset E_k \subset E_{k+1} \subset \cdots$, then $\lim_{k\to\infty} m(E_k) = m(\lim_{k\to\infty} E_k)$.
   (b). If $E_1 \supset E_2 \supset \cdots \supset E_k \supset E_{k+1} \supset \cdots$ and there exists $k_0 \geq 1$ such that $m(E_{k_0}) < \infty$, then $\lim_{k\to\infty} m(E_k) = m(\lim_{k\to\infty} E_k)$.
   (c). Find a counter-example of (ii) if such k_0 in (ii) does not exist.
7. Prove the Cantor set C is Lebesgue measurable and $m(C) = 0$.
8. Let C_p be the Cantor-like set in Problem Set 1.2, Question 5.. Prove that $C_p \in \mathcal{M}$ and compute $m(C_p)$.
9. Recall the definition of F_σ and G_δ set, and answer the following questions:
   (a). Let $\{f_n(x)\}_{n=1}^{\infty}$ be continuous on $\mathbb{R}$. Prove that $\{x \in \mathbb{R} \mid \underline{\lim}_{n\to\infty} f_n(x) > 0\}$ is F_σ-type.
   (b). Let $f(x)$ be defined on $\mathbb{R}$. Prove that $\{x \in \mathbb{R} \mid \lim_{y\to x} f(y) < \infty\}$ is G_δ-type.
10. Let $E \subset \mathbb{R}$ with finite $m^*(E) > 0$. Prove that $\forall\, a \in (0, m^*(E))$, there exists $A \subset E$ such that $m^*(A) = a$.
11. Let $A_1, A_2 \subset \mathbb{R}^n$, $A_1 \subset A_2$, $A_1 \in \mathcal{M}$, $m(A_1) = m^*(A_2) < \infty$. Prove that $A_2 \in \mathcal{M}$.
12. Prove that $E \in \mathcal{M}$ if and only if $\forall\, T \subset \mathbb{R}^n$, $m^*(T) = m^*(T \cap E) + m^*(T \cap E^c)$.
13. Let $A \in \mathcal{M}$, $B \subset \mathbb{R}^n$ with $m^*(B) < \infty$. Prove $m^*(A \cup B) + m^*(A \cap B) = m^*(A) + m^*(B)$.
14. Suppose $m^*(E) < \infty$. If $m^*(E) = \sup\{m(F) \mid F \subset E, F \text{ closed}\}$, then $E \in \mathcal{M}$.
15. Prove that if $E_k \in \mathcal{M}$ for $k \in \mathbb{N}^+$.
   (a). $m\left(\underline{\lim}_{k\to\infty} E_k\right) \leq \underline{\lim}_{k\to\infty} m(E_k)$.
   (b). If there exists $k_0 \geq 1$ such that $m\left(\cup_{k=k_0}^{\infty} E_k\right) < \infty$, then $m\left(\overline{\lim}_{k\to\infty} E_k\right) \geq \overline{\lim}_{k\to\infty} m(E_k)$.
16. Let $E_k \subset [0,1]$, $E_k \in \mathcal{M}$, $m(E_k) = 1$ for all $k \in \mathbb{N}^+$. Prove $m(\cap_{k=1}^{\infty} E_k) = 1$.
17. Let $E_i \subset [0,1]$, $E_i \in \mathcal{M}$ for all $i = 1, \ldots, k$, and $\sum_{i=1}^{k} m(E_i) > k - 1$. Prove that $m(\cap_{i=1}^{k} E_i) > 0$.
18. Let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collection of measurable sets. Prove that for any set A, $m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m^*(A \cap E_k)$.
19. Let E_k, $k \in \mathbb{N}^+$, be Lebesgue measurable, satisfying $\sum_{k=1}^{\infty} m(E_k) < \infty$. Prove that $m\left(\overline{\lim}_{k\to\infty} E_k\right) = 0$. This is called **Borel-Cantelli** lemma.
20. Give an example of an open set O such that the boundary of the closure of it has positive Lebesgue measure.
21. Let f be continuous on $[0,1]$. Prove that the graph Γ of $y = f(x)$, as a subset of $\mathbb{R}^2$, has Lebesgue measure 0.
22. Does there exists a closed proper subset F of $[0,1]$ such that $m(F) = 1$?
23. Let $E \in \mathcal{M}$ with $m(E) > 0$. Prove that there exists $x \in E$ such that for all $\delta > 0$,

$m(E \cap B_\delta(x)) > 0$, where $B_\delta(x)$ is the ball centered at x with radius $\delta > 0$.

24. Let $E \subset \mathbb{R}^n$. Prove that there exists G_δ set $G \supset E$ such that for all $A \in \mathcal{M}$, we have $m^*(E \cap A) = m(G \cap A)$.
25. Let $E \notin \mathcal{M}$. Prove that there exists $\epsilon > 0$ such that whenever $A, B \in \mathcal{M}$, $A \supset E$, $B \supset E^c$, we always have $m(A \cap B) \geq \epsilon$.
26. Let $E \subset \mathbb{R}$ and $E \in \mathcal{M}$. Suppose there exists open intervals I_k for $k \in \mathbb{N}^+$ such that $m(E \cap I_k) \geq \frac{2}{3} m(I_k)$. Prove that $m\left(E \cap \bigcup_{k=1}^\infty I_k\right) \geq \frac{1}{3} m\left(\bigcup_{k=1}^\infty I_k\right)$.

1.5 Non-Lebesgue Measurable Sets

In this section we are going to explicitly construct a type of set which is not Lebesgue measurable. However, before constructing it, we need some lemma to help us.

Lemma 1.1. Steinhaus Theorem

For all $E \in \mathcal{M}$ with $m(E) > 0$, there exists $\delta > 0$ s.t. $E - E \triangleq \{x - y \,|\, x, y \in E\} \supset B_\delta(0)$, where $B_\delta(0)$ is the open ball centered at the orgin with radius δ. ♡

Proof Since $m(E) > 0$, there exists $k \geq 1$ s.t. $m(N_k(0) \cap E) > 0$, where $N_k(0)$ is the open neighborhood of the origin with radius k. If $m(N_k(0) \cap E) = 0$ for all $k \geq 1$, since $E = \bigcup_{k=1}^\infty (N_k(0) \cap E)$, we have $m(E) \leq \sum_{k=1}^\infty m(N_k(0) \cap E) = 0$, which contradicts $m(E) > 0$. Let $F = N_k(0) \cap E$, then $m(F) < \infty$ and it suffices to show $F - F \supset B_\delta(0)$.

Claim: For all $\lambda \in (0, 1)$, there exists open rectangle R s.t. $m(F \cap R) > \lambda m(R)$. To prove this, by definition of $m^*(F)$, for all $\epsilon > 0$, there exists open R_k's s.t. $\bigcup_{k=1}^\infty R_k \supset F$ and $m(F) + \epsilon > \sum_{k=1}^\infty m(R_k)$. For a given λ, we can take $\epsilon = (\lambda^{-1} - 1)m(F) > 0$, then we will have $\lambda^{-1} m(F) > \sum_{k=1}^\infty m(R_k)$. Also, $F = \bigcup_{k=1}^\infty (F \cap R_k)$, so $m(F) \leq \sum_{k=1}^\infty m(F \cap R_k)$. This implies $\sum_{k=1}^\infty m(F \cap R_k) > \sum_{k=1}^\infty \lambda m(R_k)$. Thus, there exists at least one k_0 s.t. $m(F \cap R_{k_0}) > \lambda m(R_{k_0})$.

We can take $\lambda = \frac{3}{4}$ in the claim, and denote the rectangle we obtained as R, then we will have $m(F \cap R) > \frac{3}{4} m(R)$. Note that we only need to show $F \cap R - F \cap R$ contains $B_\delta(0)$ for some $\delta > 0$. It suffices to show there exists $\delta > 0$ s.t. for all $x \in B_\delta(0)$, $(x + F \cap R) \cap (F \cap R) \neq \varnothing$.

Take $\delta > 0$ small s.t. $m((x + R) \cap R) > \frac{1}{2} m(R)$ for all $x \in B_\delta(0)$. If so, we have

$$m((x + R) \cup R) = m(x + R) + m(R) - m((x + R) \cap R) < \frac{3}{2} m(R)$$

If $(x + F \cap R) \cap (F \cap R) = \varnothing$ for some $x \in B_\delta$, then by σ-additivity and translation invariance of Lebesgue measure,

$$m((x + F \cap R) \cup (F \cap R)) = 2m(F \cap R) > 2 \cdot \frac{3}{4} m(R) = \frac{3}{2} m(R)$$

but since $m((x + F \cap R) \cup (F \cap R)) \leq m((x + R) \cup R) < \frac{3}{2} m(R)$, we obtain a contradiction, so there exists $\delta > 0$ s.t. for all $x \in B_\delta(0)$, $(x + F \cap R) \cap (F \cap R) \neq \varnothing$, and this is enough to show the desired result. □

Remark To find δ s.t. $m((x+R)\cap R) > \frac{1}{2}m(R)$ for all $x \in B_\delta(0)$, we can take

$$\delta = \left(1 - \frac{\sqrt[n]{3}}{\sqrt[n]{4}}\right) \min\{b_k - a_k \mid k = 1, \ldots, n\}, \quad R = \prod_{k=1}^{n}(a_k, b_k)$$

because if so, for $x \in B_\delta(0)$, denote $x = (x_1, \ldots, x_n)$, we will have $|x_i| < \delta$ for all $i = 1, \ldots, n$. Also, $(x+R)\cap R = \prod_{k=1}^{n}[(a_k, b_k) \cap (a_k + x_k, b_k + x_k)]$, so

$$m((x+R)\cap R) = |(x+R)\cap R| = \prod_{k=1}^{n}(b_k - a_k - |x_k|) > \frac{3}{4}\prod_{k=1}^{n}(b_k - a_k) > \frac{1}{2}m(R)$$

After we proved the famous Steinhauss theorem, we can use it to verify the non-Lebesgue measurable set constructed below.

Theorem 1.3. Non-Lebesgue Measurable Set

In $\mathbb{R}^n$, define equivalence relation $x \sim y$ if and only if $x - y \in \mathbb{Q}^n$. This partitioned $\mathbb{R}^n$ into many equivalence classes. For each class, pick one and only one element, and collect all of the chosen elements to form a set S, then S is not Lebesgue measurable. ♡

Proof Suppose $S \in \mathcal{M}$, then there are two cases, i.e., $m(S) > 0$ or $m(S) = 0$. If $m(S) > 0$, then by Steinhauss theorem, $S - S \supset B_\delta(0)$ for some $\delta > 0$. There exists $q \in \mathbb{Q}^n \cap B_\delta(0)$, s.t. $q \neq 0$ and $q \in S - S$, i.e., there exists $x, y \in S$ s.t. $x - y = q \neq 0$. This contradicts the construction of S, so $m(S) > 0$ is impossible. If $m(S) = 0$, then for all $z \in \mathbb{R}^n$, there exists $x \in S$ and $q \in \mathbb{Q}^n$ s.t. $z = x + q$. This implies $\mathbb{R}^n = \bigcup_{m=1}^{\infty}(S + q_m)$ if we denote $\mathbb{Q}^n = \{q_m\}_{m=1}^{\infty}$. Thus, $m(\mathbb{R}^n) \leq \sum_{m=1}^{\infty} m(S + q_m) = 0$, which is impossible. □

Remark It is easy to see that $m^*(S) > 0$, because if $m^*(S)$ then $S \in \mathcal{M}$ is a contradiction. Also, one can use the same argument to prove for all $E \subset \mathbb{R}^n$ with $m^*(E) > 0$, there exists $S_E \notin \mathcal{M}$ and $S_E \subset E$.

At the end of this section, we want to resolve the problem raised in the note after Definition 1.7, that is, the outer measure of two disjoint sets may not satisfy additivity property. You have seen that if these two disjoint sets are Lebesgue measurable, then they must satisfy additivity property, so it is natural to construct some non-Lebesgue measurable sets to violate the additivity property. Before we explicitly construct them, we will first show a proposition that will help you understand the construction.

Proposition 1.2

For all $E \in \mathcal{M}$ with $m(E) < \infty$, if $E_1, E_2 \subset E$, $E_1 \cap E_2 = \varnothing$, $E = E_1 \cup E_2$, and $m(E_1 \cup E_2) = m^*(E_1) + m^*(E_2)$, then $E_1, E_2 \in \mathcal{M}$. ♠

Proof By the remark of Corollary 1.2, there exists G_δ set G_1, G_2 s.t. $G_1 \supset E$, $G_2 \supset E_2$, with $m(G_1) = m^*(E_1)$ and $m(G_2) = m^*(E_2)$. By monotonicity, $m(G_1 \cup G_2) \geq m(E)$. Also, $m(E) = m^*(E_1) + m^*(E_2)$, so $m(G_1 \cup G_2) \geq m^*(E_1) + m^*(E_2) = m(G_1) + m(G_2)$.

However, $m(G_1 \cup G_2) \le m(G_1) + m(G_2)$. This implies $m(G_1 \cup G_2) = m(G_1) + m(G_2)$. Since $G_1 \in \mathcal{M}$ and $m^*(G_2) = m^*(E_2) \le m^*(E) < \infty$, by Question 13. in Problem Set 1.4, we have $m(G_1 \cap G_2) = 0$. Since $G_1 \setminus E_1 \subset (G_1 \cup G_2 \setminus E) \cup (G_1 \cap E_2)$, by monotonicity, $m^*(G_1 \setminus E_1) \le m^*(G_1 \cup G_2 \setminus E) + m^*(G_1 \cap E_2)$. Since $G_1 \cap E_2 \subset G_1 \cap G_2$, $m^*(G_1 \cap E_2) = 0$. By subadditivity and $m(E) < \infty$, $m^*(G_1 \cup G_2 \setminus E) \le m(G_1 \cup G_2) - m(E) = 0$. Therefore, $m^*(G_1 \setminus E_1) = 0$ and $G_1 \setminus E_1 \in \mathcal{M}$. Since $G_1 \in \mathcal{M}$, $E_1 = G_1 \setminus (G_1 \setminus E_1) \in \mathcal{M}$. Also, $E_2 = E \setminus E_1 \in \mathcal{M}$. □

Conclusion *We can construct two sets $E_1, E_2 \subset \mathbb{R}^n$ s.t. $m^*(E_1 \cup E_2) < m^*(E_1) + m^*(E_2)$ easily. Take R as unit cubes in $\mathbb{R}^n$, and by the remark of Theorem 1.3 there exists $S \subset R$ s.t. $S \notin \mathcal{M}$. Simply let $E_1 = S$ and $E_2 = R \setminus E_1$, then $E_1 \cap E_2 = \varnothing$ and $E = E_1 \cup E_2$. If $m^*(E_1 \cup E_2) = m(R) = m^*(E_1) + m^*(E_2)$, then by Proposition 1.2, $E_1, E_2 \in \mathcal{M}$. This contradiction combined with subadditivity shows $m^*(E_1 \cup E_2) < m^*(E_1) + m^*(E_2)$.*

Problem Set 1.5

1. Suppose $E, F \subset \mathbb{R}$ and $E, F \in \mathcal{M}$. If $m(E) > 0$ and $m(F) > 0$, then $E + F$ contains an interval.

1.6 Non-Borel Measurable Sets

In this section we are going to explicitly construct a type of set which is Lebesgue measurable but not Borel measurable. This will directly show that Borel σ-algebra $\mathcal{B}$ is strictly contained in $\mathcal{M}$. However, before constructing it, we need to introduce the famous Cantor function.

Definition 1.11. Cantor Function

We recursively define a sequence of functions $f_k : [0,1] \mapsto [0,1]$ for $k \in \mathbb{N}$. Let $f_0(x) = x$ on $[0,1]$ and for all $k \ge 0$, define

$$f_{k+1}(x) = \begin{cases} \frac{1}{2} f_k(3x) & \text{if } 0 \le x \le \frac{1}{3} \\ \frac{1}{2} & \text{if } \frac{1}{3} \le x \le \frac{2}{3} \\ \frac{1}{2} + \frac{1}{2} f_k(3x-2) & \text{if } \frac{2}{3} \le x \le 1 \end{cases}$$

Then $f(x) = \lim_{k\to\infty} f_k(x)$ defined on $[0,1]$ is called ***Cantor function****.* ♣

Note *We need to verify $f(x)$ is well-defined, so we have two things to check:*

1. *Since at two end points $x = \frac{1}{3}$ and $x = \frac{2}{3}$, $f_k(x)$ is defined twice by using different formulae, so we need to guarantee $\frac{1}{2} f_k(3x) = \frac{1}{2}$ at $x = \frac{1}{3}$ and $\frac{1}{2} + \frac{1}{2} f_k(3x-2) = \frac{1}{2}$ at $x = \frac{2}{3}$. This is equivalent to say $f_k(1) = 1$ and $f_k(0) = 0$ for all $k \ge 0$. We can use induction to prove this. For the base case, since $f_0(x) = x$ on $[0,1]$, so it is trivial that $f_0(0) = 0$ and $f_0(1) = 1$. Now suppose for some k, $f_k(1) = 1$ and $f_k(0) = 0$, we tend to prove*

$f_{k+1}(1) = 1$ and $f_{k+1}(0) = 0$. This is also trivial because $f_{k+1}(1) = \frac{1}{2} + \frac{1}{2}f_k(1) = 1$ and $f_{k+1}(0) = \frac{1}{2}f_k(0) = 0$. Therefore, we finish the induction.

2. *After we proved each $f_k(x)$ is well-defined, we also need to prove $f(x)$ is well-defined since it is defined to be the pointwise limit of $f_k(x)$. To check the limit exists as a finite number for each x, we claim that*

$$\max_{x\in[0,1]} |f_{k+1}(x) - f_k(x)| \le \frac{1}{2} \max_{x\in[0,1]} |f_k(x) - f_{k-1}(x)|, \quad \forall\, k \ge 1$$

We can separate the LFS into three cases.

$$|f_{k+1}(x) - f_k(x)| = \frac{1}{2}|f_k(3x) - f_{k-1}(3x)|, \quad \forall\, x \in \left[0, \frac{1}{3}\right]$$

$$|f_{k+1}(x) - f_k(x)| = 0, \quad \forall\, x \in \left[\frac{1}{3}, \frac{2}{3}\right]$$

$$|f_{k+1}(x) - f_k(x)| = \frac{1}{2}|f_k(3x-2) - f_{k-1}(3x-2)|, \quad \forall\, x \in \left[\frac{2}{3}, 1\right]$$

Notice that $|f_k(3x) - f_{k-1}(3x)| \le \max_{x\in[0,1]} |f_k(x) - f_{k-1}(x)|$ for all $x \in [0, \frac{1}{3}]$. Similarly, $|f_k(3x-2) - f_{k-1}(3x-2)| \le \max_{x\in[0,1]} |f_k(x) - f_{k-1}(x)|$ for all $x \in [\frac{2}{3}, 1]$. Therefore, $|f_{k+1}(x) - f_k(x)| \le \frac{1}{2}\max_{x\in[0,1]} |f_k(x) - f_{k-1}(x)|$ for all $x \in [0,1]$, and thus our claim is proved. Since it is easy to see $|f_1(x) - f_0(x)| \le 1$ for all $x \in [0,1]$, by inductively applying our claim, $|f_{k+1}(x) - f_k(x)| \le \frac{1}{2^k}$ for all $x \in [0,1]$ and for all $k \ge 1$. Note that $f_k(x) = f_0(x) + \sum_{m=1}^{k}(f_m(x) - f_{m-1}(x))$, so to prove $\lim_{k\to\infty} f_k(x)$ exists, we only need to prove $\sum_{m=1}^{\infty}(f_m(x) - f_{m-1}(x))$ converges. By Weierstrauss M-Test, since $\sum_{m=1}^{\infty} \frac{1}{2^{m-1}} < \infty$, $\sum_{m=1}^{\infty}(f_m(x) - f_{m-1}(x)) < \infty$ on $x \in [0,1]$. This shows $f_k(x)$ converges to $\lim_{k\to\infty} f_k(x)$ uniformly on $[0,1]$.

After we verified that the Cantor function $f(x)$ defined in Definition 1.11 is valid, we are going to explore some properties of it.

Property

1. *$f(x)$ is uniformly continuous on $[0,1]$.*
2. *$f(0) = 0$ and $f(1) = 1$.*
3. *$f(x)$ is increasing on $[0,1]$.*

Proof

1. To prove this, it suffices to prove every $f_k(x)$ is continuous on $[0,1]$, because if so, since $f(x)$ is the uniform limit of $f_k(x)$, it must be continuous on $[0,1]$. Since $[0,1]$ is a compact set, $f(x)$ is uniformly continuous on $[0,1]$. To prove $f_k(x)$ is continuous on $[0,1]$ for $k \ge 0$, again we use induction. It is obvious that $f_0(x) = x$ is continuous on $[0,1]$. Suppose $f_k(x)$ is continuous on $[0,1]$ for some $k \ge 0$, then $f_k(3x)$ is continuous on $[0, \frac{1}{3}]$ and $f_k(3x-2)$ is continuous on $[\frac{2}{3}, 1]$. This shows $f_{k+1}(x)$ is continuous separately on $[0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{2}{3}]$, and $[\frac{2}{3}, 1]$. However, the continuity of $f_{k+1}(x)$ on $[0, \frac{1}{3}]$ implies $f_{k+1}(\frac{1}{3}-) = f_{k+1}(\frac{1}{3})$ where $f_{k+1}(a-)$ means the left limit of $f_{k+1}(x)$ at $x = a$ (Similarly, $f_{k+1}(b+)$ means

the right limit of $f_{k+1}(x)$ at $x = b$). Also, the continuity of $f_{k+1}(x)$ on $[\frac{1}{3}, \frac{2}{3}]$ implies $f_{k+1}(\frac{1}{3}+) = f_{k+1}(\frac{1}{3})$ and $f_{k+1}(\frac{2}{3}-) = f_{k+1}(\frac{2}{3})$. Finally, the continuity of $f_{k+1}(x)$ on $[\frac{2}{3}, 1]$ implies $f_{k+1}(\frac{2}{3}+) = f_{k+1}(\frac{2}{3})$. In conclusion, $f_{k+1}(\frac{1}{3}-) = f_{k+1}(\frac{1}{3}) = f_{k+1}(\frac{1}{3}+)$ and $f_{k+1}(\frac{2}{3}-) = f_{k+1}(\frac{2}{3}) = f_{k+1}(\frac{2}{3}+)$, so $f_{k+1}(x)$ is continuous at $x = \frac{1}{3}$ and $x = \frac{2}{3}$. This is enough to show $f_{k+1}(x)$ is continuous on $[0, 1]$ and we finish the induction.

2. Since in the note of Definition 1.11, we have shown $f_k(0) = 0$ and $f_k(1) = 1$ for all $k \geq 0$, it is trivial that $f(0) = \lim_{k\to\infty} f_k(0) = 0$ and $f(1) = \lim_{k\to\infty} f_k(1) = 1$.
3. It suffices to show that for each $k \geq 0$, $f_k(x)$ is increasing on $[0, 1]$, because if so, for any fixed $0 \leq x_1 \leq x_2 \leq 1$, $f_k(x_1) \leq f_k(x_2)$ for all $k \geq 0$. Take limit as $k \to \infty$ on both sides, we have $f(x_1) \leq f(x_2)$, and this shows $f(x)$ is increasing on $[0, 1]$. To see $f_k(x)$ is increasing on $[0, 1]$, we use induction again. For the base case, it is trivial that $f_0(x) = x$ is increasing on $[0, 1]$. Suppose for some $k \geq 0$, $f_k(x)$ is increasing on $[0, 1]$. For $x_1, x_2 \in [0, \frac{1}{3}]$, suppose $x_1 \leq x_2$, then $3x_1 \leq 3x_2$ and since $f_k(x)$ is increasing on $[0, 1]$, $f_k(3x_1) \leq f_k(3x_2)$. Since $f_k(x) \in [0, 1]$, we have $f_{k+1}(x_1) \leq f_{k+1}(x_2)$ by definition of $f_{k+1}(x)$ on $[0, \frac{1}{3}]$. This shows $f_{k+1}(x)$ is increasing on $[0, \frac{1}{3}]$ and since $f_{k+1}(\frac{1}{3}) = \frac{1}{2}$, we have $f_{k+1}(x_1) \leq f_{k+1}(x_2)$ if $x_1 \in [0, \frac{1}{3}]$ and $x_2 \in [\frac{1}{3}, \frac{2}{3}]$. Now for $x_1, x_2 \in [\frac{2}{3}, 1]$, suppose $x_1 \leq x_2$, $3x_1 - 2 \leq 3x_2 - 2$. Since $3x_1 - 2$ and $3x_2 - 2$ are both in $[0, 1]$, $f_k(3x_1 - 2) \leq f_k(3x_2 - 2)$. This shows $f_{k+1}(x_1) \leq f_{k+1}(x_2)$ for $x_1, x_2 \in [\frac{2}{3}, 1]$ and thus $f_{k+1}(x)$ is increasing on $[\frac{2}{3}, 1]$. Note that $f_{k+1}(\frac{2}{3}) = \frac{1}{2}$, so if $x_1 \in [0, \frac{2}{3}]$ and $x_2 \in [\frac{2}{3}, 1]$, we have $f_{k+1}(x_1) \leq f_{k+1}(x_2)$. In conclusion, we have shown $f_{k+1}(x)$ is increasing on $[0, 1]$ and this finishes our induction.

$\square$

Next we prove a lemma that reveals the relation between Cantor function $f(x)$ and the Cantor set C we defined in the previous section.

Lemma 1.2

Use the same notation as Definition 1.4, and define $G_k = F_{k-1} \setminus F_k$ for $k \geq 1$. Since G_n consists of 2^{n-1} disjoint subintervals, so we label them in ascending order of their left end point and denote the $m-$th subinterval as G_n^m for $m = 1, \dots, 2^{n-1}$. Then for every n and m, $f(x) = \frac{2m-1}{2^n}$ on G_n^m.

♡

Proof By the proof of Exercise 1.4, it is not hard to see for any $x \in G_k$, we can write $x = \sum_{n=1}^{\infty} a_n 3^{-n}$, where $a_n \in \{0, 2\}$ for $1 \leq n \leq k-1$, $a_k = 1$, and $a_n \in \{0, 1, 2\}$ for $n \geq k+1$ excluding the case $a_n = 0$ and the case $a_n = 2$ for all $n \geq k+1$. Then we can observe that $G_{k+1} = (\frac{1}{3}G_k) \cup (\frac{2}{3} + \frac{1}{3}G_k)$ for all $k \geq 1$. In this case, $G_{n+1}^m = \frac{1}{3}G_n^m$ if $m \leq 2^{n-1}$ and $G_{n+1}^m = \frac{2}{3} + \frac{1}{3}G_n^{m-2^{n-1}}$ if $2^{n-1}+1 \leq m \leq 2^n$. We claim that for each fixed k, $f_k(x) = \frac{2m-1}{2^n}$ for all $x \in G_n^m$ for $m = 1, \dots .2^{n-1}$, whenever $n \leq k$. We use induction on k, for $k = 1$, it is trivial that $f_1(x) = \frac{1}{2}$ for $x \in G_1^1$. Suppose this is true for some k, we need to prove $f_{k+1}(x) = \frac{2m-1}{2^n}$

for all $x \in G_n^m$ for $m = 1, \ldots .2^{n-1}$, whenever $n \leq k+1$. When $n = 1$, this is true by definition of $f_{k+1}(x)$ on $[\frac{1}{3}, \frac{2}{3}]$. When $2 \leq n \leq k+1$, for $m \leq 2^{n-2}$, any $x \in G_n^m$ is less than $\frac{1}{3}$, so $f_{k+1}(G_n^m) = \frac{1}{2} f_k(3G_n^m) = \frac{1}{2} f_k(G_{n-1}^m)$. Since $n - 1 \leq k$, we can use induction hypothesis, $f_k(G_{n-1}^m) = \frac{2m-1}{2^{n-1}}$, and thus $f_{k+1}(G_n^m) = \frac{2m-1}{2^n}$ as desired. For $2^{n-2} + 1 \leq m \leq 2^{n-1}$, any $x \in G_n^m$ is larger than $\frac{2}{3}$, so $f_{k+1}(G_n^m) = \frac{1}{2} + \frac{1}{2} f_k(3G_n - 2) = \frac{1}{2} + \frac{1}{2} f_k(G_{n-1}^{m-2^{n-2}})$. By induction hypothesis, $f_k(G_{n-1}^{m-2^{n-2}}) = \frac{2m-2^{n-1}-1}{2^{n-1}} = \frac{2m-1}{2^{n-1}} - 1$, and thus $f_{k+1}(G_n^m) = \frac{2m-1}{2^n}$ as desired. This finishes our induction and the claim is proved. Therefore, consider for any n and m, for every large enough k s.t. $n \leq k$, $f_k(G_n^m) = \frac{2m-1}{2^n}$, so we can take limit as $k \to \infty$ on both sides and we will obtain $f(G_n^m) = \frac{2m-1}{2^n}$. $\square$

We can see the Cantor function $f(x)$ is increasing but not strictly increasing, so it is not injective. We want to define an injective function which inherits its property. Thus, let $g(x) = x + f(x)$, then by the properties we proved above, $g(x)$ is continuous on $[0, 1]$, $g(0) = 0$, and $g(1) = 2$. Also, $g(x)$ is strictly increasing on $[0, 1]$. By intermediate value property of continuous function, $g([0, 1]) = [0, 2]$.

✍ **Exercise 1.8** The inverse function of g, denoted as g^{-1} exists on $[0, 2]$. Moreover, g^{-1} is continuous on $[0, 2]$.

Proof Since strictly increasing function on $[0, 1]$ must be injective and because g is surjective to $[0, 2]$, we can conclude g is bijective between $[0, 1]$ and $[0, 2]$, so g^{-1} exists on $[0, 2]$. To see g^{-1} is continuous, actually we don't need to use the continuity condition of g. We can first prove g^{-1} is also increasing. This is because for any $0 \leq y_1 < y_2 \leq 2$, there exists unique $0 \leq x_1 < x_2 \leq 1$ s.t. $g(x_1) = y_1$, $g(x_2) = y_2$, so $g^{-1}(y_1) = x_1 < x_2 = g^{-1}(y_2)$.

To prove for every $y_0 \in [0, 2]$, $g^{-1}(y)$ is continuous at $y = y_0$, we can divide y_0 into three cases, that is, $y_0 \in (0, 2)$, $y_0 = 0$ and $y_0 = 2$. If $y_0 \in (0, 2)$, denote $x_0 = g^{-1}(y_0) \in (0, 1)$, and for $\epsilon > 0$, there exists $x_1, x_2 \in [0, 1]$ s.t. $x_0 - \epsilon < x_1 < x_0 < x_2 < x_0 + \epsilon$. Since g is strictly increasing, $g(x_1) = y_1 < y_0 < y_2 = g(x_2)$. Take $\delta > 0$ small enough s.t. $y_1 < y_0 - \delta < y_0 < y_0 + \delta < y_2$. Then if $y \in [0, 2]$ and $|y - y_0| < \delta$, we have $y_1 < y < y_2$. Since g^{-1} is also increasing, we have $x_1 < g^{-1}(y) < x_2$, so $|g^{-1}(y) - g^{-1}(y_0)| < \epsilon$. This shows $g^{-1}(y)$ is continuous at y_0. If $y_0 = 0$, then $x_0 = 0$ and we only consider the RHS of it, i.e., $x_0 < x_2 < x_0 + \epsilon$. If $y_0 = 2$, then $x_0 = 1$ and we only consider the LHS of it, i.e., $x_0 - \epsilon < x_1 < x_0$. The details are omitted. $\square$

The next lemma shows this function g actually maps Cantor set (whose measure is zero by Problem Set 1.4, Question 7.) to a set with positive measure. Therefore, not all continuous function maps a set with zero measure to a set with zero measure.

Lemma 1.3

Let C be the Cantor set and $O = [0,1] \setminus C$. Then $g(O) \in \mathcal{M}$, $g(C)$ is closed and $m(g(C)) = 1$, where $f(x)$ is the Cantor function and $g(x) = x + f(x)$. ♡

Proof Use the same notation as in Lemma 1.2, $O = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} G_n^m$. Thus, $g(O) = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} g(G_n^m)$. Since $g(x) = x + f(x)$ and $f(G_n^m)$ is a constant $\frac{2m-1}{2^n}$, we have $g(G_n^m) = G_n^m + \frac{2m-1}{2^n}$. It is easy to see the translation of an open set is still an open set, so $g(G_n^m)$ is an open set. $g(O)$ is the countable union of open sets, so it is also open and measurable. Since $g([0,1]) = g(O \cup C) = g(O) \cup g(C) = [0,2]$, $g(C)$ is closed and measurable. By additivity of measurable sets, $m([0,2]) = m(g(O)) + m(g(C)) = 2$, so it suffices to show $m(g(O)) = 1$. Since G_n^m's are pairwise disjoint, and $g(x)$ is strictly increasing, so $g(G_n^m)$'s are also pairwise disjoint. Thus, by σ-subadditivity, $m(g(O)) = \sum_{n=1}^{\infty} \sum_{m=1}^{2^n-1} m(g(G_n^m)) = \sum_{n=1}^{\infty} \sum_{m=1}^{2^n-1} m(G_n^m + \frac{2m-1}{2^n})$. By translation invariance of outer measure, $m(g(O)) = \sum_{n=1}^{\infty} \sum_{m=1}^{2^n-1} m(G_n^m) = m(O) = 1 - m(C) = 1$, where $m(C) = 0$ is proved in Problem Set 1.4, Question 7.. □

The following lemma tells us a nice property of Borel measurable set and continuous function.

Lemma 1.4

Suppose $h(x)$ is continuous on $[a,b]$. Let $B \subset \mathbb{R}$ be Borel measurable set ($B \in \mathcal{B}$). Then the preimage of B under h is also Borel measurable. ♡

Proof Let $\Lambda = \{E \subset \mathbb{R} \,|\, h^{-1}(E) \in \mathcal{B}\}$, we want to show $B \in \Lambda$. By definition of continuous function, for all open sets G in $\mathbb{R}$, $h^{-1}(G)$ is open, hence Borel measurable. This shows any open set G is in Λ. Now it suffices to show Λ is a σ-algebra, because if so, then Λ is a σ-algebra containing all open sets in $\mathbb{R}$, and since $\mathcal{B}$ is the smallest σ-algebra contains all open sets in $\mathbb{R}$, $\mathcal{B}$ must be a subset of Λ. Since we always have $h^{-1}(\bigcup_{k=1}^{\infty} E_k) = \bigcup_{k=1}^{\infty} h^{-1}(E_k)$, if E_k's are in Λ, $h^{-1}(E_k) \in \mathcal{B}$ for all $k \in \mathbb{N}^+$, then since $\mathcal{B}$ is closed under countable union, $h^{-1}(\bigcup_{k=1}^{\infty} E_k) \in \mathcal{B}$, so $\bigcup_{k=1}^{\infty} E_k \in \Lambda$. By a similar idea, due to the fact that $h^{-1}(\bigcap_{k=1}^{\infty} E_k) = \bigcap_{k=1}^{\infty} h^{-1}(E_k)$, Λ is closed under intersection. Finally, for $E \in \Lambda$, $h^{-1}(E^c) = [a,b] \setminus h^{-1}(E)$. Since $[a,b]$ and $h^{-1}(E)$ are both Borel measurable, $h^{-1}(E^c) \in \mathcal{B}$. This shows Λ is also closed under complement, so it is a σ-algebra. □

Finally, after so much arduous preparation, we can construct a set which is Lebesgue measurable but not Borel measurable.

Theorem 1.4

There is a set in $\mathcal{M}$ but not in $\mathcal{B}$, i.e., $\mathcal{B} \subsetneq \mathcal{M}$. ♡

Proof Since $m(g(C)) = 1$ in Lemma 1.3, by the remark of Theorem 1.3, there exists a set $S \subset g(C)$ s.t. $S \notin \mathcal{M}$. Notice that $g^{-1}(S) \subset C$, so $m^*(g^{-1}(S)) \leq m(C) = 0$ and thus

$m^*(g^{-1}(S)) = 0$, meaning that $g^{-1}(S) \in \mathcal{M}$. We claim that $g^{-1}(S)$ is not Borel measurable. Suppose it is, since g^{-1} is continuous on $[0, 2]$ by Exercise 1.8, we can take $h = g^{-1}$ in Lemma 1.4, and thus $h^{-1}(g^{-1}(S)) = g(g^{-1}(S)) = S$ is Borel measurable. However, if S is Borel measurable then it must be Lebesgue measurable, which is a contradiction. This shows $g^{-1}(S)$ is not Borel measurable. Therefore, $g^{-1}(S)$ is the desired Lebesgue measurable set that is not Borel measurable. □

Remark Since $g^{-1}(S)$ is a Lebesgue measurable set and h^{-1} is continuous, the proof also shows that continuous function may map a Lebesgue measurable set to non-Lebesgue measurable set.

Consider the above phenomenon, a natural question is: what kind of "nicer" function will always map a set with zero measure to a set with zero measure and map a Lebesgue measurable set to Lebesgue measurable set?

Theorem 1.5

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ be Lipschitz continuous, i.e., there exists a constant $C > 0$ s.t. $|T(x) - T(y)| \leq C|x - y|$ for all $x, y \in \mathbb{R}^n$. Then $E \in \mathcal{M}$ implies $T(E) \in \mathcal{M}$ and $m(E) = 0$ implies $m(T(E)) = 0$. ♡

Proof We first prove that if $m(E) = 0$, then $m(T(E)) = 0$. Since $m(E) = 0$, for any $\epsilon > 0$, there exists L-covering $\{R_k\}_{k=1}^{\infty}$ of E s.t. $\sum_{k=1}^{\infty} |R_k| \leq \epsilon$. Notice that here R_k can be closed cubes by the second remark after Definition 1.6. Thus, $T(E) \subset \bigcup_{k=1}^{\infty} T(R_k)$. Define the diameter of a set S to be $\mathrm{diam}(S) = \sup\{|x - y| : x, y \in S\}$. Since for any $x, y \in R_k$, $|T(x) - T(y)| \leq C|x - y|$, $\mathrm{diam}(T(R_k)) \leq C\mathrm{diam}(R_k)$. Take arbitrary point x in $T(R_k)$, then the closed ball centered at x with radius $C\mathrm{diam}(R_k)$ must cover $T(R_k)$. Thus, we can construct a closed cube R_k' with edge length $2C\mathrm{diam}(R_k)$ s.t. R_k' covers the closed ball, and covers $T(R_k)$. Notice that the diameter of R_k' is $2\sqrt{n}C\mathrm{diam}(R_k)$. This shows $m^*(T(R_k)) \leq m^*(R_k') \leq (2C\mathrm{diam}(R_k))^n = (2C\sqrt{n})^n \left(\frac{\mathrm{diam}(R_k)}{\sqrt{n}}\right)^n = C'|R_k|$, where C' is a constant. Therefore, $m^*(T(E)) \leq C' \sum_{k=1}^{\infty} |R_k| \leq C'\epsilon$. Take $\epsilon \to 0$, we have $m^*(T(E)) = 0$, so $T(E)$ is measurable and $m(T(E)) = 0$.

Then we prove if $E \in \mathcal{M}$, $T(E) \in \mathcal{M}$. If A is compact, then $T(A)$ is also compact, so $T(A) \in \mathcal{M}$. If A is closed, then let $A_k = A \cap B_k(0)$ for all $k \geq 1$, then since A_k is compact, $T(A_k) \in \mathcal{M}$, we obtain $T(A) = \bigcup_{k=1}^{\infty} T(A_k) \in \mathcal{M}$. For general measurable set E, by Theorem 1.1, there exists F_σ set $A \subset E$ s.t. $m^*(Z) = 0$ for $Z = E \setminus A$. Since $A = \bigcup_{k=1}^{\infty} F_k$ where F_k is closed, $T(F_k) \in \mathcal{M}$. Note that $T(E) = T(A) \cup T(Z)$. By what we proved just now, since $m^*(Z) = 0$, $T(Z) \in \mathcal{M}$; Since $T(A) = \bigcup_{k=1}^{\infty} T(F_k) \in \mathcal{M}$, we finally have $T(E) \in \mathcal{M}$. □

Problem Set 1.6

1. Define $f : [a, b] \mapsto \mathbb{R}$ such that for all $E \subset [a, b]$ and $E \in \mathcal{M}$, we have $f(E) \in \mathcal{M}$. Prove that for all $Z \subset [a, b]$ with $m(Z) = 0$, we have $m(f(Z)) = 0$.

Chapter 2 Lebesgue Measurable Functions

2.1 Lebesgue Measurable Functions

In this section, we are going to introduce the concept of (Lebesgue) measurable function. However, for that purpose, we need to first introduce **extended real-valued functions**, that is, a function that can take $\pm\infty$ as its function values. Also, we need to make some agreement on the arithmetics of $\pm\infty$ with real numbers:

1. For all $x \in \mathbb{R}^+$, $x \cdot (+\infty) = +\infty$ and $x \cdot (-\infty) = -\infty$; for all $x \in \mathbb{R}^-$, $x \cdot (+\infty) = -\infty$, and $x \cdot (-\infty) = -\infty$.
2. $0 \cdot (+\infty) = 0$ and $0 \cdot (-\infty) = 0$.
3. $(+\infty) + (+\infty) = +\infty$ and $(-\infty) - (+\infty) = -\infty$.
4. $(+\infty) \cdot (+\infty) = +\infty$, $(+\infty) \cdot (-\infty) = -\infty$, and $(-\infty) \cdot (-\infty) = +\infty$.
5. $+\infty - (+\infty)$ and $-\infty - (-\infty)$ are not allowed and $+\infty$ can be abbreviated as ∞.

Definition 2.1. Lebesgue Measurable Function

*Let $f(x)$ be an extended real-valued function defined on a Lebesgue measurable set $E \subset \mathbb{R}^n$. We say f is **measurable** on E if for all $t \in \mathbb{R}$, $\{x \in E \,|\, f(x) > t\} \in \mathcal{M}$.* ♣

Now we list some useful and general identities, and notice that the following identities hold even if f is not measurable. The proof of them is left for you as exercise because they only involve very basic set theory knowledge.

1. $\{x \in E \,|\, f(x) \leq t\} = E \setminus \{x \in E \,|\, f(x) > t\}$
2. $\{x \in E \,|\, f(x) \geq t\} = \bigcap_{k=1}^{\infty} \{x \in E \,|\, f(x) > t - \frac{1}{k}\}$
3. $\{x \in E \,|\, f(x) < t\} = E \setminus \{x \in E \,|\, f(x) \geq t\}$
4. $\{x \in E \,|\, f(x) = t\} = \{x \in E \,|\, f(x) \geq t\} \cap \{x \in E \,|\, f(x) \leq t\}$
5. $\{x \in E \,|\, f(x) < \infty\} = \bigcup_{k=1}^{\infty} \{x \in E \,|\, f(x) < k\}$
6. $\{x \in E \,|\, f(x) = \infty\} = E \setminus \{x \in E \,|\, f(x) < \infty\}$
7. $\{x \in E \,|\, f(x) > -\infty\} = \bigcup_{k=1}^{\infty} \{x \in E \,|\, f(x) > -k\}$
8. $\{x \in E \,|\, f(x) = -\infty\} = E \setminus \{x \in E \,|\, f(x) > -\infty\}$

Theorem 2.1

If f is measurable on $E \in \mathcal{M}$, then the left hand side of the above identities are all in $\mathcal{M}$. ♡

Proof By definition, for all $t \in \mathbb{R}$, $\{x \in E \,|\, f(x) > t\} \in \mathcal{M}$. Since $E \in \mathcal{M}$, $E \setminus \{x \in E \,|\, f(x) > t\} \in \mathcal{M}$, so $\{x \in E \,|\, f(x) \leq t\} \in \mathcal{M}$. Since t is arbitrary, $\{x \in E \,|\, f(x) > t - \frac{1}{k}\} \in \mathcal{M}$, and thus $\bigcap_{k=1}^{\infty} \{x \in E \,|\, f(x) > t - \frac{1}{k}\} \in \mathcal{M}$. This shows $\{x \in E \,|\, f(x) \geq t\} \in \mathcal{M}$.

Then $E \setminus \{x \in E \,|\, f(x) \geq t\} \in \mathcal{M}$ and so $\{x \in E \,|\, f(x) < t\} \in \mathcal{M}$. Therefore, it is easy to see $\{x \in E \,|\, f(x) = t\} \in \mathcal{M}$. Again, since t is arbitrary, let $t = k$, we have $\{x \in E \,|\, f(x) < k\} \in \mathcal{M}$, so $\bigcup_{k=1}^{\infty}\{x \in E \,|\, f(x) < k\} \in \mathcal{M}$ and $\{x \in E \,|\, f(x) < \infty\} \in \mathcal{M}$. Thus, $\{x \in E \,|\, f(x) = \infty\} = E \setminus \{x \in E \,|\, f(x) < \infty\}$ is in $\mathcal{M}$. Similarly, we can prove $\{x \in E \,|\, f(x) > -\infty\}$ and $\{x \in E \,|\, f(x) = -\infty\}$ are in $\mathcal{M}$. □

Theorem 2.2

Let D be a dense subset of $\mathbb{R}$, then f is measurable on $E \in \mathcal{M}$ if and only if for all $d \in D$, $\{x \in E \,|\, f(x) > d\} \in \mathcal{M}$. ♡

Proof If f is measurable on $E \in \mathcal{M}$, then for all $t \in \mathbb{R}$, we have $\{x \in E \,|\, f(x) > t\} \in \mathcal{M}$, so the desired result is trivial. Suppose for all $d \in D$, $\{x \in E \,|\, f(x) > d\} \in \mathcal{M}$. Since D is dense, for all $t \in \mathbb{R}$, there exists decreasing sequence d_n convergent to t as $n \to \infty$. Note that $\{x \in E \,|\, f(x) > t\} = \bigcup_{n=1}^{\infty}\{x \in E \,|\, f(x) > d_n\} \in \mathcal{M}$, so f is measurable. □

Next we display some basic facts about measurable functions in the following exercises. These facts are fundamental and also very handy for you to determine whether a function is measurable or not without using definition directly.

✍ **Exercise 2.1** Suppose f is measurable on $E_1 \in \mathcal{M}$ and $E_2 \in \mathcal{M}$ separately, then f is measurable on $E_1 \cup E_2$.

Proof For all $t \in \mathbb{R}$, $\{x \in E_1 \cup E_2 \,|\, f(x) > t\} = \{x \in E_1 \,|\, f(x) > t\} \cup \{x \in E_2 \,|\, f(x) > t\}$. Therefore, by using definition, it is easy to see the desired result. □

✍ **Exercise 2.2** If f is measurable on $E \in \mathcal{M}$, then for all $A \subset E$, $A \in \mathcal{M}$, f is also measurable on A.

Proof For all $t \in \mathbb{R}$, $\{x \in A \,|\, f(x) > t\} = A \cap \{x \in E \,|\, f(x) > t\}$. Therefore, by using definition, it is easy to see the desired result. □

✍ **Exercise 2.3** Suppose f and g are measurable on $E \in \mathcal{M}$, then $\{x \in E \,|\, f(x) > g(x)\} \in \mathcal{M}$.

Proof Note that $\{x \in E \,|\, f(x) > g(x)\} = \bigcup_{n=1}^{\infty}\{x \in E \,|\, f(x) > r_n > g(x)\}$, where $\mathbb{Q} = \{r_n\}_{n=1}^{\infty}$. Furthermore,

$$\{x \in E \,|\, f(x) > r_n > g(x)\} = \{x \in E \,|\, f(x) > r_n\} \cap \{x \in E \,|\, g(x) < r_n\}$$

By Theorem 2.1, since f is measurable on E, $\{x \in E \,|\, f(x) > r_n\} \in \mathcal{M}$; since g is measurable on E, $\{x \in E \,|\, g(x) < r_n\} \in \mathcal{M}$. This shows $\{x \in E \,|\, f(x) > r_n > g(x)\} \in \mathcal{M}$ and hence $\{x \in E \,|\, f(x) > g(x)\} \in \mathcal{M}$. □

✍ **Exercise 2.4** Suppose f is measurable on $E \in \mathcal{M}$, then $cf(x)$ and $f(x) + c$ are also measurable on E, where $c \in \mathbb{R}$ is a constant.

Proof If $c = 0$, then $\{x \in E \mid 0 > t\} = E \cap \{x \in \mathbb{R}^n \mid 0 > t\}$. For every $t \in \mathbb{R}$, if $t < 0$, then $\{x \in \mathbb{R}^n \mid 0 > t\} = \mathbb{R}^n$; if $t \geq 0$, then $\{x \in \mathbb{R}^n \mid 0 > t\} = \varnothing$. Since $E \cap \mathbb{R}^n = E$ and $E \cap \varnothing = \varnothing$ are both measurable, 0 as a constant function is measurable on E. If $c \neq 0$, then for every $t \in \mathbb{R}$, $\{x \in E \mid cf(x) > t\} = \{x \in E \mid f(x) > \frac{t}{c}\} \in \mathcal{M}$, so $cf(x)$ is measurable on E. For all $t \in \mathbb{R}$, since $\{x \in E \mid f(x) + c > t\} = \{x \in E \mid f(x) > t - c\} \in \mathcal{M}$ when f is measurable on E, we know $f(x) + c$ is also measurable on E. □

✍ **Exercise 2.5** Suppose f and g are measurable on $E \in \mathcal{M}$, then $f + g$ is measurable on E.

Proof By Exercise 2.4, take $c = -1$, we can conclude $-g(x)$ is measurable. For each fixed $t \in \mathbb{R}$, $-g(x) + t$ is measurable, so by Exercise 2.3, $\{x \in E \mid f(x) > -g(x) + t\} \in \mathcal{M}$. However, $\{x \in E \mid f(x) > -g(x) + t\} = \{x \in E \mid f(x) + g(x) > t\}$, so we proved $f + g$ is measurable on E if $f + g$ is well defined on E. □

✍ **Exercise 2.6** Suppose f is measurable on $E \in \mathcal{M}$, then for any constant $p > 0$, $|f|^p$ is measurable on E.

Proof For $t < 0$, $\{x \in E \mid |f(x)|^p > t\} = E \in \mathcal{M}$; for $t \geq 0$,

$$\{x \in E \mid |f(x)|^p > t\} = \{x \in E \mid f(x) > t^{1/p}\} \cup \{x \in E \mid f(x) < -t^{1/p}\}$$

Since f is measurable on E, $\{x \in E \mid f(x) < -t^{1/p}\}$ and $\{x \in E \mid f(x) > t^{1/p}\}$ are in $\mathcal{M}$ by Definition 2.1 and Theorem 2.1. Therefore, $\{x \in E \mid |f(x)|^p > t\} \in \mathcal{M}$ for all $t \in \mathbb{R}$ and hence $|f|^p$ is measurable on E. □

✍ **Exercise 2.7** Suppose f and g are measurable on $E \in \mathcal{M}$, then $f(x)g(x)$ is measurable on E.

Proof For all $t \in \mathbb{R}$, $\{x \in E \mid f(x)g(x) > t\} = A \cup B \cup C \cup D$, where

$$A = \{x \in E \mid f(x)g(x) > t, |f(x)| < \infty, |g(x)| < \infty\}$$

$$B = \{x \in E \mid f(x)g(x) > t, |f(x)| = \infty, |g(x)| < \infty\}$$

$$C = \{x \in E \mid f(x)g(x) > t, |f(x)| < \infty, |g(x)| = \infty\}$$

$$D = \{x \in E \mid f(x)g(x) > t, |f(x)| = \infty, |g(x)| = \infty\}$$

We need to prove A, B, C, D are all in $\mathcal{M}$. For A, since $f(x)$ and $g(x)$ are finite, we can write $fg = \frac{1}{4}[(f+g)^2 - (f-g)^2]$. By Exercise 2.5 and Exercise 2.6, we can conclude fg is measurable on $A_1 = \{x \in E \mid |f(x)| < \infty\} \cap \{x \in E \mid |g(x)| < \infty\}$. By Theorem 2.1, $\{x \in E \mid |f(x)| < \infty\} = \{x \in E \mid f(x) < \infty\} \cup \{x \in E \mid f(x) > -\infty\} \in \mathcal{M}$. Similarly, $\{x \in E \mid |g(x)| < \infty\} \in \mathcal{M}$. Thus, fg is measurable on $A_1 \in \mathcal{M}$. Therefore, $A = \{x \in A_1 \mid f(x)g(x) > t\} \in \mathcal{M}$.

For D, If $f(x)g(x) = -\infty$, then it is impossible that $f(x)g(x) > t$; if $f(x)g(x) = \infty$, then

$f(x)g(x) > t$ is always true. Therefore, $D = D_1 \cup D_2$, where

$$D_1 = \{x \in E \,|\, f(x) = g(x) = \infty\} = \{x \in E \,|\, f(x) = \infty\} \cap \{x \in E \,|\, g(x) = \infty\} \in \mathcal{M}$$

$$D_2 = \{x \in E \,|\, f(x) = g(x) = -\infty\} = \{x \in E \,|\, f(x) = -\infty\} \cap \{x \in E \,|\, g(x) = -\infty\} \in \mathcal{M}$$

This finishes the proof of $D \in \mathcal{M}$.

For B and C, we only prove $B \in \mathcal{M}$ here, because $C \in \mathcal{M}$ can be proved in exactly the same way. Note $B = B_1 \cup B_2 \cup B_3$, where

$$B_1 = \{x \in E \,|\, 0 > t, |f(x)| = \infty, g(x) = 0\}$$

$$B_2 = \{x \in E \,|\, 0 > t, f(x) = \infty, g(x) > 0\}$$

$$B_3 = \{x \in E \,|\, 0 > t, f(x) = -\infty, g(x) < 0\}$$

It is easy to see $B_1, B_2, B_3 \in \mathcal{M}$. For example, B_1 can be further decomposed into

$$B_1 = \{x \in E \,|\, 0 > t\} \cap \{x \in E \,|\, |f(x)| = \infty\} \cap \{x \in E \,|\, g(x) = 0\} \in \mathcal{M}$$

Therefore, by using such decomposition, we can also prove $B_2, B_3 \in \mathcal{M}$, so $B \in \mathcal{M}$. In conclusion, A, B, C, D are all in $\mathcal{M}$ and hence $f(x)g(x)$ is measurable on E. □

Remark When we encounter a complicated set, we can decompose it into intersection or union of several simple sets, and we try to prove each simple set is in $\mathcal{M}$. If so, using the property of $\mathcal{M}$, we can prove the orginal complicated set is in $\mathcal{M}$.

✍ **Exercise 2.8** Suppose f and g are measurable on $E \in \mathcal{M}$. If $g(x) \neq 0$ on E, then $\frac{f(x)}{g(x)}$ is measurable on E.

Proof Note for all $t \in \mathbb{R}$, $\{x \in E \,|\, \frac{f(x)}{g(x)} > t\} = A \cup B$, where

$$A = \{x \in E \,|\, f(x) > tg(x), g(x) > 0\}, \quad B = \{x \in E \,|\, f(x) < tg(x), g(x) < 0\}$$

Furthermore, we can write

$$A = \{x \in E \,|\, g(x) > 0\} \cap \{x \in E \,|\, f(x) > tg(x)\} = A_1 \cap A_2$$

By Theorem 2.1, $A_1 \in \mathcal{M}$. By Exercise 2.4, $tg(x)$ is a measurable function on E for each fixed t. Thus, by Exercise 2.3, $A_2 \in \mathcal{M}$, and so $A \in \mathcal{M}$. Similarly, we can show $B \in \mathcal{M}$. Hence $\{x \in E \,|\, \frac{f(x)}{g(x)} > t\} \in \mathcal{M}$, and $\frac{f(x)}{g(x)}$ is measurable on E. □

✍ **Exercise 2.9** Suppose f is continuous on $\mathbb{R}$ and g is measurable on $E \in \mathcal{M}$. Then $(f \circ g)(x) = f(g(x))$ is measurable on E.

Proof For all $t \in \mathbb{R}$, let $A_t = f^{-1}((t, \infty))$. Since f is continuous, A_t must be an open set in $\mathbb{R}$. By Problem 1.1, $A_t = \bigcup_{k=1}^{\infty}(a_k^t, b_k^t)$, where (a_k^t, b_k^t)'s are pairwise disjoint open intervals. Notice that

$$\{x \in E \,|\, f(g(x)) > t\} = \{x \in E \,|\, g(x) \in A_t\} = \bigcup_{k=1}^{\infty} \{x \in E \,|\, a_k^t < g(x) < b_k^t\}$$

and for each $k \geq 1$,

$$\{x \in E \,|\, a_k^t < g(x) < b_k^t\} = \{x \in E \,|\, g(x) < b_k^t\} \cap \{x \in E \,|\, g(x) > a_k^t\}$$

By Theorem 2.1, $\{x \in E \,|\, g(x) < b_k^t\} \in \mathcal{M}$ and $\{x \in E \,|\, g(x) > a_k^t\} \in \mathcal{M}$, we can see $\{x \in E \,|\, a_k^t < g(x) < b_k^t\} \in \mathcal{M}$. Therefore, the fact that $\mathcal{M}$ is closed under countable union implies $\{x \in E \,|\, f(g(x)) > t\} \in \mathcal{M}$ and so $f(g(x))$ is measurable on E. □

Exercise 2.10 Define $f_+(x) = \max\{f(x), 0\}$ and $f_-(x) = \min\{0, f(x)\}$. Then $f(x)$ is measurable on $E \in \mathcal{M}$ if and only if $f_+(x)$ and $f_-(x)$ are both measurable on E.

Proof Suppose $f(x)$ is measurable on E. Notice that

$$f_+(x) = \frac{|f(x)| + f(x)}{2}, \quad f_-(x) = \frac{f(x) - |f(x)|}{2}$$

By Exercise 2.6 with $p = 1$, $|f(x)|$ is measurable on E. By Exercise 2.4, $-|f(x)|$ is measurable on E. By Exercise 2.5, $|f(x)| + f(x)$ and $f(x) - |f(x)|$ are both measurable on E. By Exercise 2.4 again, $\frac{|f(x)|+f(x)}{2}$ and $\frac{f(x)-|f(x)|}{2}$ are both measurable on E, so $f_+(x)$ and $f_-(x)$ are both measurable on E. Suppose $f_+(x)$ and $f_-(x)$ are both measurable on E. Notice that $f(x) = f_+(x) + f_-(x)$, so by Exercise 2.5, $f(x)$ is measurable on E. □

Exercise 2.11 Let $f_n(x)$ be measurable on E for all $n \geq 1$. Then $F(x) = \sup_{n\geq 1}\{f_n(x)\}$ and $G(x) = \inf_{n\geq 1}\{f_n(x)\}$ are both measurable on E.

Proof Notice that for all $t \in \mathcal{M}$,

$$\{x \in E \,|\, F(x) > t\} = \bigcup_{n=1}^{\infty} \{x \in E \,|\, f_n(x) > t\}$$

Therefore, since $f_n(x)$ is measurable, $\{x \in E \,|\, f_n(x) > t\} \in \mathcal{M}$ for all $n \geq 1$. This implies $\{x \in E \,|\, F(x) > t\} \in \mathcal{M}$ and hence $F(x)$ is measurable on E.

Note that $\inf_{n\geq 1}\{f_n(x)\} = -\sup_{n\geq 1}\{-f_n(x)\}$. By Exercise 2.4, $-f_n(x)$ is measurable on E for all $n \geq 1$. Then, by what we proved just now, $\sup_{n\geq 1}\{-f_n(x)\}$ is measurable on E. Apply Exercise 2.4 again, we can see $G(x)$ is measurable on E. □

Exercise 2.12 Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable function on $E \in \mathcal{M}$. Then $F(x) = \underline{\lim}_{n\to\infty} f_n(x)$ and $G(x) = \overline{\lim}_{n\to\infty} f_n(x)$ are measurable on E.

Proof Note that $F(x) = \inf_{m\geq 1} \sup_{n\geq m} f_n(x)$, so if we let $F_m(x) = \sup_{n\geq m} f_n(x)$, then by Exercise 2.11, $F_m(x)$ is measurable on E for each $m \geq 1$. Since $F(x) = \inf_{m\geq 1} F_m(x)$, apply Exercise 2.11 again, $F(x)$ is measurable on E. Similarly, $G(x) = \sup_{m\geq 1} \inf_{n\geq m} f_n(x)$, so we can use Exercise 2.11 twice to prove $G(x)$ is measurable on E. □

Remark In particular, if $F(x) = \lim_{n\to\infty} f_n(x)$ exists on E, then $F(x)$ is measurable on E. This because if the limit exists, then $F(x) = \overline{\lim}_{n\to\infty} f_n(x) = \underline{\lim}_{n\to\infty} f_n(x)$ and Exercise 2.12 can be applied.

Problem Set 2.1

1. Let f be defined on $E \in \mathcal{M}$, and f be finite on E. Prove that the following are equivalent:
   (a). f is measurable on E;
   (b). $f^{-1}(G) \in \mathcal{M}$ for all open set $G \subset \mathbb{R}$.
   (c). $f^{-1}(F) \in \mathcal{M}$ for all closed set $F \subset \mathbb{R}$.
   (d). $f^{-1}(B) \in \mathcal{M}$ for all Borel set $B \subset \mathbb{R}$.
2. Prove that monotone increasing function defined on $[a, b]$ is measurable.
3. Let f be defined on $[a, b]$. Suppose for all $[\alpha, \beta] \subset (a, b)$, f is measurable on $[\alpha, \beta]$. Prove f is measurable on $[a, b]$.
4. Let f be differentiable on $[a, b]$. Prove that $f'(x)$ is also measurable on $[a, b]$.
5. Define $f : \mathbb{R}^2 \mapsto \mathbb{R}$ such that $f(x, y)$ is a measurable function of $x \in \mathbb{R}$ for each fixed y. Also, for each fixed x, f is a continuous function of $y \in \mathbb{R}$. Define $F(x) = \max_{y\in[0,1]} f(x, y)$. Prove that $F(x)$ is measurable on $\mathbb{R}$.
6. Let $E \subset \mathbb{R}^n$. Prove that $E \in \mathcal{M}$ if and only if $I_E(x)$ is measurable on $\mathbb{R}^n$, where $I_E(x)$ is the indicator function (see Definition 2.3) of set E.
7. Let $f(x)$ be real-valued and measurable on $E \in \mathcal{M}$ with $m(E) < \infty$. Prove that for all $\epsilon > 0$, there exists bounded measurable function $g(x)$ defined on E such that $m(\{x \in E \,|\, f(x) \neq g(x)\}) < \epsilon$.
8. Construct an example in which f is measurable and g is continuous, but $f \circ g$ is not measurable.

2.2 Simple Approximation

Definition 2.2. Simple Function

*Let $f(x)$ be an extended real-valued function on $E \in \mathcal{M}$. If $f(E)$ is a finite set, i.e., $f(E) = \{y_1, \ldots, y_p\}$ where $y_i \neq y_j$ for $1 \leq i \neq j \leq p$, then f is a **simple function**.* ♣

Definition 2.3. Characteristic Function

*Let $E \subset \mathbb{R}^n$, then the **characteristic function** (indicator function) of E is defined to be*

$$I_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

♣

Remark Simple function can be written as linear combination of characteristic function of pairwise disjoint sets. Let $f(x)$ is a simple function on E, and $f(E) = \{y_1, \ldots, y_p\}$ where y_i's are distinct values for $i = 1, \ldots, p$. Then define $E_i = f^{-1}(y_i)$ for $i = 1, \ldots, p$, and $E = \bigcup_{i=1}^{p} E_i$ where E_i's are pairwise disjoint. Thus, $f(x) = \sum_{i=1}^{p} y_i I_{E_i}(x)$ for $x \in E$. Furthermore, on different E_i's, $f(x)$ has different values.

Definition 2.4. Measurable Simple Function

If f is measurable on E and f is a simple function, then f is called ***measurable simple function****.* ♣

Exercise 2.13 A simple function f is measurable on $E \in \mathcal{M}$ if and only if $E_i \in \mathcal{M}$, where E_i's are defined in the remark of Definition 2.3.

Proof As we shown in the remark of Definition 2.3, $f(x) = \sum_{i=1}^{p} y_i I_{E_i}(x)$, where $E_i = f^{-1}(y_i)$. If f is measurable on E, then by Theorem 2.1, $E_i = \{x \in E \,|\, f(x) = y_i\} \in \mathcal{M}$. If $E_i \in \mathcal{M}$, then by Problem Set 2.1, Question 6., $I_{E_i}(x)$ is measurable on $\mathbb{R}^n$. By Exercise 2.2, $I_{E_i}(x)$ is also measurable on any measurable subset of $\mathbb{R}^n$. Thus, $I_{E_i}(x)$ is measurable on E for every $i = 1, \ldots, p$. By Exercise 2.6, $y_i I_{E_i}(x)$ is measurable on E. By applying Exercise 2.5 $p-1$ times, we can prove $\sum_{i=1}^{p} y_i I_{E_i}(x)$ is measurable on E, so $f(x)$ is measurable on E. □

Now we show the main theorem of this section, i.e., simple approximation theorem. This theorem is a fundamental theorem in measure theory. It provides a theoretical foundation for us to use simple function to explore some property and easily extend it to general measurable function.

Theorem 2.3. Simple Approximation Theorem

Suppose $f(x)$ is measurable on $E \in \mathcal{M}$.

1. *If $f(x) \geq 0$ on E, then there exists a sequence of measurable simple functions $\{\phi_k(x)\}_{k=1}^{\infty}$ s.t. for each fixed $x \in E$, $\{\phi_k(x)\}_{k=1}^{\infty}$ is a increasing sequence and $0 \leq \phi_k(x) < \infty$ for $k \in \mathbb{N}^+$, and $\phi_k(x) \to f(x)$ pointwisely. Moreover, if $|f(x)| \leq M$ for all $x \in E$, then $\phi_k(x) \to f(x)$ uniformly on E with $|\phi_k(x)| \leq M$ for all $x \in E$ and $k \in \mathbb{N}^+$.*
2. *There exists a sequence of measurable simple functions $\{\phi_k(x)\}_{k=1}^{\infty}$ s.t. $\phi_k(x) \to f(x)$ pointwisely on E and $|\phi_k(x)| < \infty$ for all $x \in E$. Moreover, if $|f(x)| \leq M$ for all $x \in E$, then $\phi_k(x) \to f(x)$ uniformly on E with $|\phi_k(x)| \leq M$ for all $x \in E$ and $k \in \mathbb{N}^+$.* ♡

Proof

1. For each fixed $k \in \mathbb{N}^+$, let $E_{kj} = \left\{x \in E \,\middle|\, \frac{j-1}{2^k} \leq f(x) < \frac{j}{2^k}\right\}$ for $j = 1, \ldots, k2^k$ and let $E_k = \{x \in E \,|\, f(x) \geq k\}$. Then for each fixed $k \in \mathbb{N}^+$, $E = E_k \cup \left(\bigcup_{j=1}^{k2^k} E_{kj}\right)$. Let
$$\phi_k(x) = \begin{cases} \frac{j-1}{2^k} & x \in E_{kj},\ j = 1, \ldots, k2^k \\ k & x \in E_k \end{cases}$$
Therefore, $\phi_k(x)$ is a simple function. Also, $\phi_k(x) = \sum_{j=1}^{k2^k} \frac{j-1}{2^k} I_{E_{kj}}(x) + k I_{E_k}(x)$. By construction of E_{kj} and E_k, since f is measurable on E, $E_{kj} \in \mathcal{M}$ and $E_k \in \mathcal{M}$. By using a similar argument as in the proof of Exercise 2.13, we can prove $\phi_k(x)$ is measurable on E. This shows $\{\phi_k(x)\}$ is a sequence of measurable simple functions on E s.t. for each

fixed $x \in E$, $0 \le \phi_k(x) < \infty$ for $k \in \mathbb{N}^+$. Observe that $E_{kj} = E_{(k+1)(2j-1)} \cup E_{(k+1)(2j)}$, so on E_{kj}, $\phi_{k+1}(x) \ge \frac{j-1}{2^k} = \phi_k(x)$. This shows $\phi_{k+1}(x) \ge \phi_k(x)$ on E. By construction of $\phi_k(x)$, $\phi_k(x) \le f(x)$ for all $k \ge 1$.

Now it remains to show $\phi_k(x) \to f(x)$ pointwisely on E. Fix $x_0 \in E$, then $f(x_0) - \phi_k(x_0) \ge 0$. Suppose $f(x_0) < \infty$, there exists K s.t. $f(x_0) \le k$ for all $k \ge K$. Thus, $x_0 \in E_{kj}$ for some j. Since $\phi_k(x_0) = \frac{j-1}{2^k}$ and $\frac{j-1}{2^k} \le f(x_0) < \frac{j}{2^k}$, we have $|f(x_0) - \phi_k(x_0)| \le \frac{1}{2^k} \to 0$ as $k \to \infty$. This shows $\lim_{k\to\infty} \phi_k(x_0) = f(x_0)$. Suppose $f(x_0) = \infty$, then $x_0 \in E_k$ for all $k \ge 1$. Thus, $\phi_k(x_0) = k \to \infty$ as $k \to \infty$. In this case we also have $\lim_{k\to\infty} \phi_k(x_0) = f(x_0)$. Therefore, $\phi_k(x) \to f(x)$ pointwisely on E.

Finally, suppose there exists a constant $M > 0$ s.t. $|f| \le M$ on E. In this case when $k > M$, $E_k = \varnothing$, so for all $x \in E$, if $k > M$, then $x \in E_{kj}$ for some j. Then $0 \le f(x) - \phi_k(x) \le \frac{1}{2^k}$ for all $x \in E$. Since $\frac{1}{2^k}$ is independent of x, $\phi_k(x) \to f(x)$ uniformly on E. Since $0 \le \phi_k(x) \le f(x)$, it is trivial that $|\phi_k(x)| \le M$ for all $x \in E$ and $k \in \mathbb{N}^+$.

2. For general measurable function f, recall $f_+(x)$ and $f_-(x)$ defined in Exercise 2.10, and we have $f(x) = f_+(x) + f_-(x) = f_+(x) - (-f_-(x))$. Since $f_+(x)$ and $-f_-(x)$ are both nonnegative, by part 1., there exists measurable simple functions $\varphi_k(x) \to f_+(x)$ and $\psi_k(x) \to -f_-(x)$ on E pointwisely. Therefore, $\varphi_k(x) - \psi_k(x) \to f(x)$ on E pointwisely. Let $\phi_k(x) = \varphi_k(x) - \psi_k(x)$, then it is easy to see $\phi_k(x)$ is also simple and measurable. Since both $\varphi_k(x)$ and $\psi_k(x)$ are finite on E, $\phi_k(x)$ is also finite on E.

   If $|f(x)| \le M$ for all $x \in E$, then $|f_+(x)| \le M$ and $|-f_-(x)| \le M$ on E. By part 1., $\varphi_k(x) \to f_+(x)$ uniformly and $\psi_k(x) \to -f_-(x)$ uniformly. Therefore, it is easy to see $\phi_k(x) \to f(x)$ uniformly. Finally, we need to prove $|\phi_k(x)| \le |f(x)|$ for all $x \in E$. For each $x \in E$, if $f(x) > 0$, then $f(x) = f_+(x)$ and $-f_-(x) = 0$, so by construction in part 1., we know $\psi_k(x) = 0$ for all $k \ge 1$. This shows $\phi_k(x) = \varphi_k(x)$, but by part 1., $0 \le \varphi_k(x) \le f_+(x)$, we have $|\phi_k(x)| \le |f(x)|$. If $f(x) < 0$, then $|f(x)| = -f_-(x)$ and $f_+(x) = 0$. Similarly, we have $\varphi_k(x) = 0$ for all $k \ge 1$, so $\phi_k(x) = -\psi_k(x)$. By part 1., $0 \le \psi_k(x) \le -f_-(x)$ on E, so $|\phi_k(x)| \le |f(x)| \le M$ for all $x \in E$.

□

Problem Set 2.2

1. Let I be a closed, bounded interval and E a measurable subset of I. Let $\epsilon > 0$. Show that there is a step function h on I and a measurable subset F of I for which $h = I_E$ on F and $m(I \setminus F) < \epsilon$.

2.3 Egorov's Theorem

Definition 2.5. Almost Everywhere

Let $E \subset \mathbb{R}^n$. A statement $S(x)$ involving points $x \in E$ is said to be ***almost everywhere*** *(abbreviated to* ***a.e.****) on E if there exists $Z \subset E$ with $m(Z) = 0$, and $S(x)$ is true for all $x \in E \setminus Z$.* ♣

Example 2.1 Let $f(x)$ and $g(x)$ be defined on E. Suppose $m(\{x \in E \,|\, f(x) \neq g(x)\}) = 0$, then $f(x) = g(x)$ a.e. on E.

Example 2.2 Let $f(x)$ and $g(x)$ be defined on $E \in \mathcal{M}$. Suppose $f(x)$ is measurable on E and $f(x) = g(x)$ a.e. on E. Then, $g(x)$ is also measurable on E.

Proof For all $t \in \mathbb{R}$,

$$\{x \in E \,|\, g(x) > t\} \cup Z = \{x \in E \,|\, f(x) > t\} \cup Z$$

where $Z = \{x \in E \,|\, f(x) \neq g(x)\}$ with $m(Z) = 0$. Since f is measurable, we have $\{x \in E \,|\, f(x) > t\} \in \mathcal{M}$, and thus $\{x \in E \,|\, g(x) > t\} \cup Z \in \mathcal{M}$. Notice that

$$\{x \in E \,|\, g(x) > t\} \cup Z = \{x \in E \,|\, g(x) > t\} \cup (Z \setminus \{x \in E \,|\, g(x) > t\}) = A \cup B$$

where A and B are disjoint. It is easy to see $B \in \mathcal{M}$ because $m^*(B) \leq m(Z) = 0$ implies that $m^*(B) = 0$. Thus, $A = (A \cup B) \cap B^c \in \mathcal{M}$, i.e., $\{x \in E \,|\, g(x) > t\} \in \mathcal{M}$. This shows $g(x)$ is measurable on E. □

Definition 2.6. Almost Everywhere Convergence

Let $f(x)$ and $\{f_n(x)\}_{n=1}^{\infty}$ be measurable functions defined on $E \in \mathcal{M}$. If $Z \subset E$ with $m(Z) = 0$, and for all $x \in E \setminus Z$, we have $\lim_{n \to \infty} f_n(x) = f(x)$, then we say $f_n(x) \to f(x)$ ***almost everywhere*** *on E.* ♣

Remark If we only assume $f_n(x)$'s are measurable on E, then we can still show $f(x)$ is measurable on E. This is because by the remark of Exercise 2.12, $f(x)$ is measurable on $E \setminus Z$. Also, $\{x \in Z \,|\, f(x) > t\} \subset Z$ implies that $\{x \in Z \,|\, f(x) > t\} \in \mathcal{M}$ because its outer measure is zero. This shows $f(x)$ is also measurable on Z. By Exercise 2.1, $f(x)$ is measurable on $(E \setminus Z) \cup Z = E$.

Definition 2.7. Almost Uniform Convergence

Let $f(x)$ and $\{f_n(x)\}_{n=1}^{\infty}$ be measurable functions defined on $E \in \mathcal{M}$. Assume each function is finite a.e. on E, i.e., for all $n \geq 1$, there exists $Z_n \subset E$ with $m(Z_n) = 0$ and $|f_n(x)| < \infty$ for all $x \in E \setminus Z_n$. We say $f_n(x) \to f(x)$ ***almost uniformly*** *(abbreviated to* ***a.u.****) on E as $n \to \infty$ if for all $\delta > 0$, there exists $E_\delta \subset E$ with $m^*(E_\delta) < \delta$, s.t. $f_n(x) \to f(x)$ uniformly on $E \setminus E_\delta$ as $n \to \infty$.* ♣

Example 2.3 Let $f_k(x) = x^k$ where $x \in E = [0, 1]$. Denote $f(x) = 0$ for $x \in [0, 1)$ and $f(1) = 1$. Then $f_k(x) \to f(x)$ pointwisely on E as $k \to \infty$. Note that $f_k(x)$ does not converge to $f(x)$ uniformly on E because $f(x)$ is not continuous on E. However, $f_k(x) \to f(x)$ a.u. on E. For any small $\delta > 0$, let $E_\delta = [1 - \delta/2, 1]$, then $m^*(E_\delta) < \delta$. Since $|f_k(x)| \le (1 - \delta/2)^k \to 0$ on $E \setminus E_\delta$, $f_k(x) \to 0$ uniformly on $E \setminus E_\delta$, and thus we can conclude $f_k(x) \to f(x)$ almost uniformly.

Recall in elementary mathematical analysis course, we have learnt that uniform convergence implies pointwise convergence but not vice versa. Then, one may ask what is the relationship between almost uniform convergence and almost everywhere convergence?

Theorem 2.4

Let $f(x)$ and $\{f_n(x)\}_{n=1}^\infty$ be finite a.e. and measurable on $E \in \mathcal{M}$. Suppose $f_n(x) \to f(x)$ a.u. on E, then $f_n(x) \to f(x)$ a.e. on E as $n \to \infty$. ♡

Proof By definition of a.u. convergence, for all $i \ge 1$, there exists $E_i \subset E$ s.t. $m^*(E_i) < \frac{1}{i}$ and $f_n(x) \to f(x)$ uniformly on $E \setminus E_i$. Now let $E_0 = \bigcap_{i=1}^\infty E_i$, then $m^*(E_0) \le m^*(E_i)$ for all $i \ge 1$. Take $i \to \infty$, we have $m^*(E_0) = 0$. Consider any $x \in E \setminus E_0$, since

$$E \setminus E_0 = E \cap E_0^c = E \cap \left(\bigcup_{i=1}^\infty E_i^c \right) = \bigcup_{i=1}^\infty (E \cap E_i^c) = \bigcup_{i=1} (E \setminus E_i)$$

there exists at least one i_x s.t. $x \in E \setminus E_{i_x}$. Since $f_n \to f$ on $E \setminus E_{i_x}$ uniformly, $f_n(x) \to f(x)$ for this fixed x as $n \to \infty$. This shows $f_n(x) \to f(x)$ pointwisely on $E \setminus E_0$, and thus $f_n(x) \to f(x)$ a.e. on E. □

Remark In general, almost everywhere convergence cannot implies almost uniform convergence. For example, take $f_k(x) = I_{(-k,k)}(x)$ for all $k \ge 1$ and $E = \mathbb{R}$. Let $f(x) = 1$ on E. Obviously, $f_k(x) \to f(x)$ pointwisely (hence almost everywhere) on E as $k \to \infty$. However, $f_k(x)$ does not converge to $f(x)$ a.u. on E. Suppose $f_k(x) \to f(x)$ a.u. on E, then there exists $E_1 \subset E$ with $m^*(E_1) < 1$ s.t. $f_k(x) \to f(x)$ uniformly on $E \setminus E_1$. Notice that there exists a sequence $\{x_n\}_{n=1}^\infty$ s.t. $x_n \in E \setminus E_1$ and $x_n \to \infty$. To verify it, suppose not, then $\sup(E \setminus E_1) < M$ for some constant $M > 0$. Since $E = \mathbb{R}$, $(M, \infty) \subset E_1$. However, this is impossible because $m^*(E) < 1$ and $m^*((M, \infty)) = \infty$. For all $k \in \mathbb{N}^+$, there exists n_k s.t. $x_{n_k} > k$ and $|f_k(x_{n_k}) - f(x)| = 1$. This shows $f_k(x)$ does not converge to $f(x)$ uniformly on $E \setminus E_1$.

From the above theorem and its remark, we can see that just like the relation between pointwise convergence and uniform convergence, a.e. convergence is weaker than a.u. convergence. However, The more astonishing fact is that if we restrict the domain of the functions to be of finite measure, i.e., $m(E) < \infty$, then a.e. convergence is equivalent to a.u. convergence. This is proved by the following great theorem - Egorov's Theorem - which is also the title of this section.

Theorem 2.5. Egorov's Theorem

Let $f(x)$ and $\{f_n(x)\}_{n=1}^{\infty}$ be finite a.e. and measurable on $E \in \mathcal{M}$. If $m(E) < \infty$ and $f_n(x) \to f(x)$ a.e. on E, then $f_n(x) \to f(x)$ a.u. on E.

Proof Let $S = \{x \in E \,|\, |f(x)| = \infty\} \cup (\bigcup_{n=1}^{\infty}\{x \in E \,|\, |f_n(x)| = \infty\})$, then $m^*(S) = 0$. Notice that $f(x)$ and $f_n(x)$'s are all finite on $E' = E \setminus S \in \mathcal{M}$. Notice that it suffices to show $f_n(x) \to f(x)$ a.u. on E' because if so, then for all $\delta > 0$, there exists $E'_\delta \subset E'$ and $m^*(E'_\delta) < \delta$ s.t. $f_n(x) \to f(x)$ uniformly on $E' \setminus E'_\delta$. Let $E_\delta = E'_\delta \cup S$, then $m^*(E_\delta) = m^*(E'_\delta) < \delta$ and $f_n(x) \to f(x)$ uniformly on $E \setminus E_\delta = E' \setminus E'_\delta$. Also, since $f_n(x) \to f(x)$ a.e. on E, $f_n(x) \to f(x)$ a.e. on E'.

Let $Z = \{x \in E' \,|\, f_n(x) \not\to f(x)\}$, then $m(Z) = 0$. Observe for all $x \in Z$, there exists ϵ_x s.t. $|f_n(x) - f(x)| \geq \epsilon_x$ for infinitely many n's. This shows there exists an integer $l_x \geq 1$ s.t. $|f_n(x) - f(x)| \geq \frac{1}{l_x}$ for infinitely many n's. Define $E_l^n = \{x \in E' \,|\, |f_n(x) - f(x)| \geq \frac{1}{l}\}$. Since $E' \in \mathcal{M}$ and $f_n(x) - f(x)$ is measurable on E', it is easy to show $E_l^n \in \mathcal{M}$. Then for all $x \in Z$, there exists l_x s.t. $x \in E_{l_x}^n$ for infinitely many n's. By Problem Set 1.2, Question 6., $x \in \overline{\lim}_{n\to\infty} E_{l_x}^n$ for some l_x. Therefore, $Z \subset \bigcup_{l=1}^{\infty}(\overline{\lim}_{n\to\infty} E_l^n)$. Also, we can prove $\bigcup_{l=1}^{\infty}(\overline{\lim}_{n\to\infty} E_l^n) \subset Z$ by using the same argument reversely. Thus, $Z = \bigcup_{l=1}^{\infty}(\overline{\lim}_{n\to\infty} E_l^n)$.

Again, by Problem Set 1.2, Question 6., we can write $Z = \bigcup_{l=1}^{\infty}\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} E_l^n$. Denote $F_l^m = \bigcup_{n=m}^{\infty} E_l^n$, then $F_l^m \in \mathcal{M}$. Since F_l^m is decreasing w.r.t. m and $m(F_l^m) < \infty$ (because $F_l^m \subset E'$), by continuity of Lebesgue measure, $\lim_{m\to\infty} m(F_l^m) = m(\lim_{m\to\infty} F_l^m)$. Since $\lim_{m\to\infty} F_l^m = \bigcap_{m=1}^{\infty} F_l^m \subset Z$, $m(\lim_{m\to\infty} F_l^m) = 0$ for all $l \geq 1$. This shows $\lim_{m\to\infty} m(F_l^m) = 0$ for all $l \geq 1$. Therefore, for all $\delta > 0$, there exists $m_l \geq 1$ s.t. $m(F_l^{m_l}) < \frac{\delta}{2^l}$ for all $l \geq 1$. Let $E_\delta = \bigcup_{l=1}^{\infty} F_l^{m_l}$, then $m(E_\delta) < \sum_{l=1}^{\infty}\frac{\delta}{2^l} = \delta$. We claim that $f_n(x) \to f(x)$ uniformly on $E' \setminus E_\delta$. For any $x \in E' \setminus E_\delta$, since

$$E' \setminus E_\delta = E' \cap E_\delta^c = E' \cap \left(\bigcup_{l=1}^{\infty} F_l^{m_l}\right)^c = E' \cap \left(\bigcap_{l=1}^{\infty} (F_l^{m_l})^c\right)$$

we have $x \in E'$ and $x \notin F_l^{m_l}$ for all $l \geq 1$. This shows $|f_n(x) - f(x)| < \frac{1}{l}$ when $n \geq m_l$. Therefore, for all $l \geq 1$, there exists $m_l \geq 1$ s.t. $|f_n(x) - f(x)| < \frac{1}{l}$ for all $n \geq m_l$ and all $x \in E' \setminus E_\delta$. This means $f_n(x)$ converges uniformly to $f(x)$ on $E' \setminus E_\delta$. In conclusion, $f_n(x) \to f(x)$ a.u. on E'. $\square$

Problem Set 2.3

1. Let $f(x)$ be measurable and finite a.e. on E with $m(E) < \infty$. For each $\epsilon > 0$, show that there is a measurable set F contained in E and a sequence $\{\phi_n(x)\}_{n=1}^{\infty}$ of simple functions on E such that $\phi_n(x) \to f(x)$ uniformly on F and $m(E \setminus F) < \epsilon$.
2. Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of measurable functions on E that converges to a real-valued function $f(x)$ pointwisely on E. Show that $E = \bigcup_{k=1}^{\infty} E_k$, where for each k, E_k is

measurable, and $f_n(x)$ converges uniformly to $f(x)$ on each E_k if $k > 1$, and $m(E_1) = 0$.

3. Let $\{f_k(x)\}_{k=1}^{\infty}$ be measurable on $E \in \mathcal{M}$, where $m(E) < \infty$. Suppose $f_k(x) \to \infty$ a.e. on E as $k \to \infty$, then $f_k(x) \to \infty$ a.u. on E.
4. Let $\{f_n(x)\}_{n=1}^{\infty}$ be measurable on $[0,1]$ with $|f_n(x)| < \infty$ for a.e. $x \in E$. Show that there exists sequence of positive numbers c_n such that $\frac{f_n(x)}{c_n} \to 0$ a.e. on E as $n \to \infty$.
5. Let $\{f_n(x)\}_{n=1}^{\infty}$ be measurable on $\mathbb{R}$ and λ_n be a sequence of positive numbers, satisfying
$$\sum_{n=1}^{\infty} m\left(\{x \in \mathbb{R} \mid |f_n(x)| > \lambda_n\}\right) < \infty$$
Prove that $\limsup_{n\to\infty} \frac{|f_n(x)|}{\lambda_n} \leq 1$ a.e. on $\mathbb{R}$.
6. Let $f_k(x)$ be real-valued, measurable on $E \in \mathcal{M}$ for all $k \in \mathbb{N}^+$, with $m(E) < \infty$. Prove that $f_k(x) \to 0$ a.e. on E as $k \to \infty$ if and only if
$$\lim_{j\to\infty} m\left(\left\{x \in E \,\middle|\, \sup_{k \geq j} |f_k(x)| \geq \epsilon\right\}\right) = 0$$
for all $\epsilon > 0$.
7. Let $f_{k,i}(x)$ be real-valued and measurable on $[0,1]$ for all $k \in \mathbb{N}^+$ and $i \in \mathbb{N}^+$ and satisfy
   (a). For each fixed $k \geq 1$, $f_{k,i}(x) \to f_k(x)$ a.e. on $[0,1]$ as $i \to \infty$ with some real-valued $f_k(x)$ on $[0,1]$.
   (b). $f_k(x) \to g(x)$ a.e. on $[0,1]$ as $k \to \infty$, with some real-valued $g(x)$ on $[0,1]$.

   Prove that there exists k_j and i_j such that $f_{k_j,i_j}(x) \to g(x)$ a.e. on $[0,1]$ as $j \to \infty$.

2.4 Convergence In Measure

Definition 2.8. Convergence In Measure

Let $f(x)$ and $\{f_n(x)\}_{n=1}^{\infty}$ be finite a.e. and measurable on $E \in \mathcal{M}$. If for all $\sigma > 0$, $m(\{x \in E \mid |f_n(x) - f(x)| > \sigma\}) \to 0$ as $n \to \infty$, then we say $f_n(x) \to f(x)$ ***in measure*** *on E.* ♣

Remark Notice that here we can see the reason for restricting $f(x)$ and $f_n(x)$ to be finite a.e. on E, because if not, say $f_n(x) = f(x) = \infty$ on A with $m(A) > 0$, then $f_n(x) - f(x)$ is not defined on A, and the definition fails to work. However, if $f_n(x) - f(x)$ is only undefined on A with $m(A) = 0$, then as long as $m(\{x \in E \setminus A \mid |f_n(x) - f(x)| > \sigma\}) \to 0$, no matter you regard $|f_n(x) - f(x)| > \sigma$ for $x \in A$ as true or not, $m(\{x \in E \mid |f_n(x) - f(x)| > \sigma\}) \to 0$ always holds. That is to say, any set with zero measure can be ignored when we verify $f_n(x) \to f(x)$ in measure. Also, since we want to explore the relationship between convergence in measure and a.u./a.e. convergence, we also require "finite a.e." in the definition of them, although a.e. or a.u. convergence solely doesn't need this condition.

Problem 2.1 Let $f(x)$ and $\{f_n(x)\}_{n=1}^{\infty}$ be finite a.e. and measurable on $E \in \mathcal{M}$. If $f_n(x) \to f(x)$ in measure on E, then for any measurable subset $A \subset E$, $f_n(x) \to f(x)$ in measure on A.

Since convergence in measure is different from the usual convergence of sequence of numbers, so we need to first verify it is well-defined, i.e., the limiting function is unique in some sense. The following theorem shows the uniqueness of limit.

Theorem 2.6. Uniquess of Limit

Let $f(x), g(x), \{f_n(x)\}_{n=1}^{\infty}$ be finite a.e. and measurable on $E \in \mathcal{M}$. If $f_n(x) \to f(x)$ in measure and $f_n(x) \to g(x)$ in measure on E, then $f(x) = g(x)$ a.e. on E.

Proof Observe $|f(x) - g(x)| \leq |f(x) - f_n(x)| + |f_n(x) - g_n(x)|$ for all $x \in E \setminus A$, where

$$A = \{x \in E \,|\, |f(x)| = \infty\} \cup \{x \in E \,|\, |f(x)| = \infty\} \cup \left(\bigcup_{n=1}^{\infty} \{x \in E \,|\, |f_n(x)| = \infty\} \right)$$

with $m(A) = 0$. Then $\forall\, \sigma > 0$,

$$\begin{aligned} \{x \in E \setminus A \,|\, |f(x) - g(x)| > 2\sigma\} \subset \{x \in E \setminus A \,|\, |f_n(x) - f(x)| > \sigma\} \\ \cup \{x \in E \setminus A \,|\, |f_n(x) - g(x)| > \sigma\} \end{aligned}$$

Take measure (it is easy to see both sides are measurable sets) on both sides,

$$\begin{aligned} m(\{x \in E \setminus A \,|\, |f(x) - g(x)| > 2\sigma\}) \leq m(\{x \in E \setminus A \,|\, |f_n(x) - f(x)| > \sigma\}) \\ + m(\{x \in E \setminus A \,|\, |f_n(x) - g(x)| > \sigma\}) \end{aligned}$$

Since $f_n(x) \to f(x)$ in measure and $f_n(x) \to g(x)$ in measure on E, the RHS tend to zero as $n \to \infty$. Thus, $m(\{x \in E \setminus A \,|\, |f(x) - g(x)| > 2\sigma\}) = 0$ for all $\sigma > 0$. Notice that

$$\{x \in E \setminus A \,|\, f(x) \neq g(x)\} = \bigcup_{k=1}^{\infty} \left\{ x \in E \setminus A \,\middle|\, |f(x) - g(x)| > \frac{1}{k} \right\}$$

Therefore, take $\sigma = \frac{1}{2k}$, we know every set inside the union on the RHS is of measure zero. By σ-subadditivity, we have

$$m(\{x \in E \setminus A \,|\, f(x) \neq g(x)\}) \leq \sum_{k=1}^{\infty} 0 = 0$$

This shows $f(x) = g(x)$ a.e. on $E \setminus A$. However, since $m(A) = 0$, it can also imply $f(x) = g(x)$ a.e. on E. □

Now we are going to explore the relationship between a.u. convergence, a.e. convergence, and convergence in measure.

Theorem 2.7

Let $f(x)$ and $\{f_n(x)\}_{n=1}^{\infty}$ be finite a.e. and measurable on $E \in \mathcal{M}$. If $f_n(x) \to f(x)$ a.u. on E, then $f_n(x) \to f(x)$ in measure on E.

Proof By the remark of Definition 2.8, we only need to prove $f_n(x) \to f(x)$ in measure on $E' = E \setminus A$, where

$$A = \{x \in E \,|\, |f(x)| = \infty\} \cup \left(\bigcup_{n=1}^{\infty} \{x \in E \,|\, |f_n(x)| = \infty\} \right)$$

because $m(A) = 0$. For all $\delta > 0$, there exists $E_\delta \subset E$ s.t. $m^*(E_\delta) < \delta$ and $f_n(x) \to f(x)$ uniformly on $E \setminus E_\delta$. For all $\sigma > 0$, there exists $N(\sigma) \geq 1$ s.t. $|f_n(x) - f(x)| < \frac{\sigma}{2}$ for all $n \geq N(\sigma)$ and $x \in E' \setminus E_\delta$. Thus, for all $\sigma > 0$, there exists $N(\sigma)$ s.t. for all $n \geq N(\sigma)$, $\{x \in E' \,|\, |f_n(x) - f(x)| > \sigma\} \subset E_\delta$. This shows $m(\{x \in E' \,|\, |f_n(x) - f(x)| > \sigma\}) < \delta$ by subadditivity. Take $\delta \to 0$, $m(\{x \in E' \,|\, |f_n(x) - f(x)| > \sigma\}) = 0$ for $n \geq N(\sigma)$. This is even stronger than what we need, but anyway we can say for any fixed $\sigma > 0$, as $n \to \infty$, $m(\{x \in E' \,|\, |f_n(x) - f(x)| > \sigma\}) \to 0$, so $f_n(x) \to f(x)$ in measure on E'. □

Remark Note that convergence in measure cannot imply a.u. convergence because it even fails to imply a.e. convergence. For example, for all $n \in \mathbb{N}^+$, define $f_{nk}(x)$ for $k = 1, \ldots, 2^n$ by

$$f_{nk}(x) = \begin{cases} 1 & x \in (\frac{k-1}{2^n}, \frac{k}{2^n}] \\ 0 & \text{elsewhere on } (0,1] \end{cases}$$

Construct a sequence of function $\{f_i(x)\}_{i=1}^{\infty}$ by letting $f_{2^n-2+k}(x) = f_{nk}(x)$. It is obvious that when $i > 2^n$, $m(\{x \in (0,1] \,|\, |f_i(x)| > \sigma\}) = \frac{1}{2^n}$. This shows $f_i(x) \to 0$ in measure on $(0,1]$. However, $f_i(x)$ does not converge to 0 a.e., because for each $n \in \mathbb{N}^+$,

$$(0,1] = \bigcup_{i=2^n-1}^{2^{n+1}-2} \{x \in (0,1] \,|\, f_i(x) = 1\}$$

so for any fixed $x \in (0,1]$, for each n, there exists i_n s.t. $f_{i_n}(x) = 1$. This shows $f_i(x) \not\to 0$ for each fixed $x \in (0,1]$. Similarly, we can also find a subsequence of $f_i(x)$ s.t. it converges to 0 for every x. This shows $f_i(x)$ cannot converge to any function a.e. on $(0,1]$.

One may think a.e. convergence is stronger than convergence in measure. Unfortunately, this is only true when the domain is of finite measure. However, this is because Egorov's Theorem says a.e. convergence implies a.u. convergence, and it is a.u. convergence that can imply convergence in measure. Thus, to construct a counter-example, we only need to consider function defined on set with infinite measure.

Example 2.4 Let $E = \mathbb{R}$ and $f_n(x) = I_{(-n,n)}(x)$ for all $n \geq 1$. Then it is obvious that $f_n(x) \to 1$ pointwisely on $\mathbb{R}$. However, $f_n(x) \not\to 1$ in measure on $\mathbb{R}$. This is because for all $\sigma \in (0,1)$, $m(\{x \in \mathbb{R} \,|\, |f_n(x) - 1| > \sigma\}) = \infty$ for any $n \in \mathbb{N}^+$.

Notice that in practice, sometimes we want to prove $f_n(x) \to f(x)$ in measure on E for some $f(x)$, but $f(x)$ is very hard to find or cannot be found. In this case, we shall consider Cauchy criterion just like what we do in convergence of sequence of numbers.

Definition 2.9. Cauchy In Measure

*Let $\{f_n(x)\}_{n=1}^{\infty}$ be measurable on E and finite a.e. on E. We say $\{f_n(x)\}_{n=1}^{\infty}$ is **Cauchy in measure** if for every fixed $\sigma > 0$, for all $\epsilon > 0$, there exists $K \in \mathbb{N}^+$ s.t.*

$$m(\{x \in E \,|\, |f_k(x) - f_j(x)| > \sigma\}) < \epsilon, \quad \text{whenever } k, j \geq K$$

♣

Theorem 2.8

Let $f(x)$ and $\{f_n(x)\}_{n=1}^{\infty}$ be measurable and finite a.e. on E. If $f_n(x) \to f(x)$ in measure on E as $n \to \infty$, then $\{f_n(x)\}_{n=1}^{\infty}$ is Cauchy in measure.

Proof Define the same A as in Theorem 2.7, then $m(A) = 0$. Let $E' = E \setminus A$, then for all $\sigma > 0$,

$$m(\{x \in E' \,|\, |f_n(x) - f(x)| > \sigma/2\}) \to 0$$

Thus, for all $\epsilon > 0$, there exists $N \geq 1$ s.t. $m(\{x \in E' \,|\, |f_n(x) - f(x)| > \sigma/2\}) < \epsilon$ for all $n \geq N$. Observe that $|f_k(x) - f_j(x)| \leq |f_k(x) - f(x)| + |f(x) - f_j(x)|$ for all $x \in E'$. Similar to the proof of Theorem 2.6, we have

$$\begin{aligned}\{x \in E' \,|\, |f_k(x) - f_j(x)| > \sigma\} \subset \{x \in E' \,|\, |f_k(x) - f(x)| > \sigma/2\} \\ \cup \{x \in E' \,|\, |f_j(x) - f(x)| > \sigma/2\}\end{aligned}$$

Therefore, by subadditivity, we have

$$\begin{aligned}m(\{x \in E' \,|\, |f_k(x) - f_j(x)| > \sigma\}) \leq m(\{x \in E' \,|\, |f_k(x) - f(x)| > \sigma/2\}) \\ + m(\{x \in E' \,|\, |f_j(x) - f(x)| > \sigma/2\})\end{aligned}$$

When $k, j \geq N$, $m(\{x \in E' \,|\, |f_k(x) - f_j(x)| > \sigma\}) < 2\epsilon$, so $\{f_n(x)\}_{n=1}^{\infty}$ is Cauchy in measure. □

Theorem 2.9

Let $\{f_k(x)\}_{n=1}^{\infty}$ be measurable and finite a.e. on $E \in \mathcal{M}$. If $\{f_k(x)\}_{k=1}^{\infty}$ is Cauchy in measure on E, then there exists measurable function $f(x)$, finite a.e. on E and $f_k(x) \to f(x)$ in measure on E as $k \to \infty$. Moreover, there exists a subsequence $\{f_{k_i}(x)\}_{i=1}^{\infty}$ of $\{f_k(x)\}_{k=1}^{\infty}$ s.t. $f_{k_i}(x) \to f(x)$ a.u. on E as $i \to \infty$.

Proof Let $A = \bigcup_{k=1}^{\infty}\{x \in E \,|\, |f_k(x)| = \infty\}$ and $E' = E \setminus A$, then $m(A) = 0$. Since $\{f_k(x)\}_{k=1}^{\infty}$ is Cauchy in measure, for all $\epsilon > 0$, there exists $K(\epsilon) \geq 1$ s.t.

$$m(\{x \in E' \,|\, |f_k(x) - f_j(x)| > \epsilon\}) < \epsilon, \quad \forall\, k, j \geq K(\epsilon) \tag{2.1}$$

Take $\epsilon = \frac{1}{2}$, let $k_1 = K(\frac{1}{2})$; take $\epsilon = \frac{1}{2^2}$, let $k_2 = K(\frac{1}{2^2})$ s.t. $k_2 > k_1$. Keep on doing this, let $k_i = K(\frac{1}{2^i})$ s.t. $k_i > k_{i-1}$ for all $i \geq 2$. Then we will have

$$m(\{x \in E' \,|\, |f_l(x) - f_j(x)| > 2^{-i}\}) < 2^{-i}, \quad \forall\, l, j \geq k_i$$

In particular, for all $i \geq 1$, let $E_i = \{x \in E' \,|\, |f_{k_{i+1}}(x) - f_{k_i}(x)| > 2^{-i}\}$, and we have $m(E_i) < 2^{-i}$. Thus, $\sum_{i=1}^{\infty} m(E_i) < \infty$ and by Borel-Cantelli lemma, $m(\overline{\lim}_{i\to\infty} E_i) = 0$.

We first claim for all $x \in E' \setminus \overline{\lim}_{i\to\infty} E_i$, $f_{k_i}(x) \to f(x)$ pointwisely for some $f(x)$ as $i \to \infty$. Notice that if $x \notin \overline{\lim}_{i\to\infty} E_i$, then x is in at most finitely many E_i's, so there exists $I_x \geq 1$ s.t. $x \notin E_i$ for all $i \geq I_x$. This shows $|f_{k_{i+1}}(x) - f_{k_i}(x)| \leq 2^{-i}$ for all $i \geq I_x$. Thus, $\sum_{i=I_x}^{\infty}(f_{k_{i+1}}(x) - f_{k_i}(x))$ converges and $\{f_{k_i}(x)\}_{i=1}^{\infty}$ converges to some $f(x)$ pointwisely on

$E' \setminus \overline{\lim}_{i\to\infty} E_i$. It is easy to extend $f(x)$ to E' because we can just let $f(x) = 0$ for all $x \in \overline{\lim}_{i\to\infty} E_i$. In this case, $f(x)$ is measurable on $\overline{\lim}_{i\to\infty} E_i$ because $m(\overline{\lim}_{i\to\infty} E_i) = 0$. By remark of Exercise 2.12, $f(x)$ is measurable on $E' \setminus \overline{\lim}_{i\to\infty} E_i$. By Exercise 2.1, $f(x)$ is measurable on E'. Since $f(x)$ is finite on $E' \setminus \overline{\lim}_{i\to\infty} E_i$, $f(x)$ is finite a.e. on E'.

Next we claim that for each fixed $\sigma > 0$, for each fixed $k \geq 1$,

$$\{x \in E' \,|\, |f_k(x) - f(x)| > \sigma\} \subset \underline{\lim}_{i\to\infty} \{x \in E' \,|\, |f_k(x) - f_{k_i}(x)| > \sigma\} \cup \overline{\lim}_{i\to\infty} E_i$$

If $x \in$ LHS, but $x \notin \overline{\lim}_{i\to\infty} E_i$, then by our first claim, $\lim_{i\to\infty} f_{k_i}(x) = f(x)$. Then there exists $M \in \mathbb{N}^+$ s.t. $|f_{k_i}(x) - f(x)| < \frac{|f_k(x)-f(x)|-\sigma}{2}$ for all $i \geq M$. Thus,

$$|f_k(x) - f_{k_i}(x)| \geq |f_k(x) - f(x)| - |f_{k_i}(x) - f(x)| > \sigma$$

This implies $x \in \{x \in E' \,|\, |f_k(x) - f_{k_i}(x)| > \sigma\}$ for all $i \geq M$, so for every fixed $\sigma > 0$,

$$x \in \underline{\lim}_{i\to\infty} (\{x \in E' \,|\, |f_k(x) - f_{k_i}(x)| > \sigma\}), \quad \forall\, k \geq 1$$

This finishes the proof of our second claim. By subadditivity,

$$m(\{x \in E' \,|\, |f_k(x) - f(x)| > \sigma\}) \leq m\left(\underline{\lim}_{i\to\infty} \{x \in E' \,|\, |f_k(x) - f_{k_i}(x)| > \sigma\}\right)$$

By Problem Set 1.4, Question 15.,

$$m\left(\underline{\lim}_{i\to\infty} \{x \in E' \,|\, |f_k(x) - f_{k_i}(x)| > \sigma\}\right) \leq \underline{\lim}_{i\to\infty}\, m(\{x \in E' \,|\, |f_k(x) - f_{k_i}(x)| > \sigma\})$$

Note that for every fixed $\sigma > 0$, for all $\epsilon < \sigma$, there exists $I(\epsilon) \geq 1$ s.t. $k_i \geq K(\epsilon)$ if $i \geq I(\epsilon)$. Thus, for all $k \geq K(\epsilon)$ and $i \geq I(\epsilon)$, by Equation (2.1), we have

$$m(\{x \in E' \,|\, |f_k(x) - f_{k_i}(x)| > \sigma\}) \leq m(\{x \in E' \,|\, |f_k(x) - f_{k_i}(x)| > \epsilon\}) < \epsilon$$

This shows for each fixed $\sigma > 0$, $m(\{x \in E' \,|\, |f_k(x) - f(x)| > \sigma\}) < \epsilon$ for all $k \geq K(\epsilon)$. Therefore, $f_k(x) \to f(x)$ in measure on E' and hence on E.

Finally, we claim that $f_{k_i}(x) \to f(x)$ a.u. on E, so $f_{k_i}(x)$ is just the desired subsequence. For all $\delta > 0$, there exists $I(\delta) \geq 1$ s.t. $\sum_{i=I(\delta)}^{\infty} 2^{-i} < \delta$. Recall $m(E_i) < \frac{1}{2^i}$, so $m(E_\delta) < \delta$ where $E_\delta = A \cup (\bigcup_{i=I(\delta)}^{\infty} E_i)$. Now we only need to prove $f_{k_i}(x) \to f(x)$ uniformly on $E \setminus E_\delta$. For $x \in E \setminus E_\delta$, since $|f_{k_{i+1}}(x) - f_{k_i}(x)| \leq \frac{1}{2^i}$ for all $i \geq I(\delta)$. By M-test, $\sum_{i=I(\delta)}^{\infty} (f_{k_{i+1}}(x) - f_{k_i}(x))$ converges uniformly on $E \setminus E_\delta$, so $f_{k_i}(x)$ converges uniformly on $E \setminus E_\delta$ to some $g(x)$. However, we have known $f_{k_i}(x) \to f(x)$ pointwisely on $E' \setminus \overline{\lim}_{i\to\infty} E_i$ and $\overline{\lim}_{i\to\infty} E_i \subset \bigcup_{i=I(\delta)}^{\infty} E_i$, so $E \setminus E_\delta \subset E' \setminus \overline{\lim}_{i\to\infty} E_i$ and $f_{k_i}(x) \to f(x)$ pointwisely on $E \setminus E_\delta$. This shows $f(x) = g(x)$ on $E \setminus E_\delta$ and we are done. □

Conclusion *Combined with Theorem 2.8 and Theorem 2.9, we know Cauchy in measure is equivalent to convergence in measure. Furthermore, convergence in measure implies a.u. convergence for a subsequence.*

Problem Set 2.4

1. Let $E \in \mathcal{M}$, $f_k \to f$ in measure and $g_k \to g$ in measure one E as $k \to \infty$. Prove that

$f_k + g_k \to f + g$ in measure on E as $k \to \infty$.

2. Let f_∞, f_n, $n \in \mathbb{N}^+$ be measurable and finite a.e. on $E \in \mathcal{M}$, and suppose $m(E) < \infty$. Prove that if any subsequence f_{n_k} of f_n contains a subsequence $f_{n_{k_i}}$ which converges to f_∞ a.e. on E as $i \to \infty$, then $f_n \to f_\infty$ in measure on E as $n \to \infty$.
3. Let $E \in \mathcal{M}$ and $m(E) < \infty$. Suppose $f_n \to f_\infty$ and $g_n \to g_\infty$ both in measure on E. Prove that $f_n g_n \to f_\infty g_\infty$ in measure as $n \to \infty$.
4. Suppose $f_n \to f_\infty$ in measure on $E \in \mathcal{M}$; g is uniformly continuous on $\mathbb{R}$. Prove that $g \circ f_n \to g \circ f$ in measure as $n \to \infty$.
5. Let $f_{n,i} \to f_n$ in measure as $i \to \infty$ on $E \in \mathcal{M}$. Also, $f_n \to f_\infty$ in measure as $n \to \infty$. Prove that there exists subsequence $f_{n_m, i_m} \to f_\infty$ a.u. as $m \to \infty$.
6. Suppose $f_n \to f_\infty$ in measure on $E \in \mathbb{R}$, $E \in \mathcal{M}$. Assume f_n is M-Lipschitz continuous on E for all $n \geq 1$, prove that $f_n \to f_\infty$ a.e. as $n \to \infty$.

2.5 Lusin's Theorem and Littlewood's Three Principles

Up to now, you may still think measurable function is mysterious. Unlike continuous function, which is very concrete and intuitive, measurable function is abstract and intangible. In this section we will learn another very famous and great theorem which connect the continuous function and measurable function.

Theorem 2.10. Lusin's Theorem

Let $f(x)$ be measurable and finite a.e. on $E \in \mathcal{M}$. Suppose $f(x)$ is finite a.e. on E. Then for all $\delta > 0$, there exists a closed set $F_\delta \subset E$ s.t. $m(E \setminus F_\delta) < \delta$ and $f\big|_{F_\delta}(x)$ (f restricted on F_δ) is continuous on F_δ.

Proof First, we show it suffices to prove the desired result for the case when $m(E) < \infty$. If $m(E) = \infty$, then let $E_k = \{x \in E \,|\, k \leq |x| \leq k+1\}$ for $k \in \mathbb{N}$ and $E = \bigcup_{k=0}^{\infty} E_k$ with $m(E_k) < \infty$. Suppose the desired result holds for the case when $m(E) < \infty$, then for all $\delta > 0$, there exists closed $F_k \subset E_k$ s.t. $m(E_k \setminus F_k) < \frac{\delta}{2^{k+1}}$ and $f\big|_{F_k}(x)$ is continuous on F_k for all $k \in \mathbb{N}$. Let $F_\delta = \bigcup_{k=0}^{\infty} F_k$, then F_δ is closed by Problem Set 1.3, Question 4. and $f\big|_{F_\delta}(x)$ is continuous on F_δ. Also, since $E \setminus F_\delta = \bigcup_{k=0}^{\infty}(E_k \setminus F_k)$, $m(E \setminus F_\delta) \leq \sum_{k=0}^{\infty} \frac{\delta}{2^{k+1}} = \delta$. This shows that the desired result is also true when $m(E) = \infty$. Therefore, from now on, we assume $m(E) < \infty$. Also, we can assume $f(x)$ is finite on E because if we let $Z \subset E$ to be the set where $|f(x)| = \infty$, then $m(Z) = 0$. Let $E' = E \setminus Z$, and if the desired result is true on E', i.e., we find the desired $F_\delta \subset E'$, then use this F_δ, we still have the same result for E because $m(E \setminus F_\delta) = m(E' \setminus F_\delta) < \delta$.

Then we prove the desired result for the case when $f(x)$ is simple and measurable function. Let $f(x) = \sum_{i=1}^{I} y_i I_{E_i}(x)$, where $y_i, \ldots, y_I \in \mathbb{R}$, E_i's are pairwise disjoint and measurable. By Problem Set 1.4, Question 2., there exists closed $F_i \subset E_i$ s.t. $m(E_i \setminus F_i) < \frac{\delta}{I}$ for all

$i = 1, \dots, I$. Let $F_\delta = \bigcup_{i=1}^{I} F_i$, then F_δ is closed and $f\big|_{F_\delta}(x)$ is continuous on F_δ. Since $E \setminus F_\delta = \bigcup_{i=1}^{I}(E_i \setminus F_i)$, $m(E \setminus F_\delta) \le \sum_{i=1}^{I} m(E_i \setminus F_i) < \delta$.

Finally, we consider general measurable function $f(x)$. By Simple Approximation Theorem, there exists a sequence $\{\phi_k(x)\}_{k=1}^{\infty}$ of measurable simple function s.t. $\phi_k(x) \to f(x)$ pointwisely on E. Since $\phi_k(x)$ is measurable simple function on E, there exists closed $F_k \subset E$ s.t. $m(E \setminus F_k) < \frac{\delta}{2^{k+1}}$ and $\phi_k\big|_{F_k}(x)$ is continuous on F_k for all $k \ge 1$. Define $F_0 = \bigcap_{k=1}^{\infty} F_k$, then F_0 is closed, and since $E \setminus F_0 = \bigcup_{k=1}^{\infty}(E \setminus F_k)$, $m(E \setminus F_0) \le \sum_{k=1}^{\infty} m(E \setminus F_k) < \frac{\delta}{2}$. It is also obvious that $\phi_k\big|_{F_0}(x)$ is continuous on F_0 for all $k \ge 1$. Since $\phi_k(x) \to f(x)$ pointwisely on F_0 and $m(F_0) < \infty$, by Egorov's theorem, $\phi_k(x) \to f(x)$ a.u. on F_0. Thus, there exists $\tilde{F}_1 \subset F_0$ s.t. $m(F_0 \setminus \tilde{F}_1) < \frac{\delta}{4}$ and $\phi_k(x) \to f(x)$ uniformly on $\tilde{F}_1$. Notice that we can assume $\tilde{F}_1 \in \mathcal{M}$ because in proof of Egorov's theorem, the "E_δ" we construct is indeed measurable. Since $\phi_k\big|_{F_0}(x)$ is continuous on F_0, $\phi_k\big|_{\tilde{F}_1}(x)$ is continuous on $\tilde{F}_1$, and thus its uniform limit $f\big|_{\tilde{F}_1}(x)$ is continuous on $\tilde{F}_1$. Since $\tilde{F}_1 \in \mathcal{M}$, by Problem Set 1.4, Question 2., there exists closed $F_\delta \subset \tilde{F}_1$ s.t. $m(\tilde{F}_1 \setminus F_\delta) < \frac{\delta}{4}$. Notice that

$$m(E \setminus F_\delta) = m((E \setminus F_0) \cup (F_0 \setminus \tilde{F}_1) \cup (\tilde{F}_1 \setminus F_\delta)) < \frac{\delta}{2} + \frac{\delta}{4} + \frac{\delta}{4} = \delta$$

and $f\big|_{F_\delta}(x)$ is continuous on F_δ because $f\big|_{\tilde{F}_1}(x)$ is continuous on $\tilde{F}_1$. □

The last topic in this chapter is the famous Littlewood's three principles of real analysis. The statement of three principles is high-level idea of some theorem we have already proved.

Theorem 2.11. Littlewood's Three Principles

1. *Every measurable set is **nearly** the union of a finite collection of disjoint open intervals.*
2. *Every measurable function is **nearly** continuous.*
3. *Every pointwise convergent sequence of functions is **nearly** uniformly convergent.* ♡

Proof

1. See Theorem 1.1, part 5.. Notice that closed cubes can be replaced by open cubes and the results will not change.
2. See Lusin's theorem.
3. See Egorov's theorem.

□

Remark The spirit of these principles lies in the word "nearly". It means an approximation of a general, abstract object by a simple, concrete object with desirable properties.

Example 2.5 Let $f(x)$ be real-valued on $\mathbb{R}$ s.t. $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Suppose $f(x)$ is measurable on $\mathbb{R}$, then $f(x)$ is continuous on $\mathbb{R}$.

Proof Notice that $f(0) = f(0 + 0) = f(0) + f(0)$ implies $f(0) = 0$. If we want to prove $f(x)$

is continuous at x, we need to verify for all $\epsilon > 0$, there exists $\delta > 0$ s.t. $|f(x+h) - f(x)| < \epsilon$ for all $|h| < \delta$. Since $|f(x+h) - f(x)| = |f(h) - f(0)|$, it suffices to show $f(x)$ is continuous at $x = 0$. Apply Lusin's theorem on $E = [-1, 1]$, and take $\delta = \frac{1}{2}$, there exists closed $F_\delta \subset E$ s.t. $m(E \backslash F_\delta) < \frac{1}{2}$. Then $m(F_\delta) > 0$ and $f\big|_{F_\delta}(x)$ is continuous on F_δ. Since F_δ is compact, $f\big|_{F_\delta}(x)$ is uniform continuous on F_δ. Thus, for all $\epsilon > 0$, there exists $\delta_1 > 0$ s.t. $|f(x) - f(y)| < \epsilon$ for all $x, y \in F_\delta$ and $|x - y| < \delta_1$. Since $m(F_\delta) > 0$, there exists a neighborhood $(-\delta_2, \delta_2) \subset F_\delta - F_\delta$ by Steinhauss theorem. Let $\delta_0 = \min\{\delta_1, \delta_2\}$, then for all $z \in \mathbb{R}$ s.t. $|z| < \delta_0$, there exists $x, y \in F_\delta$ and $z = x - y$ s.t. $|f(z)| = |f(x-y)| = |f(x) - f(y)| < \epsilon$. This shows $f(z) \to 0 = f(0)$ as $z \to 0$. Thus, $f(x)$ is continuous at $x = 0$. □

Example 2.6 Prove that there exists a closed $F \subset [0, 1]$ s.t. $m(F) > 0$ and $F \cap \mathbb{Q} = \varnothing$.

Proof Consider Dirichlet function $D(x)$ defined on $E = [0, 1]$ by

$$D(x) = \begin{cases} 0 & x \in [0,1] \setminus \mathbb{Q} \\ 1 & x \in [0,1] \cap \mathbb{Q} \end{cases}$$

Let $f(x) = 1 - D(x)$, then it is obvious that $f(x)$ is measurable on $E = [0, 1]$ because $f(x) = I_{[0,1]\setminus\mathbb{Q}}(x)$ where $[0, 1] \setminus \mathbb{Q} \in \mathcal{M}$. By Lusin's theorem, take $\delta = \frac{1}{2}$, then there exists closed $F_1 \subset E$ s.t. $m(E \setminus F_1) < \frac{1}{2}$. Thus, $m(F_1) > \frac{1}{2}$ and $f\big|_{F_1}(x)$ is continuous on F_1. Note $(f\big|_{F_1})^{-1}(\{1\}) = F_1 \setminus \mathbb{Q}$, and since $\{1\}$ is closed, $F_1 \setminus \mathbb{Q}$ is also closed. Let $F = F_1 \setminus \mathbb{Q}$, then $m(F) = m(F_1) > 0$, so F is the desired set. □

Problem Set 2.5

1. Let f be real-valued and defined on $E \in \mathbb{R}^n$, $E \in \mathcal{M}$, satisfying $\forall\, \delta > 0$, there exists closed $F_\delta \subset E$ s.t. $m(E \setminus F_\delta) < \delta$ and $f\big|_{F_\delta}$ is continuous. Prove f is measureable on E.
2. Let f be real-valued, measurable on a finite interval $[a, b]$. Prove that there exists sequence h_k s.t. $h_k \to 0$, $f(x + h_k) \to f(x)$ for a.e. $x \in [a, b]$ as $k \to \infty$.

Chapter 3 Lebesgue Integration

3.1 Lebesgue Integrals of Nonnegative Measurable Functions

In this section, we are going to define a new type of integral which is different from Riemann integral. Before we do that, let's first do some review on Riemann integral and get enough motivation to introduce a brand new integral.

Definition 3.1. Riemann Integral

*Let $f(x)$ be defined and bounded on $[a, b]$ where $a, b \in \mathbb{R}$. Define a **partition** $P = \{x_0, \ldots, x_n\}$ of $[a, b]$. Denote $m_i = \inf_{[x_{i-1}, x_i]} f(x)$, $M_i = \sup_{[x_{i-1}, x_i]} f(x)$, and $\Delta x_i = x_i - x_{i-1}$ for $i = 1, \ldots, n$. Then the **lower sum** is $\sum_{i=1}^{n} m_i \Delta x_i$ and **upper sum** is $\sum_{i=1}^{n} M_i \Delta x_i$. If $\lim_{\Delta \to 0} \sum_{i=1}^{n} m_i \Delta x_i$ and $\lim_{\Delta \to 0} \sum_{i=1}^{n} M_i \Delta x_i$ both exist and are equal, where $\Delta = \max\{\Delta x_i\}_{i=1}^{n}$, then this limit is called **Riemann integral** of $f(x)$ on $[a, b]$, denoted as $\int_a^b f(x)\, dx$.* ♣

Remark In mathematical analysis, the Lebesgues Criterion for Riemann integrability says that if f is Riemann integrable on $[a, b]$, then f is continuous a.e. on $[a, b]$.

It seems that Riemann integral has strong relation with continuity a function. However, since we define a wider category of functions - measurable functions, we want to define an integration that also works for some measurable but not continuous function, like Dirichlet function.

Definition 3.2. Lebesgue Integrals of Measurable Simple Functions

*Let $f(x) \geq 0$ be measurable simple function on $E \in \mathcal{M}$. Then $f(x) = \sum_{i=1}^{I} y_i I_{E_i}(x)$, $y_i \geq 0$ (y_i can be ∞), E_i's pairwise disjoint and measurable, and $E = \bigcup_{i=1}^{I} E_i$. We define the **Lebesgue integral** of $f(x)$ on E as $\int_E f(x)\, dx = \sum_{i=1}^{I} y_i m(E_i)$.* ♣

Example 3.1 Consider the function $f(x)$ defined in the proof Example 2.6) on $E = [0, 1]$. Let $E_1 = E \cap \mathbb{Q}$ and $E_2 = E \cap \mathbb{Q}^c$, then $f(x) = 0 \cdot I_{E_1}(x) + I_{E_2}(x)$. Since $E_1 \cap E_2 = \varnothing$ and $E = E_1 \cup E_2$, by Definition 3.2, Lebesgue integral $\int_E f(x)\, dx = 0 \cdot m(E_1) + m(E_2) = 1$.

✍ **Exercise 3.1** Notice that for the same set E, there are infinitely many possible cases for E_i's in $E = \bigcup_{i=1}^{I} E_i$. For example, let $E = [0, 1]$ and $f(x) = I_E(x)$, then $f(x) = I_{E_1}(x) + I_{E_2}(x)$. We can take $E_1 = [0, \frac{1}{2})$ and $E_2 = [\frac{1}{2}, 1]$, but we can also take $E_1 = [0, \frac{1}{3})$ and $E_2 = [\frac{1}{3}, 1]$. Is the Lebesgue integral defined in Definition 3.2 unique? Please verify that Lebesgue integral is well-defined.

Proof Suppose $f(x) = \sum_{i=1}^{I} y_i I_{E_i}(x) = \sum_{j=1}^{J} z_j I_{F_j}(x)$, where $E = \bigcup_{i=1}^{I} E_i = \bigcup_{j=1}^{J} F_j$, E_i's are pairwise disjoint, F_j's are pairwise disjoint, and $E_i, F_j \in \mathcal{M}$ for all $i = 1, \ldots, I$ and $j = 1, \ldots, J$. It suffices to show $\sum_{i=1}^{I} y_i m(E_i) = \sum_{j=1}^{J} z_j m(F_j)$. Notice that $E_i = \bigcup_{j=1}^{J} (E_i \cap F_j)$

for all $i = 1, \dots I$ and $F_j = \bigcup_{i=1}^{I}(E_i \cap F_j)$ for all $j = 1, \dots, J$. Thus,

$$\sum_{i=1}^{I} y_i m(E_i) = \sum_{i=1}^{I} y_i m\left(\bigcup_{j=1}^{J}(E_i \cap F_j)\right) = \sum_{i=1}^{I} y_i \left(\sum_{j=1}^{J} m(E_i \cap F_j)\right) = \sum_{i=1}^{I}\sum_{j=1}^{J} y_i m(E_i \cap F_j)$$

$$\sum_{j=1}^{J} z_j m(F_j) = \sum_{j=1}^{J} z_j m\left(\bigcup_{i=1}^{I}(E_i \cap F_j)\right) = \sum_{j=1}^{J} z_j \left(\sum_{i=1}^{I} m(E_i \cap F_j)\right) = \sum_{j=1}^{J}\sum_{i=1}^{I} z_j m(E_i \cap F_j)$$

Since both of them are finite sum, we can exchange the order of summation, and it suffices to show $\sum_{i=1}^{I}\sum_{j=1}^{J} y_i m(E_i \cap F_j) = \sum_{i=1}^{I}\sum_{j=1}^{J} z_j m(E_i \cap F_j)$. There are two cases, $E_i \cap F_j = \varnothing$ and $E_i \cap F_j \neq \varnothing$. If $E_i \cap F_j = \varnothing$, then $z_j m(E_i \cap F_j) = y_i m(E_i \cap F_j) = 0$ is always true. If $E_i \cap F_j \neq \varnothing$, then there exists $x_0 \in E_i \cap F_j$. By definition of E_i and F_j (see remark of Definition 2.3), $f(x_0) = y_i$ and $f(x_0) = z_j$, so $z_j = y_i$. Since i, j are arbitrary, $\sum_{i=1}^{I}\sum_{j=1}^{J} y_i m(E_i \cap F_j) = \sum_{i=1}^{I}\sum_{j=1}^{J} z_j m(E_i \cap F_j)$ is proved. □

Property *Let $f(x) \geq 0$ be measurable simple function on $E \in \mathcal{M}$. Then,*

1. *If $f(x) \leq g(x)$ on E, then $\int_E f(x)\,dx \leq \int_E g(x)\,dx$.*
2. *If $A \subset B \subset E$ and $A, B \in \mathcal{M}$, then $\int_A f(x)\,dx \leq \int_B f(x)\,dx$.*
3. *If $c > 0$ is a constant, then $\int_E cf(x)\,dx = c\int_E f(x)\,dx$.*
4. *If $f(x) = 0$ on E, then $\int_E f(x)\,dx = 0$ even if $m(E) = \infty$.*
5. *If $m(E) = 0$, then $\int_E f(x)\,dx = 0$ even if $f(x) = \infty$ on E.*
6. *$\int_E f(x)\,dx = \int_{\mathbb{R}^n} I_E(x)f(x)\,dx$.*

Proof

1. Suppose $f(x) = \sum_{i=1}^{I} y_i I_{E_i}(x)$ and $g(x) = \sum_{j=1}^{J} z_j I_{F_j}(x)$ where $E = \bigcup_{i=1}^{I} E_i = \bigcup_{j=1}^{J} F_j$, E_i's are pairwise disjoint, F_j's pariwise disjoint, and $E_i, F_j \in \mathcal{M}$ for all $i = 1, \dots, I$ and $j = 1, \dots, J$. By the same argument as in the proof of Exercise 3.1,
$$\int_E f(x)\,dx = \sum_{i=1}^{I}\sum_{j=1}^{J} y_i m(E_i \cap F_j), \qquad \int_E g(x)\,dx = \sum_{i=1}^{I}\sum_{j=1}^{J} z_j m(E_i \cap F_j)$$
For (i, j) pair s.t. $E_i \cap F_j = \varnothing$, $y_i m(E_i \cap F_j) = z_j m(E_i \cap F_j)$. For (i, j) pair s.t. $E_i \cap F_j \neq \varnothing$, there exists $x_0 \in E_i \cap F_j$ s.t. $f(x_0) = y_i$ and $g(x_0) = z_j$. Since $f(x_0) \geq g(x_0) \geq 0$, $y_i \geq z_j \geq 0$. Thus, $y_i m(E_i \cap F_j) \geq z_j m(E_i \cap F_j)$ for all (i, j) pair. This shows
$$\int_E f(x)\,dx = \sum_{i=1}^{I}\sum_{j=1}^{J} y_i m(E_i \cap F_j) \geq \sum_{i=1}^{I}\sum_{j=1}^{J} z_j m(E_i \cap F_j) = \int_E g(x)\,dx$$
2. Suppose $f(x) = \sum_{i=1}^{I} y_i I_{E_i}(x)$ where $E = \bigcup_{i=1}^{I} E_i$, E_i's are pairwise disjoint, and $E_i \in \mathcal{M}$ for all $i = 1, \dots, I$. Now let $A_i = E_i \cap A$ and $B_i = E_i \cap B$ for all $i = 1, \dots, I$. Then $A = \bigcup_{i=1}^{I} A_i$ and $B = \bigcup_{i=1}^{I} B_i$. Furthermore, A_i's are pairwise disjoint, B_i's are

pairwise disjoint, and $f(A_i) = f(B_i) = \{y_i\}$. Therefore,

$$\int_A f(x)\ dx = \sum_{i=1}^{I} y_i m(A_i), \quad \int_B f(x)\ dx = \sum_{i=1}^{I} y_i m(B_i)$$

Since $A \subset B$, $m(A_i) \leq m(B_i)$, thus $\int_A f(x)\ dx \leq \int_B f(x)\ dx$.

3. Suppose $f(x) = \sum_{i=1}^{I} y_i I_{E_i}(x)$ where $E = \bigcup_{i=1}^{I} E_i$, E_i's are pairwise disjoint, and $E_i \in \mathcal{M}$ for all $i = 1, \ldots, I$. Then $cf(x) = \sum_{i=1}^{I}(cy_i)I_{E_i}(x)$. Therefore,

$$\int_E cf(x)\ dx = \sum_{i=1}^{I}(cy_i)m(E_i), \quad c\int_E f(x)\ dx = c\sum_{i=1}^{I} y_i m(E_i)$$

Thus, $\int_E cf(x)\ dx = c\int_E f(x)\ dx$ when $c > 0$.

4. If $f(x) = 0$ on E, then $f(x) = 0 \cdot I_E(x)$, and thus $\int_E f(x)\ dx = 0 \cdot m(E) = 0$ even if $m(E) = \infty$.

5. Suppose $f(x) = \sum_{i=1}^{I} y_i I_{E_i}(x)$ where $E = \bigcup_{i=1}^{I} E_i$, E_i's are pairwise disjoint, and $E_i \in \mathcal{M}$ for all $i = 1, \ldots, I$. Then $m(E_i) = 0$ for all $i = 1, \ldots, I$ because $m(E) = 0$. Therefore, $\int_E f(x)\ dx = \sum_{i=1}^{I} y_i m(E_i) = 0$ even if $y_i = \infty$ for all $i = 1, \ldots, I$.

6. Suppose $f(x) = \sum_{i=1}^{I} y_i I_{E_i}(x)$ where $E = \bigcup_{i=1}^{I} E_i$, E_i's are pairwise disjoint, and $E_i \in \mathcal{M}$ for all $i = 1, \ldots, I$. Then let $g(x) = I_E(x)f(x)$ be defined on $\mathbb{R}^n$, and we have $g(x) = f(x)$ on E and $g(x) = 0$ on E^c. This implies $g(x) = \sum_{i=1}^{I+1} y_i I_{E_i}(x)$ where $y_{I+1} = 0$ and $E_{I+1} = E^c$. Thus,

$$\int_{\mathbb{R}^n} g(x)\ dx = \sum_{i=1}^{I+1} y_i m(E_i) = \sum_{i=1}^{I} y_i m(E_i) = \int_E f(x)\ dx$$

□

Definition 3.3. Lebesgue Integrals of Nonnegative Measurable Functions

*Let $f(x)$ be measurable and nonnegative on $E \in \mathcal{M}$. The **Lebesgue integral** of $f(x)$ on E is defined by $\int_E f(x)\ dx = \sup(S(f;E))$, where set $S(f;E)$ is*

$$S(f;E) = \left\{ \int_E s(x)\ dx \,\middle|\, 0 \leq s(x) \leq f(x),\ s(x) \text{ measurable simple on } E \right\}$$

♣

✍ **Exercise 3.2** Use Definition 3.3 to generalize all six properties of Lebesgue integral for nonnegative measurable simple functions to Lebesgue integral for nonnegative measurable functions.

Proof

1. By Definition 3.3, we claim $S(f;E) \subset S(g;E)$. If the claim is true, then

$$\int_E f(x)\ dx = \sup(S(f;E)) \leq \sup(S(g;E)) = \int_E g(x)\ dx$$

To prove the claim, for all $L \in S(f;E)$, there exists nonnegative measurable simple $s(x)$ on E s.t. $s(x) \leq f(x)$ and $\int_E s(x)\ dx = L$. Since $f(x) \leq g(x)$ on E, $s(x) \leq g(x)$ on E, so $L \in S(g;E)$. Therefore, $S(f;E) \subset S(g;E)$ and we are done.

2. Take arbitrary element $L \in S(f;A)$, then there exists nonnegative measurable simple

function $s(x)$ on A s.t. $s(x) \leq f(x)$ and $\int_A s(x)\ dx = L$. Consider $\tilde{s}(x)$ on B defined by $\tilde{s}(x) = s(x)$ on A and $\tilde{s}(x) = 0$ on $B \setminus A$. Then $\tilde{s}(x)$ is a nonegative measurable simple function on B s.t. $\tilde{s}(x) \leq f(x)$. Notice that

$$L = \int_A s(x)\ dx = \int_A \tilde{s}(x)\ dx \leq \int_B \tilde{s}(x)\ dx$$

where the last inequality is due to property of Lebesgue integral for measurable simple functions. This shows that for all $L \in S(f;A)$, there exists $L' \geq L$ s.t. $L' \in S(f;B)$. This implies $\sup(S(f;A)) \leq \sup(S(f;B))$, so $\int_A f(x)\ dx \leq \int_B f(x)\ dx$.

3. Take arbitrary $L \in S(cf;E)$, then there exists nonnegative measurable simple function $s(x)$ on E s.t. $s(x) \leq cf(x)$ and $\int_E s(x)\ dx = L$. Since $c > 0$, $\frac{s(x)}{c} \leq f(x)$. Note that $\tilde{s}(x) = \frac{s(x)}{c}$ is also nonnegative measurable simple function on E. By property of measurable simple function, $\int_E \tilde{s}(x)\ dx = \frac{1}{c}\int_E s(x)\ dx = \frac{L}{c}$. Thus, $\frac{L}{c} \in S(f;E)$, and so $\frac{L}{c} \leq \int_E f(x)\ dx$. Since L is arbitrary, we have $\int_E cf(x)\ dx \leq c\int_E f(x)\ dx$. Similarly, take arbitrary $L' \in S(f;E)$, then there exists nonnegative measurable function $t(x)$ on E s.t. $t(x) \leq f(x)$ and $\int_E t(x)\ dx = L'$. Then $\tilde{t}(x) = ct(x) \leq cf(x)$ is also nonnegative measurable simple function on E and $\int_E \tilde{t}(x)\ dx = c\int_E t(x)\ dx = cL'$. Thus, $cL' \in S(cf;E)$ and $cL' \leq \int_E cf(x)\ dx$. Since L' is arbitrary, $c\int_E f(x)\ dx \leq \int_E cf(x)\ dx$. This shows $\int_E cf(x)\ dx = c\int_E f(x)\ dx$.
4. If $f(x) = 0$ on E, then $f(x)$ must be measurable simple function, so this one is the same as the proof of the property for measurable simple function.
5. If $m(E) = 0$, then $S(f;E) = \{0\}$, so $\int_E f(x)\ dx = \sup\{0\} = 0$.
6. Let $g(x) = I_E(x)f(x)$ be defined on $\mathbb{R}^n$, so $g(x) = f(x)$ on E and $g(x) = 0$ on E^c. This shows $\int_E g(x)\ dx = \int_E f(x)\ dx$. By part 2., we have

$$\int_{\mathbb{R}^n} I_E(x)f(x)\ dx = \int_{\mathbb{R}^n} g(x)\ dx \geq \int_E g(x)\ dx = \int_E f(x)\ dx$$

Now it suffices to show $\int_{\mathbb{R}^n} I_E(x)f(x)\ dx \leq \int_E f(x)\ dx$. For arbitrary $L \in S(g;\mathbb{R}^n)$, there exists nonnegative measurable simple function $s(x)$ on $\mathbb{R}^n$ s.t. $s(x) \leq g(x)$ and $\int_{\mathbb{R}^n} s(x)\ dx = L$. Notice that $s\big|_E(x)$ on E is also nonnegative measurable simple function s.t. $s\big|_E(x) \leq f(x)$, and by property of measurable simple function, $\int_E s\big|_E(x)\ dx = \int_{\mathbb{R}^n} I_E(x)s\big|_E(x)\ dx$. Notice that $I_E(x)s\big|_E(x) = 0$ on E^c and $I_E(x)s\big|_E(x) = s(x)$ on E, but $s(x) = 0$ on E^c because $g(x) = 0$ on E^c. This shows $I_E(x)s\big|_E(x) = s(x)$ on $\mathbb{R}^n$, so $\int_E s\big|_E(x)\ dx = \int_{\mathbb{R}^n} s(x)\ dx = L$. Thus, $L \in S(f;E)$ and $S(g;\mathbb{R}^n) \subset S(f;E)$. Take supremum, and we obtain $\int_{\mathbb{R}^n} g(x)\ dx \leq \int_E f(x)\ dx$.

□

Conclusion *From property of Lebesgue integrals of nonnegative measurable simple functions to Exercise 3.2, we can see that some properties of Lebesgue integrals can be generalized from integrals for nonnegative simple functions to general nonnegative functions by definition. However, in the next section, we will see that if we want to generalize some other properties, like* *linearity or integration term by term of Lebesgue integrals, from nonnegative simple functions to nonnegative functions, we need to use the so called monotone convergence theorem (baby version).*

Problem Set 3.1

1. Let $f(x)$ be measurable and nonnegative on $E \in \mathcal{M}$. Suppose $\int_E f(x)\,dx = 0$. Prove that $f = 0$ a.e. on E.
2. Let $f(x) \geq 0$ be measurable on $E \in \mathcal{M}$, and positive a.e. on E with $\int_E f(x)\,dx = 0$. Prove that $m(E) = 0$.
3. Let $f(x)$ be measurable on $[0,1]$ s.t. $f(x) > 0$, for all $x \in [0,1]$. Prove that for all $q \in (0,1)$, there exists $\delta > 0$ s.t. $\int_E f(x)\,dx > \delta$, whenever $E \subset [0,1]$, $E \in \mathcal{M}$ and $m(E) \geq q$.

3.2 Monotone Convergence Theorem

In this section, we continue exploring properties of Lebesgue integrals for nonnegative measurable functions. Similar to the last section, we first show the properties are true for nonnegative measurable simple functions.

Exercise 3.3 If $f(x)$ and $g(x)$ are nonnegative measurable simple functions on E, then
$$\int_E [f(x) + g(x)]\,dx = \int_E f(x)\,dx + \int_E g(x)\,dx$$

Proof Let $f(x) = \sum_{i=1}^I y_i I_{E_i}(x)$, where E_i's are pairwise disjoint and $E = \bigcup_{i=1}^I E_i$; $g(x) = \sum_{j=1}^J z_j I_{F_j}(x)$, where F_j's are pairwise disjoint and $E = \bigcup_{j=1}^J F_j$. By the same argument in the proof of the first property of Lebesgue integrals for measurable simple function,
$$\int_E f(x)\,dx = \sum_{i=1}^I \sum_{j=1}^J y_i m(E_i \cap F_j), \qquad \int_E g(x)\,dx = \sum_{i=1}^I \sum_{j=1}^J z_j m(E_i \cap F_j)$$
Notice that $I_{E_i}(x) = \sum_{j=1}^J I_{E_i \cap F_j}(x)$ and $I_{F_j}(x) = \sum_{i=1}^I I_{E_i \cap F_j}(x)$ for any $i = 1, \ldots, I$ and $j = 1, \ldots, J$, so we have
$$f(x) + g(x) = \sum_{i=1}^I y_i \sum_{j=1}^I I_{E_i \cap F_j}(x) + \sum_{j=1}^J z_j \sum_{i=1}^I I_{E_i \cap F_j}(x)$$
This verifies that $f(x) + g(x)$ is indeed a simple function, and thus,
$$\int_E [f(x) + g(x)]\,dx = \sum_{i=1}^I \sum_{j=1}^J (y_i + z_j) m(E_i \cap F_j) = \int_E f(x)\,dx + \int_E g(x)\,dx$$
□

Exercise 3.4 Suppose $E_1, E_2 \in \mathcal{M}$, $E = E_1 \cup E_2$ and E_1, E_2 are disjoint. If $f(x)$ is nonnegative measurable simple function on E, then $\int_E f(x)\,dx = \int_{E_1} f(x)\,dx + \int_{E_2} f(x)\,dx$.

Proof By Exercise 3.2, part 6., we can write $\int_E f(x)\ dx = \int_{\mathbb{R}^n} I_E(x)f(x)\ dx$ and

$$\int_{E_1} f(x)\ dx = \int_{\mathbb{R}^n} I_{E_1}(x)f(x)\ dx, \quad \int_{E_2} f(x)\ dx = \int_{\mathbb{R}^n} I_{E_2}(x)f(x)\ dx$$

Notice that $I_E(x)f(x) = I_{E_1}(x)f(x) + I_{E_2}(x)f(x)$ on $\mathbb{R}^n$, so by Exercise 3.3,

$$\int_{\mathbb{R}^n} I_E(x)f(x)\ dx = \int_{\mathbb{R}^n} I_{E_1}(x)f(x)\ dx + \int_{\mathbb{R}^n} I_{E_2}(x)f(x)\ dx$$

This also shows $\int_E f(x)\ dx = \int_{E_1} f(x)\ dx + \int_{E_2} f(x)\ dx$. □

Theorem 3.1. Monotone Convergence Theorem I (MCT-I)

Let $\{f_n(x)\}_{n=1}^{\infty}$ be measurable simple and nonnegative on $E \in \mathcal{M}$. For each fixed $x \in E$, $\{f_n(x)\}_{n=1}^{\infty}$ is an increasing sequence in n. Let $f(x) = \lim_{n\to\infty} f_n(x)$, then

$$\lim_{n\to\infty} \int_E f_n(x)\ dx = \int_E f(x)\ dx = \int_E \lim_{n\to\infty} f_n(x)\ dx$$

♡

Proof Since $f_n(x)$ is increasing for each fixed x, the pointwise limit of $f_n(x)$ always exists (perhaps equal to infinity), so $f(x)$ is nonnegative measurable function (may not be simple). Also, $f_n(x) \le f(x)$ for all $n \ge 1$ on E, so by Exercise 3.2, part 1., $\int_E f_n(x)\ dx \le \int_E f(x)\ dx$ for all $n \ge 1$. Take $n \to \infty$ on both sides, we have $\lim_{n\to\infty} \int_E f_n(x)\ dx \le \int_E f(x)\ dx$. Notice that this limit also exists because $\int_E f_n(x)\ dx$ is also an increasing sequence in n by applying Exercise 3.2, part 1. to $f_n(x) \le f_{n+1}(x)$ for $n \ge 1$.

Now we only need to show $\lim_{n\to\infty} \int_E f_n(x)\ dx \ge \int_E f(x)\ dx$. It suffices to show $\lim_{n\to\infty} \int_E f_n(x)\ dx \ge L$ for all $L \in S(f;E)$. For each L, there exists nonnegative measurable simple function $s(x)$ on E s.t. $s(x) \le f(x)$ and $\int_E s(x)\ dx = L$. Let $s(x) = \sum_{i=1}^{I} y_i I_{E_i}(x)$ where E_i's are pairwise disjoint and $E = \bigcup_{i=1}^{I} E_i$. By repeatedly applying Exercise 3.4, we have $\int_E s(x)\ dx = \sum_{i=1}^{I} \int_{E_i} s(x)\ dx$. Similarly, $\int_E f_n(x)\ dx = \sum_{i=1}^{I} \int_{E_i} f_n(x)\ dx$ for all $n \ge 1$. Therefore, we need to prove $\lim_{n\to\infty} \int_{E_i} f_n(x)\ dx \ge \int_{E_i} s(x)\ dx = y_i m(E_i)$ for all $i = 1, \dots, I$. We discuss three cases:

1. If $y_i = 0$, then $\lim_{n\to\infty} \int_{E_i} f_n(x)\ dx \ge 0 = y_i m(E_i)$ is obvious.
2. If $0 < y_i < \infty$, then for all $\epsilon > 0$, let $A_n^{\epsilon} = \{x \in E_i \mid f_n(x) \ge y_i - \epsilon\} \in \mathcal{M}$. Since $f_n(x)$ is increasing, $A_1^{\epsilon} \subset \cdots \subset A_n^{\epsilon} \subset \cdots$. Notice that $f(x) \ge s(x)$, and on E_i, $s(x) = y_i$, so there exists large N s.t. for all $n \ge N$, $f_n(x) \ge y_i - \epsilon$. This shows $E_i = \lim_{n\to\infty} A_n^{\epsilon}$. By continuity of Lebesgue measure, $m(E_i) = \lim_{n\to\infty} m(A_n^{\epsilon})$. By Exercise 3.2, part 2. and part 1., we have $\int_{E_i} f_n(x)\ dx \ge \int_{A_n^{\epsilon}} f_n(x)\ dx \ge \int_{A_n} (y_i - \epsilon)\ dx = (y_i - \epsilon) m(A_n)$. Take $n \to \infty$ on both sides, we obtain $\lim_{n\to\infty} \int_{E_i} f_n(x)\ dx \ge (y_i - \epsilon) m(E_i)$. Take $\epsilon \to 0$, we have the desired result.
3. If $y_i = \infty$, then for all $M \ge 1$, define $B_n^M = \{x \in E_i \mid f_n(x) \ge M\}$. Since $f_n(x)$ is increasing, $B_1^M \subset \cdots \subset B_n^M \subset \cdots$. Note that $f(x) \ge s(x) = \infty$, so there exists large N s.t. for all $n \ge N$, $f_n(x) \ge M$. Thus, $E_i = \lim_{n\to\infty} B_n^M$. By continuity of Lebesgue

measure, $m(E_i) = \lim_{n\to\infty} m(B_n^M)$. By Exercise 3.2, part 2. and part 1., we have

$$\int_{E_i} f_n(x)\ dx \ge \int_{B_n^M} f_n(x)\ dx \ge \int_{B_n^M} M\ dx = Mm(B_n^M)$$

Take $n \to \infty$, we have $\lim_{n\to\infty} \int_{E_i} f_n(x)\ dx \ge Mm(E_i)$. If $m(E_i) > 0$, then by taking $M \to \infty$, we can obtain $\lim_{n\to\infty} \int_{E_i} f_n(x)\ dx \ge \infty = y_i m(E_i)$. If $m(E_i) = 0$, then $y_i m(E_i) = 0$, so $\lim_{n\to\infty} \int_{E_i} f_n(x)\ dx \ge y_i m(E_i)$ is trivial.

Combine all three cases, we finish the proof of this theorem. □

✍ **Exercise 3.5** Prove the result in Exercise 3.3 and Exercise 3.4 is also true for general nonnegative measurable functions.

Proof For nonnegative measurable functions $f(x)$ and $g(x)$, by simple approximation theorem, there exists sequences of nonnegative measurable simple functions $\{\phi_k\}_{k=1}^\infty$ and $\{\psi_k\}_{k=1}^\infty$ s.t. $\phi_k(x) \to f(x)$ and $\psi_k(x) \to g(x)$ pointwisely on E. By Exercise 3.3, for all $k \ge 1$,

$$\int_E [\phi_k(x) + \psi_k(x)]\ dx = \int_E \phi_k(x)\ dx + \int_E \psi_k(x)\ dx$$

Notice that $\phi_k(x)$ and $\psi_k(x)$ are increasing in k for each fixed x, so $\phi_k(x) + \psi_k(x)$ is also increasing and converges to $f(x) + g(x)$ pointwisely. Thus, by MCT-I, we have

$$\begin{aligned}\int_E [f(x) + g(x)]\ dx &= \lim_{k\to\infty} \int_E [\phi_k(x) + \psi_k(x)]\ dx \\ &= \lim_{k\to\infty} \int_E \phi_k(x)\ dx + \lim_{k\to\infty} \int_E \psi_k(x)\ dx \\ &= \int_E f(x)\ dx + \int_E g(x)\ dx\end{aligned}$$

To prove the result in Exercise 3.4 for general nonnegative measurable function, we can adopt the same method as in the proof of Exercise 3.4. The details are omitted. □

✍ **Exercise 3.6** Let $f(x)$ be nonnegative measurable function on $E \in \mathcal{M}$. If set Z satisfies $m(Z) = 0$, then $\int_E f(x)\ dx = \int_{E\setminus Z} f(x)\ dx$.

Proof By Exercise 3.2, part 5., we have $\int_Z f(x)\ dx = 0$. Since Z and $E \setminus Z$ are disjoint, by Exercise 3.5, $\int_E f(x)\ dx = \int_Z f(x)\ dx + \int_{E\setminus Z} f(x)\ dx = \int_{E\setminus Z} f(x)\ dx$. □

Problem 3.1 Let $f(x)$ and $g(x)$ be nonnegative measurable functions on $E \in \mathcal{M}$. Use the result in Exercise 3.6 to prove

1. If $f(x) = g(x)$ a.e. on E, then $\int_E f(x)\ dx = \int_E g(x)\ dx$.
2. If $f(x) = 0$ a.e. on E, then $\int_E f(x)\ dx = 0$.

✍ **Exercise 3.7** Let $f(x)$ be nonnegative measurable functions on $E \in \mathcal{M}$. If $\int_E f(x)\ dx < \infty$, then $f(x)$ is finite a.e. on E.

Proof Let $B = \{x \in E \mid f(x) = \infty\}$. By Exercise 3.2, part 2., we have

$$\int_B f(x)\ dx \le \int_E f(x)\ dx < \infty$$

Since $f(x)$ on B is a constant, so it is a simple function on B, and $\int_B f(x)\ dx = \infty \cdot m(B) < \infty$. If $m(B) > 0$, we will have $\infty \cdot m(B) = \infty$, so this contradiction shows $m(B) = 0$. □

Exercise 3.8 Let $f(x)$ be nonnegative measurable functions on $E \in \mathcal{M}$. Prove that for all $\alpha \in (0, \infty)$, $m(E_\alpha) \leq \frac{1}{\alpha} \int_E f(x)\ dx$, where $E_\alpha = \{x \in E \,|\, f(x) > \alpha\}$. This statement corresponds to the famous **Markov's inequality** in probability theory.

Proof By Exercise 3.2, part 2., we have $\int_{E_\alpha} f(x)\ dx \leq \int_E f(x)\ dx$. Since on E_α, $f(x) > \alpha$, by Exercise 3.2, part 1., $\int_{E_\alpha} f(x)\ dx \geq \int_{E_\alpha} \alpha\ dx = \alpha m(E_\alpha)$. Thus, $m(E_\alpha) \leq \frac{1}{\alpha} \int_E f(x)\ dx$. □

Recall we proved the linearity property of Lebesgue integrals for nonnegative measurable functions in Exercise 3.5. By using induction, we can easily see that

$$\int_E \sum_{i=1}^n f_i(x)\ dx = \sum_{i=1}^n \int_E f_i(x)\ dx$$

for nonnegative measurable functions $\{f_i(x)\}_{i=1}^n$ on $E \in \mathcal{M}$. Now we want to prove this is also true for sequence of nonnegative measurable functions $\{f_i(x)\}_{i=1}^\infty$. Such property is usually called integration term by term property.

Theorem 3.2. Integration Term by Term I (ITT-I)

Let $\{f_n(x)\}_{n=1}^\infty$ be a sequence of nonnegative measurable functions on $E \in \mathcal{M}$. Let $f(x) = \sum_{n=1}^\infty f_n(x)$ on E. Then $\int_E f(x)\ dx = \sum_{n=1}^\infty \int_E f_n(x)\ dx$.

Proof For all fixed $k \geq 1$, $\sum_{n=1}^k f_n(x) \leq f(x)$ on E, so $\int_E \sum_{n=1}^k f_n(x)\ dx \leq \int_E f(x)\ dx$ by Exercise 3.2, part 1.. By linearity property of Lebesgue integrals for nonnegative measurable functions in Exercise 3.5, $\sum_{n=1}^k \int_E f_n(x)\ dx \leq \int_E f(x)\ dx$. Take $k \to \infty$, we have $\sum_{n=1}^\infty \int_E f_n(x)\ dx \leq \int_E f(x)\ dx$. To verify the other direction, by simple approximation theorem, there exists nonnegative measurable simple functions $\{f_{kj}(x)\}_{j=1}^\infty$ s.t. $f_{kj}(x) \to f_k(x)$ pointwisely and $f_{kj}(x)$ is increasing in j on E for each fixed $k \geq 1$. Let $S_k(x) = \sum_{i=1}^k f_{ik}(x)$, then $S_k(x)$ is nonnegative measurable simple function and $S_k(x) \leq S_{k+1}(x)$ on E for all $k \geq 1$. Also, $f(x) \geq S_k(x)$, and we have $\lim_{k\to\infty} S_k(x) \leq f(x)$. However, for each fixed $m \geq 1$, if $m \leq k$, $S_k(x) \geq \sum_{i=1}^m f_{ik}(x)$. Take $k \to \infty$ on both sides, $\lim_{k\to\infty} S_k(x) \geq \sum_{i=1}^m f_i(x)$. Take $m \to \infty$, we have $\lim_{k\to\infty} S_k(x) \geq f(x)$. Therefore, we have $\lim_{k\to\infty} S_k(x) = f(x)$. By MCT-I, we have $\lim_{k\to\infty} \int_E S_k(x)\ dx = \int_E f(x)\ dx$. Since $S_k(x) \leq \sum_{i=1}^k f_i(x)$, by Exercise 3.2, part 1., $\int_E S_k(x)\ dx \leq \int_E \sum_{i=1}^k f_i(x)\ dx$. By linearity property in in Exercise 3.5, $\int_E \sum_{i=1}^k f_i(x)\ dx = \sum_{i=1}^k \int_E f_i(x)\ dx$. In conclusion,

$$\int_E f(x)\ dx = \lim_{k\to\infty} \int_E S_k(x)\ dx \leq \lim_{k\to\infty} \sum_{i=1}^k \int_E f_i(x)\ dx = \sum_{i=1}^\infty \int_E f_i(x)\ dx$$

Therefore, $\int_E f(x)\ dx \leq \sum_{n=1}^\infty \int_E f_n(x)\ dx$ and the proof is finished. □

Corollary 3.1

Suppose $E = \bigcup_{k=1}^{\infty} E_k$, $E_k \in \mathcal{M}$, E_k's are pairwise disjoint, and $f(x)$ is a nonnegative measurable function on E. Then $\int_E f(x)\ dx = \sum_{k=1}^{\infty} \int_{E_k} f(x)\ dx$.

Proof Notice that this is a further generalization of Exercise 3.4 because it allows union of infinitely many sets. In this case, we can adopt the same method as in Exercise 3.4. Let $f_k(x) = I_{E_k}(x) f(x)$ be defined on E, then $f(x) = \sum_{k=1}^{\infty} f_k(x)$. By ITT-I, we have

$$\int_E f(x)\ dx = \sum_{k=1}^{\infty} \int_E f_k(x)\ dx = \sum_{k=1}^{\infty} \int_E I_{E_k}(x) f(x)\ dx$$

By Exercise 3.2, part 6., we have $\int_E I_{E_k}(x) f(x)\ dx = \int_{\mathbb{R}^n} I_E(x) I_{E_k}(x) f(x)\ dx$. Notice that $I_E(x) I_{E_k}(x) = I_{E_k}(x)$, so $\int_{\mathbb{R}^n} I_E(x) I_{E_k}(x) f(x)\ dx = \int_{\mathbb{R}^n} I_{E_k}(x) f(x)\ dx$. Apply Exercise 3.2, part 6. again, we have $\int_{\mathbb{R}^n} I_{E_k}(x) f(x)\ dx = \int_{E_k} f(x)\ dx$. Thus, we obtain the desired result, i.e., $\int_E f(x)\ dx = \sum_{k=1}^{\infty} \int_{E_k} f(x)\ dx$. □

Now we are going to see an interesting application of the above corollary. Recall in the definition of Riemann integral, we partition the domain into many subintervals and define the limit of the upper sum and lower sum to be the integral value. In fact, we can define Lebesgue integral in a similar way as Riemann integral, just with domain partition replaced by codomain partition. The following example shows the details and that such kind of definition is equivalent to our previous definition of Lebesgue integral for nonnegative measurable function.

Example 3.2 Let $f(x) \geq 0$ and measurable on $E \in \mathcal{M}$. Also, let $f(x)$ be finite a.e. on E with $m(E) < \infty$. Let $y_0 = 0 < y_1 < \cdots < y_k < \cdots$ with $y_{k+1} - y_k < \delta$ for all $k \geq 0$ and $y_k \to \infty$ as $k \to \infty$. Let $E_k = \{x \in E \,|\, y_k \leq f(x) < y_{k+1}\}$, then

1. $\int_E f(x)\ dx < \infty$ if and only if $\sum_{k=0}^{\infty} y_k m(E_k) < \infty$.
2. $\lim_{\delta \to 0} \sum_{k=0}^{\infty} y_k m(E_k) = \int_E f(x)\ dx$.

Proof Let $Z = \{x \in E\ f(x) = \infty\}$, then $m(Z) = 0$. Observe that $E \setminus Z = \bigcup_{k=0}^{\infty} E_k$ where E_k's are disjoint and measurable. By Corollary 3.1, $\int_{E \setminus Z} f(x)\ dx = \sum_{k=0}^{\infty} \int_{E_k} f(x)\ dx$. By Exercise 3.6, $\int_E f(x)\ dx = \int_{E \setminus Z} f(x)\ dx$. By Exercise 3.2, part 1., we have

$$y_k m(E_k) = \int_{E_k} y_k\ dx \leq \int_{E_k} f(x)\ dx \leq \int_{E_k} y_{k+1}\ dx = y_{k+1} m(E_k)$$

Take summation on both sides, we have

$$\sum_{k=0}^{\infty} y_k m(E_k) \leq \int_E f(x)\ dx \leq \sum_{k=0}^{\infty} y_{k+1} m(E_k) \leq \sum_{k=0}^{\infty} (\delta + y_k) m(E_k)$$

By σ-additivity, we have

$$\sum_{k=0}^{\infty} y_k m(E_k) \leq \int_E f(x)\ dx \leq \delta m(E) + \sum_{k=0}^{\infty} y_k m(E_k)$$

If $\int_E f(x)\ dx < \infty$, $\sum_{k=0}^{\infty} y_k m(E_k) < \infty$; if $\sum_{k=0}^{\infty} y_k m(E_k) < \infty$, then since $\delta m(E) < \infty$, $\int_E f(x)\ dx < \infty$. Furthermore, take $\overline{\lim}_{\delta \to 0}$ on both sides of the first inequality and then take

$\overline{\lim}_{\delta\to 0}$ on both sides of the second inequality (upper and lower limit always exists, although may be infinite), we have

$$\overline{\lim_{\delta\to 0}}\sum_{k=0}^{\infty} y_k m(E_k) \le \int_E f(x)\ dx \le 0 + \underline{\lim_{\delta\to 0}}\sum_{k=0}^{\infty} y_k m(E_k)$$

Since we always have upper limit no less than lower limit, all of the inequality above becomes equality and thus the limit exists and is equal to $\int_E f(x)\ dx$. □

Remark This example shows that when $m(E) < \infty$, we can define Lebesgue integral in a similar way as Riemann integral. The y_i's can be regarded as a partition on codomain. However, in fact, even if $m(E) = \infty$, we can still prove the same result (see Problem Set 3.2, Question 6.).

Now, we are ready to prove the monotone convergence theorem for general nonnegative measurable functions (MCT-II).

Theorem 3.3. Monotone Convergence Theorem II (MCT-II)

Let $\{f_n(x)\}_{n=1}^{\infty}$ be measurable and nonnegative on $E \in \mathcal{M}$. For each fixed $x \in E$, $\{f_n(x)\}_{n=1}^{\infty}$ is an increasing sequence in n. Let $f(x) = \lim_{n\to\infty} f_n(x)$, then

$$\lim_{n\to\infty}\int_E f_n(x)\ dx = \int_E f(x)\ dx = \int_E \lim_{n\to\infty} f_n(x)\ dx$$

♡

Proof If there exists $k_0 \ge 1$ s.t. $m(A_0) > 0$ where $A_0 = \{x \in E \mid f_{k_0}(x) = \infty\}$. Notice that $f(x)$ is also nonnegative measurable function on E, so $m(A) > 0$ where $A = \{x \in E \mid f(x) = \infty\}$. Then by Exercise 3.2, part 2., $\int_E f(x)\ dx \ge \int_A f(x)\ dx = \infty\cdot m(A) = \infty$. Since $\int_E f_n(x)\ dx$ is an increasing sequence in n, for all $n \ge k_0$, we have $\int_E f_n(x)\ dx \ge \int_E f_{k_0}(x)\ dx$. Take limit on both sides, we have $\lim_{n\to\infty}\int_E f_n(x)\ dx \ge \int_E f_{k_0}(x)\ dx$. By Exercise 3.2, part 2., $\int_E f_{k_0}(x)\ dx \ge \int_{A_0} f_{k_0}(x)\ dx = \infty\cdot m(A_0) = \infty$. This shows $\lim_{n\to\infty}\int_E f_n(x)\ dx = \infty$, so the desired property holds.

If for all $n \ge 1$, $m(E_n) = 0$ where $E_n = \{x \in E \mid f_n(x) = \infty\}$, then let $E_\infty = \bigcup_{n=1}^{\infty} E_n$, and by σ-subadditivity, $m(E_\infty) = 0$. Denote $E' = E \setminus E_\infty$, then since we have Exercise 3.6, it suffices to show $\lim_{n\to\infty}\int_{E'} f_n(x)\ dx = \int_{E'} f(x)\ dx$. For $x \in E'$, let $g_1(x) = f_1(x)$ and $g_n(x) = f_n(x) - f_{n-1}(x)$ for all $n \ge 2$, then $g_n(x)$ is nonnegative measurable on E' for $n \ge 1$. By ITT-I, $\int_{E'}\sum_{n=1}^{\infty} g_n(x)\ dx = \sum_{n=1}^{\infty}\int_{E'} g_n(x)\ dx$. Observe that $\sum_{n=1}^{m} g_n(x) = f_m(x)$, so $\sum_{n=1}^{\infty} g_n(x) = f(x)$ and $\int_{E'} f(x)\ dx = \lim_{m\to\infty}\sum_{n=1}^{m}\int_{E'} g_n(x)\ dx$. By linearity property in Exercise 3.5, $\sum_{n=1}^{m}\int_{E'} g_n(x)\ dx = \int_{E'}\sum_{n=1}^{m} g_n(x)\ dx = \int_{E'} f_m(x)\ dx$. Thus, we obtain $\int_{E'} f(x)\ dx = \lim_{m\to\infty}\int_{E'} f_m(x)\ dx$. □

Example 3.3 Let $f(x)$ be nonnegative measurable function on $\mathbb{R}$. Then

$$\lim_{n\to\infty}\int_{-n}^{n} f(x)\ dx = \int_{-\infty}^{\infty} f(x)\ dx$$

Notice that here all integrals are Lebesgue integrals.

Proof Let $f_n(x) = I_{E_n}(x) f(x)$ on $\mathbb{R}$, where $E_n = (-n, n)$. Then it obvious that $\{f_n(x)\}_{n=1}^{\infty}$

is an increasing sequence of nonnegative measurable function on $\mathbb{R}$. Moreover, $f_n(x) \to f(x)$ pointwisely on $\mathbb{R}$. By MCT-II, $\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\ dx = \int_{\mathbb{R}} f(x)\ dx$. By Exercise 3.2, part 6., $\int_{E_n} f(x)\ dx = \int_{\mathbb{R}} f_n(x)\ dx$. Thus, we have $\lim_{n\to\infty} \int_{E_n} f(x)\ dx = \int_{\mathbb{R}} f(x)\ dx$. □

At the end of this pretty long section, we are going to introduce a very famous and handy lemma of Lebesgue integral for nonnegative measurable functions, the so called Fatou's lemma.

Lemma 3.1. Fatou's Lemma

Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of nonnegative measurable functions on $E \in \mathcal{M}$. Then $\int_E \underline{\lim}_{n\to\infty} f_n(x)\ dx \le \underline{\lim}_{n\to\infty} \int_E f_n(x)\ dx$. ♡

Proof Let $g_k(x) = \inf_{n\ge k} f_n(x)$, then $g_k(x)$ is nonnegative measurable function on E for all $k \ge 1$. Also, $g_k(x)$ is increasing in k for each $x \in E$ and $g_k(x) \to \underline{\lim}_{n\to\infty} f_n(x)$ pointwisely as $k \to \infty$. Apply MCT-II to $\{g_k\}_{k=1}^{\infty}$, we have $\int_E \lim_{k\to\infty} g_k(x)\ dx = \lim_{k\to\infty} \int_E g_k(x)\ dx$. Thus, $\int_E \underline{\lim}_{n\to\infty} f_n(x) = \lim_{k\to\infty} \int_E g_k(x)\ dx$. Since $g_k(x) \le f_k(x)$, by Exercise 3.2, part 1., $\int_E g_k(x)\ dx \le \int_E f_k(x)\ dx$. Hence, $\underline{\lim}_{k\to\infty} \int_E g_k(x)\ dx \le \underline{\lim}_{k\to\infty} \int_E f_k(x)\ dx$. In conclusion, we obtain $\int_E \underline{\lim}_{n\to\infty} f_n(x) \le \underline{\lim}_{k\to\infty} \int_E f_k(x)\ dx$. □

Example 3.4 Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of nonnegative measurable function on E s.t. $\int_E f_n(x)\ dx \to \int_E f(x)\ dx < \infty$ and $f_n(x) \to f(x)$ pointwisely. Then for all $A \subset E$, $A \in \mathcal{M}$, $\int_A f_n(x)\ dx \to \int_A f(x)\ dx$.

Proof By Fatou's lemma, we have

$$\underline{\lim}_{n\to\infty} \int_A f_n(x)\ dx \ge \int_A \underline{\lim}_{n\to\infty} f_n(x)\ dx = \int_A f(x)\ dx$$

$$\int_{E\setminus A} f(x)\ dx = \int_{E\setminus A} \underline{\lim}_{n\to\infty} f_n(x)\ dx \le \underline{\lim}_{n\to\infty} \int_{E\setminus A} f_n(x)\ dx$$

Thus, by Exercise 3.5, $\int_A f(x)\ dx = \int_E f(x)\ dx - \int_{E\setminus A} f(x)\ dx$. Notice that this is valid because $\int_{E\setminus A} f(x)\ dx \le \int_E f(x)\ dx < \infty$ by Exercise 3.2, part 2.. Combined the above inequalities, we obtain

$$\underline{\lim}_{n\to\infty} \int_A f_n(x)\ dx \ge \int_E f(x)\ dx - \underline{\lim}_{n\to\infty} \int_{E\setminus A} f_n(x)\ dx$$

Recall that $\overline{\lim}_{n\to\infty}(a_n + b_n) \le \overline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n$ for any two sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$. Also $\overline{\lim}_{n\to\infty}(-a_n) = -\underline{\lim}_{n\to\infty} a_n$. Thus, we have

$$\overline{\lim}_{n\to\infty} \left(\int_E f_n(x)\ dx - \int_{E\setminus A} f_n(x)\ dx \right) \le \int_E f(x)\ dx - \underline{\lim}_{n\to\infty} \int_{E\setminus A} f_n(x)\ dx$$

Combine all inequalities above, we have

$$\underline{\lim}_{n\to\infty} \int_A f_n(x)\ dx \ge \int_A f(x)\ dx \ge \overline{\lim}_{n\to\infty} \int_A f_n(x)\ dx$$

Since upper limit is always no less than lower limit, all inequalities can be changed to equalities and upper or lower limit can be changed to limit. This shows $\lim_{n\to\infty} \int_A f_n(x)\ dx = \int_A f(x)\ dx$.

□

Problem Set 3.2

1. Let $f(x)$ be nonnegative, measurable on $E \in \mathcal{M}$ satisfying $\int_E f(x)\,dx < \infty$. Let $E_k = \{x \in E \,|\, f(x) \geq k\}$, $k \geq 1$. Prove that $\sum_{k=1}^{\infty} m(E_k) < \infty$.
2. Let $f(x) \geq 0$ be measurable on $E \in \mathcal{M}$, where $m(E) < \infty$. Prove $\int_E f(x)\,dx < \infty$ if and only if $\sum_{k=0}^{\infty} 2^k m(E_{2^k}) < \infty$, where $E_k = \{x \in E \,|\, f(x) \geq k\}$ for all $k \geq 0$.
3. Let $f_k(x)$ be nonnegative and measurable on $[0,1]$ s.t. $f_k(x) \to \infty$ a.e. on $[0,1]$. Prove that $\int_0^1 f_k(x)\,dx \to \infty$.
4. Let $f_k(x)$ be nonnegative and measurable on $E \in \mathcal{M}$, $f_k(x) \to f_\infty(x)$ in measure on E. Prove that $\int_E f_\infty(x)\,dx \leq \underline{\lim}_{k\to\infty} \int_E f_k(x)\,dx$.
5. Let $E_k \subset [0,1]$, $E_k \in \mathcal{M}$, for all $k \geq 1$ s.t. $m(E_k) \geq \delta > 0$ where δ is a constant. Assume for a sequence a_k we have $\sum_{k=1}^{\infty} |a_k| I_{E_k}(x) < \infty$ a.e. on $[0,1]$. Prove that $\sum_{k=1}^{\infty} |a_k| < \infty$.
6. Prove that Example 3.2 is true even if $m(E) = \infty$.

3.3 Lebesgue Integrals of Measurable Functions

In the previous two sections, we have explored many useful properties of Lebesgue integrals of nonnegative measurable functions. Now we are going to finish our goal to define a new integral for all Lebesgue measurable functions in this section. Recall that in Exercise 2.10, we decompose any measurable functions $f(x)$ into positive part $f_+(x)$ and negative part $f_-(x)$ and we proved they are both measurable. This implies that we can write $f(x) = f_+(x) - (-f_-(x))$ where $f_+(x)$ and $-f_-(x)$ are both nonnegative. Thus, it is natural to use the integrals of $f_+(x)$ and $f_-(x)$ to define the Lebesgue Integrals of general measurable functions.

Definition 3.4. Lebesgue Integrals of Measurable Functions

Let $f(x)$ be measurable on $E \in \mathcal{M}$. If at least one of $\int_E f_+(x)\,dx$ or $\int_E(-f_-(x))\,dx$ is finite, then the Lebesgue Integral of $f(x)$ on E exists and is defined as

$$\int_E f(x)\,dx = \int_E f_+(x)\,dx - \int_E (-f_-(x))\,dx$$

If both $\int_E f_+(x)\,dx$ and $\int_E(-f_-(x))\,dx$ are finite, then we say $f(x)$ is ***Lebesgue integrable*** *on E and denote $f \in L^1(E)$, where $L^1(E)$ is the set of all Lebesgue integrable functions on E.* ♣

Remark In particular, if $f(x)$ is nonnegative measurable, then $f(x) \in L^1(E)$ is equivalent to $\int_E f(x)\,dx < \infty$. Notice that $f_+(x) = f(x)$ and $f_-(x) = 0$ on E, so $\int_E(-f_-(x))\,dx = 0$, thus finite automatically.

✍ **Exercise 3.9** Let $f(x)$ be measurable function on $E \in \mathcal{M}$. Prove that $f \in L^1(E)$ if and only if $|f| \in L^1(E)$, that is, $\int_E |f(x)|\,dx < \infty$.

Proof If $f \in L^1(E)$, then $\int_E f_+(x)\ dx < \infty$ and $\int_E(-f_-(x))\ dx < \infty$. Since $f_+, -f_-$ are both nonnegative measurbale, by remark of Definition 3.4, $f_+, -f_- \in L^1(E)$. Recall that $|f(x)| = f_+(x) + (-f_-(x))$, by linearity of nonnegative measurable functions,

$$\int_E |f(x)|\ dx = \int_E f_+(x)\ dx + \int_E (-f_-(x))\ dx < \infty$$

Since $|f(x)|$ is also nonnegative, by remark of Definition 3.4, $|f| \in L^1(E)$. □

Now we explore some basic properties of Lebesgue integral for general measurable function in the following exercises. Notice that many properties have occurred in the previous sections, but they are generalized from nonnegative measurable functions to general measurable functions.

✍ **Exercise 3.10** If $f(x) = 0$ a.e. on $E \in \mathcal{M}$, then $f(x)$ is measurable on E and $\int_E f(x)\ dx = 0$.

Proof For $t \in \mathbb{R}$, we want to prove $E_t = \{x \in E \,|\, f(x) > t\} \in \mathcal{M}$. If $t \geq 0$, since $E_t \subset \{x \in E \,|\, f(x) \neq 0\}$, $m^*(E_t) \leq m(\{x \in E \,|\, f(x) \neq 0\}) = 0$, and thus $E_t \in \mathcal{M}$. If $t < 0$, then $E \setminus E_t = \{x \in E \,|\, f(x) \leq t\} \subset \{x \in E \,|\, f(x) \neq 0\}$. Similarly, we will have $E \setminus E_t \in \mathcal{M}$ because it has zero outer measure. Since $E \in \mathcal{M}$, $E_t = E \setminus (E \setminus E_t) \in \mathcal{M}$. Thus, $f(x)$ is measurable on E. If $f(x) = 0$ a.e. on E, then it is easy to verify $f_+(x) = 0$ a.e. on E. Since $f_+(x)$ is nonnegative measurable, by Problem 3.1, $\int_E f_+(x)\ dx = 0$. Similarly, we can prove $\int_E(-f_-(x))\ dx = 0$ because $-f_-(x) = 0$ a.e. on E. Thus,

$$\int_E f(x)\ dx = \int_E f_+(x)\ dx - \int_E (-f_-(x))\ dx = 0 - 0 = 0$$

□

✍ **Exercise 3.11** If $f(x)$ is measurable on $E \in \mathcal{M}$ and $f \in L^1(E)$, then $f(x)$ is finite a.e. on E.

Proof By Exercise 3.9, $|f| \in L^1(E)$. Let $E_\infty = \{x \in E \,|\, |f(x)| = \infty\}$. Suppose $f(x)$ is not finite a.e. on E, then $m(E_\infty) > 0$. Thus, $\int_{E_\infty} |f(x)|\ dx = \infty \cdot m(E_\infty) = \infty$. By Exercise 3.2, part 2., $\int_E |f(x)|\ dx \geq \int_{E_\infty} |f(x)|\ dx = \infty$. Therefore, $f(x)$ is finite a.e. on E. □

Problem 3.2 Let f and g be measurable on $E \in \mathcal{M}$. If $g \in L^1(E)$ and $|f(x)| \leq g(x)$ for all $x \in E$, then $f \in L^1(E)$. In particular, if $m(E) < \infty$ and $|f(x)| \leq M$ on E, then $f \in L^1(E)$.

✍ **Exercise 3.12** Let $f(x)$ and $g(x)$ be measurable on $E \in \mathcal{M}$ and $c \in \mathbb{R}$. Suppose $f \in L^1(E)$ and $g \in L^1(E)$, then

1. $cf \in L^1(E)$ and $\int_E (cf(x))\ dx = c\int_E f(x)\ dx$.
2. $f + g \in L^1(E)$ and $\int_E [f(x) + g(x)]\ dx = \int_E f(x)\ dx + \int_E g(x)\ dx$.

Proof Compared with Exercise 3.5 and Exercise 3.2, part 3., the result we want to prove is a more general version of linearity property for Lebesgue integrals.

1. If $c = 0$, then $cf(x) = 0$ is a simple function on E, and thus we have

$$\int_E (cf(x))\ dx = 0 \cdot m(E) = 0 = 0 \cdot \int_E f(x)\ dx = c\int_E f(x)\ dx$$

Since $cf(x)$ is also nonnegative, by Exercise 3.9, $cf(x) \in L^1(E)$.

If $c > 0$, then we have $(cf(x))_+ = cf_+(x)$. Since $f_+(x)$ is nonnegative, by Exercise 3.2, part 3., $\int_E cf_+(x)\ dx = c\int_E f_+(x)\ dx$. Since $f \in L^1(E)$, $\int_E f_+(x)\ dx < \infty$, and thus $\int_E cf_+(x)\ dx < \infty$. This shows $\int_E (cf(x))_+\ dx < \infty$. Similarly, we have $-(cf(x))_- = c(-f_-(x))$. Since $-f_-(x)$ is nonnegative, by Exercise 3.2, part 3., $\int_E c(-f_-(x))\ dx = c\int_E (-f_-(x))\ dx$. $\int_E(-f_-(x))\ dx < \infty$ because $f \in L^1(E)$. Thus, $\int_E [-(cf(x))_-]\ dx = \int_E c(-f_-(x))\ dx < \infty$. By Definition 3.4, we have

$$\begin{aligned}\int_E (cf(x))\ dx &= c\int_E f_+(x)\ dx - c\int_E (-f_-(x))\ dx \\ &= c\left(\int_E f_+(x)\ dx - \int_E (-f_-(x))\ dx\right) = c\int_E f(x)\ dx\end{aligned}$$

This shows $cf \in L^1(E)$ at the same time.

If $c < 0$, then $(cf(x))_+ = cf_-(x) = (-c)(-f_-(x))$ and $-(cf(x))_- = -cf_+(x)$. Notice that $-f_-(x)$ and $f_+(x)$ are nonnegative, by Exercise 3.2, part 3., we have

$$\int_E (cf(x))_+\ dx = (-c)\int_E (-f_-(x))\ dx, \quad \int_E [-(cf(x))_-]\ dx = (-c)\int_E f_+(x)\ dx$$

Since $f \in L^1(E)$, $\int_E(-f_-(x))\ dx < \infty$ and $\int_E f_+(x)\ dx < \infty$, so

$$\begin{aligned}\int_E (cf(x))\ dx &= (-c)\int_E (-f_-(x))\ dx + c\int_E f_+(x)\ dx \\ &= c\left(\int_E f_+(x)\ dx - \int_E (-f_-(x))\ dx\right) = c\int_E f(x)\ dx\end{aligned}$$

This shows $cf \in L^1(E)$ at the same time.

2. Notice that $|f(x)+g(x)| \le |f(x)| + |g(x)|$,

$$\int_E |f(x)+g(x)|\ dx \le \int_E [|f(x)| + |g(x)|]\ dx = \int_E |f(x)|\ dx + \int_E |g(x)|\ dx < \infty$$

where the first inequality is by Exercise 3.2, part 1., the equality is by Exercise 3.5, and the second inequality is by the fact that $|f| \in L^1(E)$ and $|g| \in L^1(E)$. Thus, we obtain $f+g \in L^1(E)$. Observe that $f+g = (f+g)_+ + (f+g)_-$, $f = f_+ + f_-$ and $g = g_+ + g_-$, so we have $(f+g)_+ + (-f_-) + (-g_-) = f_+ + g_+ + [-(f+g)_-]$. Take integration on both sides over E, since each term on both sides is nonnegative, by Exercise 3.5, we obtain

$$\begin{aligned}&\int_E (f(x)+g(x))_+\ dx + \int_E (-f_-(x))\ dx + \int_E (-g_-(x))\ dx \\ &\qquad = \int_E f_+(x)\ dx + \int_E g_+(x)\ dx + \int_E [-(f(x)+g(x))_-]\ dx\end{aligned}$$

Notice that the six terms above are all finite because of $f, g, f+g \in L^1(E)$, so by manipulating these terms, we have

$$\begin{aligned}&\int_E (f(x)+g(x))_+\ dx - \int_E [-(f(x)+g(x))_-]\ dx \\ &= \int_E f_+(x)\ dx - \int_E (-f_-(x))\ dx + \int_E g_+(x)\ dx - \int_E (-g_-(x))\ dx\end{aligned}$$

which is exactly equivalent to $\int_E [f(x)+g(x)]\ dx = \int_E f(x)\ dx + \int_E g(x)\ dx$.

$\square$

✍ **Exercise 3.13** Suppose $f(x)$ and $g(x)$ are measurable on $E \in \mathcal{M}$ and $g \in L^1(E)$. If $f(x) = g(x)$ a.e. on E, then $f \in L^1(E)$ and $\int_E f(x)\, dx = \int_E g(x)\, dx$.

Proof Since $f(x) = g(x)$ a.e. on E, $f(x) - g(x) = 0$ a.e. on E. By Exercise 3.10, $\int_E [f(x) - g(x)]\, dx = 0$, and thus $f - g \in L^1(E)$. Since $g \in L^1(E)$, by Exercise 3.12, we have $(f - g) + g = f \in L^1(E)$. Also,

$$\begin{aligned}\int_E f(x)\, dx &= \int_E [(f(x) - g(x)) + g(x)]\, dx \\ &= \int_E [f(x) - g(x)]\, dx + \int_E g(x)\, dx = \int_E g(x)\, dx\end{aligned}$$

□

✍ **Exercise 3.14** Suppose $f(x)$ and $g(x)$ are measurable on $E \in \mathcal{M}$ and $f, g \in L^1(E)$. If $f(x) \leq g(x)$ on E, $\int_E f(x)\, dx \leq \int_E g(x)\, dx$.

Proof Take $c = -1$ in Exercise 3.12, we have $-f \in L^1(E)$, thus $g - f = g + (-f) \in L^1(E)$. Since $g = g - f + f$, we have $\int_E g(x)\, dx = \int_E [(g(x) - f(x)) + f(x)]\, dx$. By Exercise 3.12 again, $\int_E g(x)\, dx = \int_E [g(x) - f(x)]\, dx + \int_E f(x)\, dx$. Notice that $g(x) - f(x) \geq 0$, so by Exercise 3.2, part 1., $\int_E [g(x) - f(x)]\, dx \geq 0$. This shows $\int_E g(x)\, dx \geq \int_E f(x)\, dx$. □

✍ **Exercise 3.15** Let $f(x)$ and $g(x)$ be measurable on $E \in \mathcal{M}$. Suppose $f \in L^1(E)$ and $g(x)$ is bounded on E. Prove that $f \cdot g \in L^1(E)$.

Proof Note that there exists $M > 0$ s.t. $|g(x)| \leq M$ on E, and thus $|f(x)g(x)| \leq M|f(x)|$ on E. Since $|f(x)g(x)|$ and $M|f(x)|$ are both nonnegative measurable function on E, by Exercise 3.2, part 1. and part 3., $\int_E |f(x)g(x)|\, dx \leq \int_E M|f(x)|\, dx = M \int_E |f(x)|\, dx$. Since $f \in L^1(E)$, $\int_E |f(x)|\, dx < \infty$ and $\int_E |f(x)g(x)|\, dx < \infty$, so $f \cdot g \in L^1(E)$. □

✍ **Exercise 3.16** Let $f(x)$ be measurable on $E \in \mathcal{M}$. Suppose $f \in L^1(E)$, then similar to Riemann integral, we have $\left|\int_E f(x)\, dx\right| \leq \int_E |f(x)|\, dx$.

Proof Notice that $\pm f(x) \leq |f(x)|$ and $\pm f \in L^1(E)$ (by Exercise 3.12), so by applying Exercise 3.14, we have $\int_E \pm f(x)\, dx \leq \int_E |f(x)|\, dx$. By Exercise 3.12, $\int_E \pm f(x)\, dx = \pm \int_E f(x)\, dx$. Thus, we have $\pm \int_E f(x)\, dx \leq \int_E |f(x)|\, dx$, which is equivalent to the desired result. □

✍ **Exercise 3.17** Let $f(x)$ be measurable on $E \in \mathcal{M}$. Suppose $\int_E f(x)\, dx$ exists and $E = A \cup B$ where A, B are disjoint measurable sets. Prove $\int_E f(x)\, dx = \int_A f(x)\, dx + \int_B f(x)\, dx$.

Proof Since $\int_E f(x)\, dx = \int_E f_+(x)\, dx - \int_E (-f_-(x))\, dx$, by Exercise 3.5,

$$\int_E f_+(x)\, dx = \int_A f_+(x)\, dx + \int_B f_+(x)\, dx \tag{1}$$

$$\int_E (-f_-(x))(x)\, dx = \int_A (-f_-(x))\, dx + \int_B (-f_-(x))\, dx \tag{2}$$

Since $\int_E f(x)\, dx$ exists, either $\int_E f_+(x)\, dx$ or $\int_E (-f_-(x))\, dx$ is finite. If $\int_E f_+(x)\, dx$ is finite, then both $\int_A f_+(x)\, dx$ and $\int_B f_+(x)\, dx$ are finite . Thus, $\int_A f(x)\, dx$ and $\int_B f(x)\, dx$

exist. If $\int_E (-f_-(x))\ dx$ is finite, then both $\int_A (-f_-(x))\ dx$ and $\int_B (-f_-(x))\ dx$ are finite. Thus, $\int_A f(x)\ dx$ and $\int_B f(x)\ dx$ also exist. In any case, we can use equation (1) minus equation (2), and we will obtain the desired result. ☐

✍ **Exercise 3.18** Let $f(x)$ be measurable on $E \in \mathcal{M}$. Suppose $\int_E f(x)\ dx$ exists. Prove $\int_E f(x)\ dx = \int_{E\setminus Z} f(x)\ dx$ where $Z \in \mathcal{M}$ and $m(Z) = 0$.

Proof By Definition 3.4, we have $\int_E f(x)\ dx = \int_E f_+(x)\ dx - \int_E -f_-(x)\ dx$. Since $f_+(x)$ and $-f_-(x)$ are both nonnegative measurable, by Exercise 3.6,

$$\int_E f_+(x)\ dx = \int_{E\setminus Z} f_+(x)\ dx, \quad \int_E -f_-(x)\ dx = \int_{E\setminus Z} -f_-(x)\ dx$$

Since $\int_E f(x)\ dx$ exists, either $\int_E f_+(x)\ dx$ or $\int_E -f_-(x)\ dx$ is finite, so either $\int_{E\setminus Z} f_+(x)\ dx$ or $\int_{E\setminus Z} -f_-(x)\ dx$ is finite. This shows $\int_{E\setminus Z} f(x)\ dx$ exists and

$$\begin{aligned}\int_{E\setminus Z} f(x)\ dx &= \int_{E\setminus Z} f_+(x)\ dx - \int_{E\setminus Z} -f_-(x)\ dx \\ &= \int_E f_+(x)\ dx - \int_E -f_-(x)\ dx = \int_E f(x)\ dx\end{aligned}$$

☐

Remark Notice that $f(x)$ is measurable on E if and only if $f(x)$ is measurable on $E \setminus Z$ when $E \in \mathcal{M}$ and $m(Z) = 0$. Thus, we can use exactly the same argument to prove if $\int_{E\setminus Z} f(x)\ dx$ exists, then $\int_E f(x)\ dx$ also exists and $\int_E f(x)\ dx = \int_{E\setminus Z} f(x)\ dx$.

✍ **Exercise 3.19** Let $f(x)$ be measurable on $E \subset \mathbb{R}^n$, $E \in \mathcal{M}$.

1. Suppose $\int_E f(x)\ dx$ exists. Prove that $\int_{\mathbb{R}^n} I_E(x)f(x)\ dx$ exists and
$$\int_{\mathbb{R}^n} I_E(x)f(x)\ dx = \int_E f(x)\ dx$$
2. Suppose $\int_{\mathbb{R}^n} I_E(x)f(x)\ dx$ exists. Prove that $\int_E f(x)\ dx$ exists and
$$\int_E f(x)\ dx = \int_{\mathbb{R}^n} I_E(x)f(x)\ dx$$

Proof

1. By Definition 3.4, we have $\int_E f(x)\ dx = \int_E f_+(x)\ dx - \int_E -f_-(x)\ dx$. Since $f_+(x)$ and $-f_-(x)$ are both nonnegative measurable, by Exercise 3.2, part 6.,
$$\int_E f_+(x)\ dx = \int_{\mathbb{R}^n} I_E(x)f_+(x)\ dx, \quad \int_E -f_-(x)\ dx = \int_{\mathbb{R}^n} -f_-(x)I_E(x)\ dx$$
Similar to the proof of Exercise 3.18, either $\int_{\mathbb{R}^n} I_E(x)f_+(x)\ dx$ or $\int_{\mathbb{R}^n} -f_-(x)I_E(x)\ dx$ is finite, so $\int_{\mathbb{R}^n} f(x)I_E(x)\ dx$ exists and
$$\begin{aligned}\int_{\mathbb{R}^n} f(x)I_E(x)\ dx &= \int_{\mathbb{R}^n} I_E(x)f_+(x)\ dx - \int_{\mathbb{R}^n} -f_-(x)I_E(x)\ dx \\ &= \int_E f_+(x)\ dx - \int_E -f_-(x)\ dx = \int_E f(x)\ dx\end{aligned}$$
2. Since $\int_{\mathbb{R}^n} I_E(x)f(x)\ dx$ exists, either $\int_{\mathbb{R}^n} [I_E(x)f(x)]_+\ dx$ or $\int_{\mathbb{R}^n} -[I_E(x)f(x)]_-\ dx$ is finite. Notice that, $[I_E(x)f(x)]_+ = I_E(x)f_+(x)$ and $-[I_E(x)f(x)]_- = -f_-(x)I_E(x)$

on $\mathbb{R}^n$. Thus, either $\int_{\mathbb{R}^n} I_E(x)f_+(x)\ dx$ or $\int_{\mathbb{R}^n} -f_-(x)I_E(x)\ dx$ is finite. Notice that $f_+(x)$ and $-f_-(x)$ are nonnegative measurable functions, by Exercise 3.2, part 6.,

$$\int_E f_+(x)\ dx = \int_{\mathbb{R}^n} I_E(x)f_+(x)\ dx, \quad \int_E -f_-(x)\ dx = \int_{\mathbb{R}^n} -f_-(x)I_E(x)\ dx$$

Therefore, either $\int_E f_+(x)\ dx$ or $\int_E -f_-(x)\ dx$ is finite, so $\int_E f(x)\ dx$ exists and

$$\begin{aligned}\int_E f(x)\ dx &= \int_E f_+(x)\ dx - \int_E -f_-(x)\ dx \\ &= \int_{\mathbb{R}^n} I_E(x)f_+(x)\ dx - \int_{\mathbb{R}^n} -f_-(x)I_E(x)\ dx \\ &= \int_{\mathbb{R}^n} [I_E(x)f(x)]_+\ dx - \int_{\mathbb{R}^n} -[I_E(x)f(x)]_-\ dx = \int_{\mathbb{R}^n} f(x)I_E(x)\ dx\end{aligned}$$

□

Problem 3.3 Let $f(x)$ be function defined on Z. If $Z \in \mathcal{M}$ with $m(Z) = 0$, then $f(x)$ is measurable on Z and $\int_Z f(x)\ dx = 0$.

Problem 3.4 Let $f(x)$ be measurable function on $E \in \mathcal{M}$ and $f \in L^1(E)$. Suppose $A \subset E$ and $A \in \mathcal{M}$, then $f \in L^1(A)$.

Problem Set 3.3

1. Let $f(x)$ be nonnegative measurable on $[0,1]$. Prove that if there exists constant $A < \infty$ s.t. $\int_0^1 f^k(x)\ dx = A$ for all $k \geq 1$, then $f(x) = I_E(x)$ a.e. on $[0,1]$ for some $E \subset [0,1]$.
2. Suppose $f \in L^1(\mathbb{R})$, $f(0) = 0$, $f'(0)$ exists. Prove that $\frac{f(x)}{x} \in L^1(\mathbb{R})$.
3. Let $f(x)$ be measurable on $\mathbb{R}$, $c \in \mathbb{R} \setminus \{0\}$ and $a \in \mathbb{R}$. Suppose $f \in L^1(\mathbb{R})$. Prove that $f(cx+a) \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} f(cx+a)\ dx = \frac{1}{|c|}\int_{\mathbb{R}} f(y)\ dy$.
4. Let $E \subset \mathbb{R}$ and $E \in \mathcal{M}$. Suppose $f(x)$ is measurable on E and $f \in L^1(E)$. Prove $\int_{\frac{E-a}{c}} f(cx+a)\ dx = \frac{1}{|c|}\int_E f(y)\ dy$ for all $c \in \mathbb{R} \setminus \{0\}$, $a \in \mathbb{R}$.
5. Let $f \in L^1(\mathbb{R})$, and $a > 0$. Define $F(x) = \sum_{n=-\infty}^{\infty} f(x/a + n)$. Prove the series converges absolutely for almost all $x \in \mathbb{R}$, $F \in L^1([0,a])$ and F is periodic with period a.

3.4 Dominated Convergence Theorem

In this section we are going to introduce another fundamental theorem in real analysis: dominated convergence theorem (DCT). In addition, we are going to explore many useful properties of Lebesgue integral induced by DCT. Finally, we are going to introduce another mode of convergence - L^1-convergence - and the relation between it and other modes of convergence discussed before.

Theorem 3.4. Dominated Convergence Theorem

Let $F(x)$ and $\{f_k(x)\}_{k=1}^{\infty}$ be measurable functions on $E \in \mathcal{M}$ s.t. $|f_k(x)| \le F(x)$ a.e. on E. Suppose $F \in L^1(E)$ and $f_k(x) \to f(x)$ a.e. on E for some $f(x)$. Then $\int_E |f_k(x) - f(x)|\, dx \to 0$ as $k \to \infty$. In particular, $\lim_{k\to\infty} \int_E f_k(x)\, dx = \int_E f(x)\, dx$. ♡

Proof Let $E_1 = \{x \in E \,|\, f_k(x) \not\to f(x)\}$ and $E_2^k = \{x \in E \,|\, |f_k(x)| > F(x)\}$ for all $k \ge 1$. Also, let $E_2 = \bigcup_{k=1}^{\infty} E_2^k$, then we can show $m(E_1) = m(E_2) = 0$. Since $F \in L^1(E)$, by Exercise 3.11, $F(x)$ is finite a.e. on E. Thus, if $E_3 = \{x \in E \,|\, |F(x)| = \infty\}$, $m(E_3) = 0$. Denote $E' = E_1 \cup E_2 \cup E_3$, and we have $m(E') = 0$. Now, it suffices to show $\int_{E \setminus E'} |f_k(x) - f(x)|\, dx \to 0$ as $k \to \infty$ because of Exercise 3.6. Notice that on $E \setminus E'$, $|f_k(x)| \le F(x)$ everywhere and since $F(x)$ is finite everywhere, each $f_k(x)$ is finite everywhere. Furthermore, $f_k(x) \to f(x)$ pointwisely on $E \setminus E'$.

Let $A = E \setminus E'$, then $A \in \mathcal{M}$. Since $F \in L^1(E)$, by Problem 3.4, $F \in L^1(A)$. Notice that $|f_k(x)| \le F(x)$ on A for all $k \ge 1$, so by taking $k \to \infty$, $|f(x)| \le F(x)$ on A. By Problem 3.2, $f_k \in L^1(A)$ and $f \in L^1(A)$. Let $g_k(x) = |f_k(x) - f(x)|$ for each $k \ge 1$ on A, then $g_k(x) \to 0$ pointwisely on A. Also, it is easy to show $g_k(x)$'s are nonnegative measurable on A with $g_k(x) \le 2F(x)$. By Exercise 3.12, $2F \in L^1(A)$. Thus, by Problem 3.2 again, $g_k \in L^1(A)$ for all $k \ge 1$. Apply Fatou's lemma to $2F(x) - g_k(x) \ge 0$ on A, we obtain

$$\varliminf_{k\to\infty} \int_A (2F(x) - g_k(x))\, dx \ge \int_A \varliminf_{k\to\infty} (2F(x) - g_k(x)) = \int_A 2F(x)\, dx$$

By Exercise 3.12,

$$\begin{aligned}\varliminf_{k\to\infty} \int_A (2F(x) - g_k(x))\, dx &= \varliminf_{k\to\infty} \left(\int_A 2F(x)\, dx - \int_A g_k(x)\, dx \right) \\ &= \int_A 2F(x)\, dx - \varlimsup_{k\to\infty} \int_A g_k(x)\, dx\end{aligned}$$

Thus, we obtain $\varlimsup_{k\to\infty} \int_A g_k(x)\, dx \le 0$. This implies $\lim_{k\to\infty} \int_A g_k(x)\, dx = 0$. Therefore, $\int_{E\setminus E'} |f_k(x) - f(x)|\, dx \to 0$ as $k \to \infty$. Since $m(E') = 0$, by Exercise 3.6, we obtain $\int_E |f_k(x) - f(x)|\, dx \to 0$ as $k \to \infty$.

To prove the claim after "In particular", observe that

$$\pm \left(\int_E f_k(x)\, dx - \int_E f(x)\, dx \right) = \int_E \pm (f_k(x) - f(x))\, dx \le \int_E |f_k(x) - f(x)|\, dx$$

where the equality is by Exercise 3.12 and the inequality is by Exercise 3.14. Thus we have $\left| \int_E f_k(x)\, dx - \int_E f(x)\, dx \right| \to 0$ as $k \to \infty$, which is equivalent to the desired result. □

Example 3.5 Suppose $f(x)$ is measurable on $E \subset \mathbb{R}^n$, $E \in \mathcal{M}$, and $f \in L^1(E)$, prove that

$$\lim_{k\to\infty} \int_{E \cap B_k} f(x)\, dx = \int_E f(x)\, dx$$

where B_k is the open ball with radius k centered at the origin.

Proof Since $f \in L^1(E)$, and $E \cap B_k \subset E$ is measurable, by Problem 3.4, $f \in L^1(E \cap B_k)$.

Thus, $\int_{E\cap B_k} f(x)\,dx$ exists. By Exercise 3.19,

$$\int_{E\cap B_k} f(x)\,dx = \int_{\mathbb{R}^n} I_{E\cap B_k}(x)f(x)\,dx$$

Since $\int_{\mathbb{R}^n} I_{E\cap B_k}(x)f(x)\,dx$ exists, by Exercise 3.17,

$$\int_{\mathbb{R}^n} I_{E\cap B_k}(x)f(x)\,dx = \int_E I_{E\cap B_k}(x)f(x)\,dx + \int_{\mathbb{R}^n\setminus E} I_{E\cap B_k}(x)f(x)\,dx$$

Notice that $I_{E\cap B_k}(x)f(x) = 0$ on $\mathbb{R}^n \setminus E$, so

$$\int_{E\cap B_k} f(x)\,dx = \int_{\mathbb{R}^n} I_{E\cap B_k}(x)f(x)\,dx = \int_E I_{E\cap B_k}(x)f(x)\,dx$$

Denote $f_k(x) = I_{E\cap B_k}(x)f(x)$, then $f_k(x)$ is measurable, $|f_k(x)| \le |f(x)|$, and $f_k(x) \to f(x)$ pointwisely on E. By DCT with $f(x)$ as dominating function, we have

$$\lim_{k\to\infty} \int_{E\cap B_k} f(x)\,dx = \lim_{k\to\infty} \int_E f_k(x)\,dx = \int_E f(x)\,dx$$

□

Proposition 3.1. Differentiation Under the Integral Sign

Let $f(x,y)$ be defined on $E \times (a,b)$, where $x \in E \in \mathcal{M}$, $y \in (a,b)$ with $a,b \in \mathbb{R}$. If

- *for each fixed $y \in (a,b)$, $f(x,y)$ is in $L^1(E)$;*
- *for each fixed $x \in E$, $\frac{\partial f}{\partial y}(x,y)$ exists for all $y \in (a,b)$;*
- *there exists $g \in L^1(E)$, $y_0 \in (a,b)$ and $\delta > 0$ s.t. $\left|\frac{\partial f}{\partial y}(x,y)\right| \le g(x)$ for each fixed $x \in E$ and all $y \in (y_0-\delta, y_0+\delta) \subset (a,b)$.*

Then we can exchange the order of differentiation and integration, i.e.,

$$\left[\frac{d}{dy}\int_E f(x,y)\,dx\right]_{y=y_0} = \int_E \left.\frac{\partial f}{\partial y}(x,y)\right|_{y=y_0} dx$$

Proof For small enough $h \in \mathbb{R}^+$, since for fixed y, $f \in L^1(E)$, by Exercise 3.12,

$$\frac{\int_E f(x,y_0+h)\,dx - \int_E f(x,y_0)\,dx}{h} = \int_E \frac{f(x,y_0+h)-f(x,y_0)}{h}\,dx$$

Since $\frac{\partial f}{\partial y}$ exists on (a,b) for each fixed x, $f(x,y)$ is continuous on $[y_0-\delta, y_0+\delta]$ for small $\delta > 0$ s.t. $h < \delta$. Then we can apply mean value theorem, i.e. there exists $\theta \in (0,1)$ s.t. $\theta h < \delta$ and

$$\int_E \frac{f(x,y_0+h)-f(x,y_0)}{h}\,dx = \int_E \frac{\partial f}{\partial y}(x,y_0+\theta h)\,dx$$

Pick arbitrary sequence $h_k \to 0$ with $h_k \ne 0$ for all $k \ge 1$. Let $u_k(x) = \frac{f(x,y_0+h_k)-f(x,y_0)}{h_k}$, then $u_k(x)$ is measurable on E and $|u_k(x)| \le g(x)$ on E for all $k \ge 1$. Since $g \in L^1(E)$ and $u_k(x) \to \frac{\partial f}{\partial y}(x,y_0)$ pointwisely on E, we can apply DCT to $u_k(x)$. Thus,

$$\lim_{k\to\infty} \frac{\int_E f(x,y_0+h_k)\,dx - \int_E f(x,y_0)\,dx}{h_k} = \lim_{k\to\infty}\int_E u_k(x)\,dx = \int_E \frac{\partial f}{\partial y}(x,y_0)\,dx$$

By definition of limit, this implies the desired result. □

Note *Since the definition of $f \in L^1(E)$ includes the condition that $f(x)$ is measurable on $E \in \mathcal{M}$, sometimes we only say $f \in L^1(E)$ with $E \in \mathcal{M}$ and omit the measurablity condition.*

Theorem 3.5. Riemann Integral is Lebesgue Integral

If $f(x)$ is Riemann integrable on bounded interval $[a,b]$, then $f \in L^1([a,b])$ and

$$(\mathcal{R}) \int_a^b f(x)\ dx = (\mathcal{L}) \int_a^b f(x)\ dx$$

where $(\mathcal{R})$ stands for Riemann and $(\mathcal{L})$ stands for Lebesgue. ♡

Proof Recall Lebesgue's Criterion for integrablility, $f(x)$ is Riemann integrable if and only if $f(x)$ is bounded and continuous a.e. on $[a,b]$. Let B be the set of discontinuous points of $f(x)$ on $[a,b]$, then $m(B) = 0$. This implies $f(x)$ is continuous on $[a,b] \setminus B \in \mathcal{M}$, so $f(x)$ is measurable on $[a,b] \setminus B$, hence measurable on $[a,b]$. Since $f(x)$ is bounded on $[a,b]$, by Problem 3.2, $f \in L^1([a,b])$. Let $P_k = \{x_0^k, \ldots, x_{n_k}^k\}$ be a sequence of partition of $[a,b]$ s.t. $\min_{i=1}^{n_k} |x_i^k - x_{i-1}^k| \to 0$ as $k \to \infty$. Since $f(x)$ is Riemann integrable, the Riemann sum converges, i.e., $\sum_{i=0}^{n_k} f(x_i^k)(x_{i+1}^k - x_i^k) \to (\mathcal{R}) \int_a^b f(x)\ dx$ as $k \to \infty$. Let

$$f_k(x) = \begin{cases} f(x_0^k) & x \in [x_0^k, x_1^k) \\ \vdots & \vdots \\ f(x_{n_k}^k) & x \in [x_{n_k-1}^k, x_{n_k}^k] \end{cases}$$

Then $f_k(x)$ is measurable simple and bounded by M on $[a,b]$. Also, $f_k(x) \to f(x)$ a.e. on $[a,b]$ because for $x \in [a,b] \setminus B$, $f_k(x) \to f(x)$. We can verify this by using $f(x)$ is continuous on $[a,b] \setminus B$, i.e., for all $\epsilon > 0$, there exists $\delta > 0$ s.t. for all $|y - x| < \delta$, $|f(y) - f(x)| < \epsilon$. Thus, for all $\epsilon > 0$, we can find K s.t. for all $k \geq K$, $\min_{i=1}^{n_k} |x_i^k - x_{i-1}^k| < \delta$, then $f_k(x) = f(y)$ where $|y - x| < \delta$, and thus $|f_k(x) - f(x)| = |f(y) - f(x)| < \epsilon$. This shows $f_k(x) \to f(x)$ on $[a,b] \setminus B$. Therefore, by DCT, $(\mathcal{L}) \int_a^b f_k(x)\ dx \to (\mathcal{L}) \int_a^b f(x)\ dx$. Notice that by definition of Lebesgue integral for measurable simple function, $(\mathcal{L}) \int_a^b f_k(x)\ dx = \sum_{i=0}^{n_k} f(x_i^k)(x_{i+1}^k - x_i^k)$, so the two limits $(\mathcal{L}) \int_a^b f(x)\ dx$ and $(\mathcal{R}) \int_a^b f(x)\ dx$ coincides with each other. □

Theorem 3.6. Integration Term by Term II (ITT-II)

Let $f_k(x)$ be measurable on $E \in \mathcal{M}$ and $f_k \in L^1(E)$ for all $k \geq 1$. Suppose $\sum_{k=1}^{\infty} \int_E |f_k(x)|\ dx < \infty$, then $\sum_{k=1}^{\infty} f_k(x)$ converges a.e. on E and $\sum_{k=1}^{\infty} f_k \in L^1(E)$. Furthermore,

$$\int_E \sum_{k=1}^{\infty} f_k(x)\ dx = \sum_{k=1}^{\infty} \int_E f_k(x)\ dx$$

♡

Proof Let $g(x) = \sum_{k=1}^{\infty} |f_k(x)|$, by ITT-I, $\int_E g(x)\ dx = \sum_{k=1}^{\infty} \int_E |f_k(x)|\ dx < \infty$. Thus, $g \in L^1(E)$ and by Exercise 3.11, $g(x)$ is finite a.e. on E. Thus, $\sum_{k=1}^{\infty} |f_k(x)|$ converges a.e. on E, and so does $\sum_{k=1}^{\infty} f_k(x)$. Since $|\sum_{k=1}^{\infty} f_k(x)| \leq \sum_{k=1}^{\infty} |f_k(x)| = g(x)$, by Problem 3.2, $\sum_{k=1}^{\infty} f_k \in L^1(E)$. Let $h_m(x) = \sum_{k=1}^{m} f_k(x)$, then $h_m(x) \to \sum_{k=1}^{\infty} f_k(x)$ a.e. on E. Also, $|h_m(x)| \leq g(x)$ on E, so by DCT, $\int_E h_m(x)\ dx \to \int_E \sum_{k=1}^{\infty} f_k(x)\ dx$ as $m \to \infty$. Notice that by Exercise 3.12, $\int_E h_m(x)\ dx = \sum_{k=1}^{m} \int_E f_k(x)\ dx$, so the desired property holds. □

At the end of this section, we introduce another mode of convergence besides a.e. convergence, a.u. convergence, and convergence in measure, that is, L^1-convergence. We will not go deep into it because in the next chapter we are going to study L^p-space systematically, and at that time, we will generalize L^1-convergence to L^p-convergence and explore more properties of it.

Definition 3.5. L^1-convergence

Let $f(x)$ and $f_k(x)$ be measurable on $E \in \mathcal{M}$. Suppose $f \in L^1(E)$ and $f_k \in L^1(E)$ for all $k \geq 1$. We say $f_k(x) \to f(x)$ in $L^1(E)$ if $\int_E |f_k(x) - f(x)|\, dx \to 0$ as $k \to \infty$. ♣

Note *Notice that the conclusion in DCT can be regarded as L^1-convergence, so in short, DCT says if a sequence of function is bounded by Lebesgue integrable function, then a.e. convergence implies L^1-convergence.*

Theorem 3.7

Let $f(x)$, $g(x)$, and $f_k(x)$ be measurable function on E for all $k \geq 1$.

1. *If $f_k(x) \to f(x)$ in $L^1(E)$ as $k \to \infty$, then $f_k(x) \to f(x)$ in measure.*
2. *If $|f_k(x)| \leq g(x)$ on E where $g \in L^1(E)$, then $f_k(x) \to f(x)$ a.e. implies $f_k(x) \to f(x)$ a.u..* ♡

Proof

1. For all $\sigma > 0$, recall Markov's inequality in Exercise 3.8, we have
$$m(\{x \in E \,|\, |f_k(x) - f(x)| > \sigma\}) \leq \frac{1}{\sigma}\int_E |f_k(x) - f(x)|\, dx$$
Since $f_k(x) \to f(x)$ in $L^1(E)$, the RHS converges to zero, so LHS also converges to zero, and this means $f_k(x) \to f(x)$ in measure on E.
2. Since $g \in L^1(E)$, by Exercise 3.11, $g(x)$ is finite a.e. on E, and since $|f_k(x)| \leq g(x)$, $f_k(x)$ is also finite a.e. on E. Thus, we can observe that this statement is quite similar to Egorov's theorem, so we want to prove this statement by using the proof of Egorov's theorem. Notice that the only missing condition is that Egorov's theorem needs $m(E)$ to be finite, so we need to scrutinize the proof of Egorov's theorem, find out at which step we used $m(E) < \infty$ and try to obtain the same conclusion without using $m(E) < \infty$. In fact we use $m(E) < \infty$ only once in the whole proof of Egorov's theorem, that is, when we use the continuity of Lebesgue measure to prove $\lim_{m\to\infty} m(F_l^m) = 0$. Thus, if we can prove that $m(F_l^1) < \infty$ without using $m(E) < \infty$, but by some new conditions in this question, i.e., $f_k(x)$ is bounded by Lebesgue integrable function for all $k \geq 1$, then we are done.

   Now we adopt all notations in the proof of Egorov's theorem. Since $|f_k(x)| \leq g(x)$ for all $k \geq 1$, and $f_k(x) \to f(x)$ pointwisely on $E' \setminus Z$, so $|f(x)| \leq g(x)$ on $E' \setminus Z$. Since $m(Z) = 0$, it suffices to show $m(F_l^1 \setminus Z) < \infty$. For all $x \in F_l^1 \setminus Z$, there exists

i_x s.t. $|f_{i_x}(x) - f(x)| > \frac{1}{l}$. Since $|f_{i_x} - f(x)| \leq |f_{i_x}| + |f(x)| \leq 2g(x)$, we have $x \in \{x \in E \,|\, g(x) > \frac{1}{2l}\}$. Thus, $F_l^1 \setminus Z \subset \{x \in E \,|\, g(x) > \frac{1}{2l}\}$. By Markov's inequality,

$$m(F_l^1 \setminus Z) \leq m\left(\left\{x \in E \,|\, g(x) > \frac{1}{2l}\right\}\right) \leq 2l \int_E |g(x)|\, dx < \infty$$

so $m(F_l^1) = m(F_1^1 \setminus Z) < \infty$ and we can use exactly the same proof of Egorov's theorem.

□

Problem Set 3.4

1. Let $f_k(x)$ be measurable on $E \in \mathcal{M}$ s.t. $|f_k(x)| \leq F(x)$ a.e. on E, where $F \in L^1(E)$ and $f_k(x) \to f_\infty(x)$ in measure on E. Prove that $\int_E |f_k(x) - f_\infty(x)|\, dx \to 0$ as $k \to \infty$. In particular, $\int_E f_k(x)\, dx \to \int_E f_\infty(x)\, dx$ as $k \to \infty$.
2. Let $f_k(x)$ be measurable and nonnegative on $E \in \mathcal{M}$, where $m(E) < \infty$. Prove that $f_k(x) \to 0$ in measure on E iff $\int_E \frac{f_k(x)}{1+f_k(x)}\, dx \to 0$.
3. Let $f_k(x)$ be nonnegative measurable on $E \in \mathcal{M}$. Let $f \in L^1(E)$ s.t. $f_k(x) \to f(x)$ in measure on E and $\int_E f_k(x)\, dx \to \int_E f(x)\, dx$. Prove that $\int_E |f_k(x) - f(x)|\, dx \to 0$.
4. Suppose $f \in L^1(E)$, $E \in \mathcal{M}$. $E = \bigcup_{k=1}^\infty E_k$, $E_k \in \mathcal{M}$, pairwise disjoint. Prove that $\int_E f(x)\, dx = \sum_{k=1}^\infty \int_{E_k} f(x)\, dx$.
5. Prove that for all $f \in L^1(E)$, $E \in \mathcal{M}$, there exists a sequence $f_k(x) \in L^1(E)$, s.t. f_k is bounded on E and $f_k \to f$ in $L^1(E)$ as $k \to \infty$.
6. Prove that for all $f \in L^1(E)$, $E \in \mathcal{M}$, there exists simple functions $f_k(x) \in L^1(E)$ s.t. $f_k \to f$ in $L^1(E)$.
7. Use "$\Longrightarrow$" to denote "implies" and "$\longrightarrow$" to denote "after passing to a subsequence implies", complete the following diagram

$\boxed{\text{converge a.u.}}$

$\boxed{\text{converge a.e.}}$ $\qquad$ $\boxed{\text{converge in measure}}$

$\boxed{\text{converge in } L^1(E)}$

in general case, special case when $m(E) < \infty$, and special case when $|f_k| \leq g \in L^1(E)$ respectively.
8. Suppose $f \in L^1(E)$. Prove that for all $\epsilon > 0$, there exists $\delta > 0$ s.t. for all $e \subset E$, $e \in \mathcal{M}$, with $m(e) < \delta$, we have $\int_e |f(x)|\, dx < \epsilon$.
9. Let $f_k \in L^1(E)$ be s.t. $f_k \to f_\infty$ a.e. on E. Suppose $m(E) < \infty$. Prove that $f_\infty \in L^1(E)$ and $f_k \to f_\infty$ in $L^1(E)$ if and only if for all $\epsilon > 0$, there exists $\delta > 0$ s.t. $\int_e |f_k(x)|\, dx < \epsilon$ for all $k \geq 1$ whenever $e \subset E$, $e \in \mathcal{M}$ and $m(E) < \delta$.
10. Recall there are two types of improper integral. One type is $(\mathcal{I}) \int_a^b f(x)\, dx$, which

can be regarded as $\lim_{c\to a^+}(\mathcal{R})\int_c^b f(x)\,dx$. If such a limit exists as a finite number, then we say the improper integral $(\mathcal{I})\int_a^b f(x)\,dx$ is convergent. Also, the other type is $(\mathcal{I})\int_{-\infty}^{\infty} f(x)\,dx$, which can be regarded as $\lim_{a\to-\infty,b\to\infty}(\mathcal{R})\int_a^b f(x)\,dx$. If such a limit exists as a finite number, then we say the improper integral $(\mathcal{I})\int_{-\infty}^{\infty} f(x)\,dx$ is convergent.

(a). Suppose the improper integral $(\mathcal{I})\int_a^b f(x)\,dx$ is absolutely convergent. Prove that $f\in L^1([a,b])$ and $(\mathcal{L})\int_a^b f(x)\,dx = (\mathcal{I})\int_a^b f(x)\,dx$.

(b). Suppose $(\mathcal{I})\int_a^b f(x)\,dx$ is an improper integral and $f\in L^1([a,b])$. Prove that $(\mathcal{I})\int_a^b f(x)\,dx$ is absolutely convergent.

(c). Prove the same result for improper integral $(\mathcal{I})\int_{-\infty}^{\infty} f(x)\,dx$ as in (a). and (b)..

11. Let $\alpha > -1$. Define $\Gamma(\alpha) = (\mathcal{L})\int_0^\infty e^{-t}t^{\alpha+1}\,dt$. Prove Lebesgue integral

$$(\mathcal{L})\int_0^\infty \frac{e^{-x}}{1-e^{-x}}x^{\alpha+1}\,dx = \Gamma(\alpha)\sum_{n=1}^{\infty}\frac{1}{n^{\alpha+2}}$$

Is the improper integral $(\mathcal{I})\int_0^\infty \frac{e^{-x}}{1-e^{-x}}\,x^{\alpha+1}\,dx$ convergent absolutely?

3.5 Fubini-Tonelli Theorem

Recall in calculus, for Riemann integrable function $f(x,y)$ defined on $[a,b]\times[c,d]$ where $a,b,c,d\in\mathbb{R}$, we can calculate the double integral by the iterated integral, i.e.,

$$(\mathcal{R})\iint_{[a,b]\times[c,d]} f(x,y)\,d(x,y) = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy = \int_a^b\left(\int_c^d f(x,y)\,dy\right)dx$$

However, this property is too restricted because $f(x,y)$ needs to be bounded and a.e. continuous on a closed rectangle. To make it more handy in practice, we want to generalize this property to any Lebesgue integrable functions.

Note *Throughout this section, we let $\mathbb{R}^n = \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, where $n = n_1+n_2$. Denote point $\tilde{x}\in\mathbb{R}^n$ as $\tilde{x} = (x,y)$, where $x\in\mathbb{R}^{n_1}$ and $y\in\mathbb{R}^{n_2}$.*

Definition 3.6. Fubini Condition

*Let $f(x,y)$ be nonnegative measurable on $\mathbb{R}^n$. $f(x,y)$ satisfies **Fubini condition** if*

(a). For almost every fixed $x\in\mathbb{R}^{n_1}$, $f(x,y)$ is measurable on $\mathbb{R}^{n_2}$.

(b). Let $g(x) = \int_{\mathbb{R}^{n_2}} f(x,y)\,dy$, then $g(x)$ is measurable on $\mathbb{R}^{n_1}$.

(c). $\int_{\mathbb{R}^{n_1}} g(x)\,dx = \int_{\mathbb{R}^n} f(x,y)\,d(x,y)$.

Furthermore, the set of all nonnegative measurable functions on $\mathbb{R}^n$ satisfying Fubini condition is denoted as $\mathcal{F}$. ♣

Remark Notice that for the second part, for those x s.t. $f(x,y)$ is not measurable on $\mathbb{R}^{n_2}$, $g(x)$ is not well-defined by the formula because Lebesgue integral is only defined for measurable function. To resolve this problem, we can simply define $g(x) = 0$ for those x and it will not affect the value of $\int_{\mathbb{R}^{n_1}} g(x)\,dx$ by Exercise 3.6.

Lemma 3.2

1. *If $f \in \mathcal{F}$, then $c \cdot f \in \mathcal{F}$ for all constant $c \geq 0$.*
2. *If $f_1, f_2 \in \mathcal{F}$, then $f_1 + f_2 \in \mathcal{F}$.*
3. *If $f_1, f_2 \in \mathcal{F}$, $f_2 \in L^1(\mathbb{R}^n)$, and $f_1 - f_2 \geq 0$ on $\mathbb{R}^n$, then $f_1 - f_2 \in \mathcal{F}$.*
4. *Suppose $f_k \in \mathcal{F}$ for all $k \in \mathbb{N}^+$ and $f_k(x, y)$ is increasing in k for all fixed $(x, y) \in \mathbb{R}^n$. If $f_k(x, y) \to f(x, y)$ pointwisely on $\mathbb{R}^n$, then $f \in \mathcal{F}$.*
5. *Suppose $f_k \in \mathcal{F}$ for all $k \in \mathbb{N}^+$ and $f_k(x, y)$ is decreasing in k for all fixed $(x, y) \in \mathbb{R}^n$. If $f_k(x, y) \to f(x, y)$ pointwisely on $\mathbb{R}^n$ and there exists $k_0 \geq 1$ s.t. $f_{k_0} \in L^1(\mathbb{R}^n)$, then $f \in \mathcal{F}$.*

♡

Proof

1. Since $f \in \mathcal{F}$, $f(x)$ is nonnegative measurable function on $\mathbb{R}^n$. By Exercise 2.4, $cf(x)$ is also nonnegative measurable function on $\mathbb{R}^n$. To prove $cf \in \mathcal{F}$, it remains to check the three conditions in Definition 3.6.
   (a). Since $f \in \mathcal{F}$, for almost every fixed $x \in \mathbb{R}^{n_1}$, $f(x, y)$ is measurable on $\mathbb{R}^{n_2}$. Denote $A = \{x \in \mathbb{R}^{n_1} \mid f(x, y) \text{ is not measurable on } \mathbb{R}^{n_2}\}$, then $m(A) = 0$. By Exercise 2.4, $cf(x, y)$ is also measurable on $\mathbb{R}^{n_2}$ for $x \in \mathbb{R}^{n_1} \setminus A$.
   (b). Let $g_1(x) = \int_{\mathbb{R}^{n_2}} cf(x, y)\ dy$ and $g(x) = \int_{\mathbb{R}^{n_2}} f(x, y)\ dy$, then $g(x)$ and $g_1(x)$ is well-defined on $\mathbb{R}^{n_1} \setminus A$. Since $f(x, y)$ is nonnegative, by Exercise 3.2, part 3., $g_1(x) = cg(x)$ on $\mathbb{R}^{n_1} \setminus A$. Notice that $g(x)$ is measurable on $\mathbb{R}^{n_1}$, so by Exercise 2.2, $g(x)$ is measurable on $\mathbb{R}^{n_1} \setminus A$. By Exercise 2.4, $g_1(x)$ is measurable on $\mathbb{R}^{n_1} \setminus A$. Since $m(A) = 0$, by Problem 3.3, $g_1(x)$ is measurable on A. Therefore, by Exercise 2.1, $g_1(x)$ is measurable on $\mathbb{R}^{n_1}$.
   (c). Since $g(x)$ is nonnegative on $\mathbb{R}^{n_1}$, by Exercise 3.2, part 3.,
   $$\begin{aligned} \int_{\mathbb{R}^{n_1}} g_1(x)\ dx &= \int_{\mathbb{R}^{n_1}} cg(x)\ dx = c\int_{\mathbb{R}^{n_1}} g(x)\ dx \\ &= c\int_{\mathbb{R}^n} f(x, y)\ d(x, y) = \int_{\mathbb{R}^n} cf(x, y)\ d(x, y) \end{aligned}$$
   where the third equality is because $f \in \mathcal{F}$; the last equality is because $f(x, y)$ is nonnegative measurable on $\mathbb{R}^n$ and thus Exercise 3.2, part 3. applies.
2. See Problem Set 3.5, Question 1..
3. Since $f_1, f_2 \in \mathcal{F}$, $f_1(x, y)$ and $f_2(x, y)$ are nonnegative measurable on $\mathbb{R}^n$. Notice that $f_1 - f_2 \geq 0$ is well-defined on $\mathbb{R}^n$, then by Exercise 2.5, $f_1(x, y) - f_2(x, y)$ is also nonnegative measurable on $\mathbb{R}^n$. To prove $f_1 - f_2 \in \mathcal{F}$, it remains to check the three conditions in Definition 3.6.
   (a). Since $f_1, f_2 \in \mathcal{F}$, for almost all $x \in \mathbb{R}^{n_1}$, $f_1(x, y)$ and $f_2(x, y)$ are measurable functions on $\mathbb{R}^{n_2}$. Denote $A_j = \{x \in \mathbb{R}^{n_1} \mid f_j(x, y) \text{ is not measurable on } \mathbb{R}^{n_2}\}$ for $j = 1, 2$, then $m(A_1) = m(A_2) = 0$. Let $A = A_1 \cup A_2$, then $m(A) = 0$. Notice that $f_1 - f_2$ is well defined on $\mathbb{R}^n$, so by Exercise 2.4 & 2.5, $f_1(x, y) - f_2(x, y)$ is

also measurable function on $\mathbb{R}^{n_2}$ for $x \in \mathbb{R}^{n_1} \setminus A$.

(b). Let $g_j(x) = \int_{\mathbb{R}^{n_2}} f_j(x, y)\, dy$ for $j = 1, 2$ and $g(x) = \int_{\mathbb{R}^{n_2}} [f_1(x, y) - f_2(x, y)]\, dy$. Since $f_2 \in \mathcal{F}$, $\int_{\mathbb{R}^{n_1}} g_2(x)\, dx = \int_{\mathbb{R}^n} f_2(x, y)\, d(x, y)$. Combined with the assumption $f_2 \in L^1(\mathbb{R}^n)$, we have $\int_{\mathbb{R}^{n_1}} g_2(x)\, dx < \infty$. By Exercise 3.11, $g_2(x)$ is finite a.e. on $\mathbb{R}^{n_1}$. Let $B = \{x \in \mathbb{R}^{n_1} \mid g_2(x) = \infty\} \cup A$, then $m(B) = 0$ and $g_1(x) - g_2(x)$ is well-defined on $\mathbb{R}^{n_1} \setminus B$. Since $g_1(x)$ and $g_2(x)$ are measurable on $\mathbb{R}^{n_1}$, by Exercise 2.2, they are measurable on $\mathbb{R}^{n_1} \setminus B$. By Exercise 2.4 & 2.5, $g_1(x) - g_2(x)$ is also measurable on $\mathbb{R}^{n_1} \setminus B$. Now we want to prove $g(x) = g_1(x) - g_2(x)$ on $\mathbb{R}^{n_1} \setminus B$. Write $f_1 = (f_1 - f_2) + f_2$, then since $f_1 - f_2 \geq 0$ and $f_2 \geq 0$ on $\mathbb{R}^{n_2}$ for each fixed $x \in \mathbb{R}^{n_1} \setminus B$, by Exercise 3.5,

$$\int_{\mathbb{R}^{n_2}} f_1(x, y)\, dy = \int_{\mathbb{R}^{n_2}} [f_1(x, y) - f_2(x, y)]\, dy + \int_{\mathbb{R}^{n_2}} f_2(x, y)\, dy$$

which is exactly $g_1(x) = g(x) + g_2(x)$. On $\mathbb{R}^{n_1} \setminus B$, since $g_2(x)$ is finite, we have $g(x) = g_1(x) - g_2(x)$. Thus, $g(x)$ is measurable on $\mathbb{R}^{n_1} \setminus B$. Since $m(B) = 0$, by Problem 3.3, $g(x)$ is measurable on B. By Exercise 2.1, $g(x)$ is measurable on $\mathbb{R}^{n_1}$.

(c). Write $g_1 = (g_1 - g_2) + g_2$. Since $g_1 - g_2 \geq 0$ and $g_2 \geq 0$ on $x \in \mathbb{R}^{n_1} \setminus B$, by Exercise 3.5,

$$\int_{\mathbb{R}^{n_1} \setminus B} g_1(x)\, dx = \int_{\mathbb{R}^{n_1} \setminus B} [g_1(x) - g_2(x)]\, dx + \int_{\mathbb{R}^{n_1} \setminus B} g_2(x)\, dx$$

Notice that $m(B) = 0$, so by Exercise 3.6,

$$\int_{\mathbb{R}^{n_1}} g_1(x)\, dx = \int_{\mathbb{R}^{n_1}} [g_1(x) - g_2(x)]\, dx + \int_{\mathbb{R}^{n_1}} g_2(x)\, dx$$

Since $\int_{\mathbb{R}^{n_1}} g_2(x)\, dx < \infty$, we can move it to the LHS, and we will have

$$\int_{\mathbb{R}^{n_1}} [g_1(x) - g_2(x)]\, dx = \int_{\mathbb{R}^{n_1}} g_1(x)\, dx - \int_{\mathbb{R}^{n_1}} g_1(x)\, dx \tag{3.1}$$

Since $f_1, f_2 \in \mathcal{F}$, we have

$$\int_{\mathbb{R}^{n_1}} g_1(x)\, dx - \int_{\mathbb{R}^{n_1}} g_1(x)\, dx = \int_{\mathbb{R}^n} f_1(x, y)\, d(x, y) - \int_{\mathbb{R}^n} f_2(x, y)\, d(x, y) \tag{3.2}$$

Write $f_1 = (f_1 - f_2) + f_2$. Since $f_1 - f_2 \geq 0$ and $f_2 \geq 0$ on $\mathbb{R}^n$, by Exercise 3.5,

$$\int_{\mathbb{R}^n} f_1(x, y)\, d(x, y) = \int_{\mathbb{R}^n} [f_1(x, y) - f_2(x, y)]\, d(x, y) + \int_{\mathbb{R}^n} f_2(x, y)\, d(x, y)$$

Since $\int_{\mathbb{R}^n} f_2(x, y)\, d(x, y) < \infty$, we can move it to the LHS, and thus we have

$$\int_{\mathbb{R}^n} f_1(x, y)\, d(x, y) - \int_{\mathbb{R}^n} [f_1(x, y) - f_2(x, y)]\, d(x, y) = \int_{\mathbb{R}^n} f_2(x, y)\, d(x, y) \tag{3.3}$$

Combine Equation (3.1), (3.2), and (3.3), we have

$$\int_{\mathbb{R}^{n_1}} [g_1(x) - g_2(x)]\, dx = \int_{\mathbb{R}^n} [f_1(x, y) - f_2(x, y)]\, d(x, y)$$

Recall $g(x) = g_1(x) - g_2(x)$ on $\mathbb{R}^{n_1} \setminus B$, so by applying Exercise 3.6 twice, we have

$$\int_{\mathbb{R}^{n_1}} g(x)\, dx = \int_{\mathbb{R}^{n_1}} [g_1(x) - g_2(x)]\, dx$$

This shows the third condition holds, and so $f_1 - f_2 \in \mathcal{F}$.

4. See Problem Set 3.5, Question 2..
5. Notice that $f(x,y)$ is also nonnegative measurable on $\mathbb{R}^n$ by the remark of Exercise 2.12. It remains to show the three conditions in Definition 3.6.
   (a). Since $f_k \in \mathcal{F}$, for almost every fixed $x \in \mathbb{R}^{n_1}$, $f_k(x,y)$ is measurable on $\mathbb{R}^{n_2}$. Let $A_k = \{x \in \mathbb{R}^{n_1} \mid f_k(x,y) \text{ is not measurable on } \mathbb{R}^{n_2}\}$, then $m(A_k) = 0$ for all $k \geq 1$. Denote $A = \bigcup_{k=1}^{\infty} A_k$, then $m(A) = 0$ and $f_k(x,y)$ is measurable on $\mathbb{R}^{n_2}$ for all $x \in \mathbb{R}^{n_1} \setminus A$. By the remark of Exercise 2.12 again, $f(x,y)$ is measurable on $\mathbb{R}^{n_2}$ for $x \in \mathbb{R}^{n_1} \setminus A$.
   (b). Let $g_k(x) = \int_{\mathbb{R}^{n_2}} f_k(x,y)\, dy$ and $g(x) = \int_{\mathbb{R}^{n_2}} f(x,y)\, dy$, then $g_k(x)$ is measurable on $\mathbb{R}^{n_1}$. Since $f_{k_0} \in \mathcal{F}$, $\int_{\mathbb{R}^{n_1}} g_{k_0}(x)\, dx = \int_{\mathbb{R}^n} f_{k_0}(x,y)\, d(x,y)$. Combined with the assumption $f_{k_0} \in L^1(\mathbb{R}^n)$, $g_{k_0} \in L^1(\mathbb{R}^{n_1})$. By Exercise 3.11, $g_{k_0}(x)$ is finite a.e. on $\mathbb{R}^{n_1}$. Let $B = \{x \in \mathbb{R}^{n_1} \mid g_{k_0}(x) = \infty\} \cup A$, then $m(B) = 0$. Thus, for each fixed $x \in \mathbb{R}^{n_1} \setminus B$, $f_{k_0}(x,y)$ is in $L^1(\mathbb{R}^{n_2})$. Since $0 \leq f_k(x,y) \leq f_{k_0}(x,y)$ on $\mathbb{R}^{n_2}$ for all $k \geq k_0$ for each fixed $x \in \mathbb{R}^{n_1} \setminus B$, and $f_k(x,y) \to f(x,y)$ pointwisely on $\mathbb{R}^{n_2}$ for each fixed $x \in \mathbb{R}^{n_1} \setminus B$, by DCT, $g_k(x) \to g(x)$ pointwisely on $\mathbb{R}^{n_1} \setminus B$. Therefore, $g(x)$ is also measurable on $\mathbb{R}^{n_1} \setminus B$ by the remark of Exercise 2.12. Note that $m(B) = 0$, so by Problem 3.3, $g(x)$ is measurable on B. By Exercise 2.1, $g(x)$ is measurable on $\mathbb{R}^{n_1}$.
   (c). Since $0 \leq f_k(x,y) \leq f_{k_0}(x,y)$ on $\mathbb{R}^{n_2}$ for each fixed $x \in \mathbb{R}^{n_1} \setminus A$, by Exercise 3.2, part 1., $0 \leq g_k(x,y) \leq g_{k_0}(x,y)$ on $\mathbb{R}^{n_1} \setminus A$. Since $g_k(x) \to g(x)$ a.e. on $\mathbb{R}^{n_1}$ and $g_{k_0} \in L^1(\mathbb{R}^{n_1})$, by DCT, $\int_{\mathbb{R}^{n_1}} g_k(x)\, dx \to \int_{\mathbb{R}^{n_1}} g(x)\, dx$. Now consider $f_k(x,y) \to f(x,y)$ pointwisely on $\mathbb{R}^n$ and $0 \leq f_k(x,y) \leq f_{k_0}(x,y)$ on $\mathbb{R}^n$ for all $k \geq k_0$ with $f_{k_0} \in L^1(\mathbb{R}^n)$, we can apply DCT to $f_k(x,y)$, and we will obtain $\int_{\mathbb{R}^n} f_k(x,y)\, d(x,y) \to \int_{\mathbb{R}^n} f(x,y)\, d(x,y)$. Notice that $f_k \in \mathcal{F}$ implies $\int_{\mathbb{R}^{n_1}} g_k(x)\, dx = \int_{\mathbb{R}^n} f_k(x,y)\, d(x,y)$, so $\int_{\mathbb{R}^{n_1}} g(x)\, dx = \int_{\mathbb{R}^n} f_k(x,y)\, d(x,y)$.

□

Lemma 3.3

If $E \subset \mathbb{R}^n$ and $E \in \mathcal{M}$, then $I_E(x,y)$ is in $\mathcal{F}$ where $x \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$. ♡

Proof Notice that $I_E(x,y)$ is always measurable on $\mathbb{R}^n$ because of Problem Set 2.1, Question 6.. We are going to divide the whole proof in five steps.

1. Suppose $E = R_1 \times R_2$, where R_j is closed rectangle in $\mathbb{R}^{n_j}$ for $j = 1, 2$. In this case we can write $I_E(x,y) = I_{R_1}(x) I_{R_2}(y)$. Now it remains to prove the three conditions in Definition 3.6.
   (a). For each fixed $x \in \mathbb{R}^{n_1}$, $I_E(x,y) = I_{R_2}(y)$ or $I_E(x,y) = 0$ on $\mathbb{R}^{n_2}$. Thus, $I_E(x,y)$ is measurable on $\mathbb{R}^{n_2}$ for each fixed $x \in \mathbb{R}^{n_1}$. This proves the first condition.
   (b). Let $g(x) = \int_{\mathbb{R}^{n_2}} I_E(x,y)\, dy$, then $g(x)$ is well-defined on $\mathbb{R}^{n_1}$. By Exercise 3.2, part 3., $g(x) = I_{R_1}(x) \int_{\mathbb{R}^{n_2}} I_{R_2}(y)\, dy = |R_2| I_{R_1}(x)$. Since $I_{R_1}(x)$ is measurable on

$\mathbb{R}^{n_1}$, by Exercise 2.4, $g(x)$ is measurable on $\mathbb{R}^{n_1}$. This proves the second condition.

(c). By Exercise 3.2, part 3., $\int_{\mathbb{R}^{n_1}} g(x)\ dx = |R_2| \int_{\mathbb{R}^{n_1}} I_{R_1}(x)\ dx = |R_2||R_1| = |E|$. Since E is also a rectangle in $\mathbb{R}^n$, $\int_{\mathbb{R}^n} I_E(x,y)\ d(x,y) = m(E) = |E|$. This proves the third condition.

2. Suppose E is open. Then by Exercise 1.3, $E = \bigcup_{k=1}^{\infty} c_k$, where c_k's are almost disjoint closed cubes. Thus, $I_E(x,y) = \sum_{i=1}^{\infty} I_{c_k}(x,y)$ a.e. on $\mathbb{R}^n$ (the equality may not hold on the boundary of each c_k). Since c_k is closed cubes, it can be written as $c_k = R_1 \times R_2$ where R_j's are closed rectangles in $\mathbb{R}^{n_j}$ for $j = 1, 2$. Thus, by step one, $I_{c_k} \in \mathcal{F}$ for all $k \geq 1$. By Lemma 3.2, part 2., $g_m = \sum_{k=1}^{m} I_{c_k} \in \mathcal{F}$ for all $m \geq 1$. Notice that $g_m(x,y) \to I_E(x,y)$ pointwisely on $\mathbb{R}^n$ and $g_m(x,y)$ is increasing in m for all fixed $(x,y) \in \mathbb{R}^n$, so by Lemma 3.2, part 4., $I_E(x,y) \in \mathcal{F}$.
3. Suppose E is G_δ set, then $E = \bigcap_{i=1}^{\infty} G_k$, where G_i is open. If G_1 is bounded, let $F_k = \bigcap_{i=1}^{k} G_i$. Notice that F_k decreases to E as k increases to ∞, so $I_{F_k}(x,y) \to I_E(x,y)$ pointwisely on $\mathbb{R}^n$ and $I_{F_k}(x,y)$ is decreasing for each fixed $(x,y) \in \mathbb{R}^n$. Since F_k is open, by step two, $F_k \in \mathcal{F}$ for all $k \geq 1$. Also, F_1 is bounded, so $I_{F_1} \in L^1(\mathbb{R}^n)$. Thus, by Lemma 3.2, part 5., $I_E \in \mathcal{F}$. If G_1 is not bounded, let $G_i^m = G_i \cap B_m$, where B_m is the open ball centered at the orgin with radius m. Denote $E_m = \bigcap_{i=1}^{\infty} G_i^m$, since E_m is a bounded G_δ set, $I_{E_m} \in \mathcal{F}$ for all $m \geq 1$. Notice that $I_{E_m}(x,y)$ is increasing to $I_E(x,y)$ as $m \to \infty$ for each fixed $(x,y) \in \mathbb{R}^n$, so by Lemma 3.2, part 4., $I_E \in \mathcal{F}$.
4. Suppose $m(E) = 0$. By Theorem 1.1, there exists G_δ set $H \supset E$ and $m(H \setminus E) = 0$. It is easy to see $m(H) = 0$, so $\int_{\mathbb{R}^{n_1}} I_H(x,y)\ d(x,y) = m(H) = 0$. By step three, $I_H \in \mathcal{F}$. By Definition 3.6, $\int_{\mathbb{R}^{n_1}} g_H(x)\ dx = \int_{\mathbb{R}^{n_1}} I_H(x,y)\ d(x,y) = 0$, where $g_H(x) = \int_{\mathbb{R}^{n_2}} I_H(x,y)\ dy$. Notice that $g_H(x)$ is nonnegative, so by Problem Set 3.1, Question 1., $g_H(x) = 0$ a.e. on $\mathbb{R}^{n_1}$. Let $A = \{x \in \mathbb{R}^{n_1} \mid g_H(x) \neq 0\}$, then $m(A) = 0$. Since $I_H(x,y)$ is also nonnegative, for all $x \in \mathbb{R}^{n_1} \setminus A$, $I_H(x,y) = 0$ a.e. on $\mathbb{R}^{n_2}$ by using Problem Set 3.1, Question 1. again. Note that $I_E(x,y) \leq I_H(x,y)$ on $\mathbb{R}^n$, so for all $x \in \mathbb{R}^{n_1} \setminus A$, $I_E(x,y) = 0$ a.e. on $\mathbb{R}^{n_2}$. Then we check the conditions in Definition 3.6.

   (a). For each fixed $x \in \mathbb{R}^{n_1} \setminus A$, $I_E(x,y) = 0$ a.e. on $\mathbb{R}^{n_1}$. Thus, for each fixed $x \in \mathbb{R}^{n_1} \setminus A$, by Exercise 3.10, $I_E(x,y)$ is measurable on $\mathbb{R}^{n_1}$

   (b). Let $g(x) = \int_{\mathbb{R}^{n_2}} I_E(x,y)\ dy$, then for each fixed $x \in \mathbb{R}^{n_1} \setminus A$, $g(x) = 0$. This means $g(x) = 0$ a.e. on $\mathbb{R}^{n_1}$, so by Exercise 3.10, $g(x)$ is measurable on $\mathbb{R}^{n_1}$.

   (c). Since $m(E) = 0$, $\int_{\mathbb{R}^n} I_E(x,y)\ d(x,y) = m(E) = 0$. Since $g(x) = 0$ a.e. on $\mathbb{R}^{n_1}$, by Exercise 3.10, $\int_{\mathbb{R}^{n_1}} g(x)\ dx = 0$.

   Therefore, we have proved $I_E \in \mathcal{F}$ when $m(E) = 0$.
5. Suppose $E \in \mathcal{M}$. By Theorem 1.1, we can take G_δ set H s.t. $H \supset E$ and $m(H \setminus E) = 0$. Write $E = H \setminus (H \setminus E)$, then $I_E(x,y) = I_H(x,y) - I_{H \setminus E}(x,y)$. By step three, $I_H \in \mathcal{F}$. By step four, $I_{H \setminus E} \in \mathcal{F}$. Also, $I_{H \setminus E} \in L^1(\mathbb{R}^n)$. By Lemma 3.2, part 3., $I_E \in \mathcal{F}$.

□

Theorem 3.8. Fubini-Tonelli Theorem I (FTT-I)

If $f(x,y)$ is nonnegative and measurable on $\mathbb{R}^n$, then $f \in \mathcal{F}$. In particular,

$$\int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} f(x,y)\, dy \right) dx = \int_{\mathbb{R}^n} f(x,y)\, d(x,y) = \int_{\mathbb{R}^{n_2}} \left(\int_{\mathbb{R}^{n_1}} f(x,y)\, dx \right) dy$$

♡

Proof By simple approximation theorem, there exists measurable simple functions $\phi_k(x,y)$ s.t. $\phi_k(x,y) \to f(x,y)$ pointwisely on $\mathbb{R}^n$ as $k \to \infty$ and $\phi_k(x,y)$ is increasing in k for each fixed $(x,y) \in \mathbb{R}^n$. Since every simple function can be written as a finite linear combination of indicator function of measurable sets, by Lemma 3.3 and Lemma 3.2, part 1. & 2., $\phi_k \in \mathcal{F}$ for all $k \geq 1$. By Lemma 3.2, part 4., $f \in \mathcal{F}$.

By Definition of $\mathcal{F}$, it is easy to see $\int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} f(x,y)\, dy \right) dx = \int_{\mathbb{R}^n} f(x,y)\, d(x,y)$. To prove the second equality, we only need to exchange the "character" of x and y, i.e., regard y here as the x in Definition 3.6 and x here as the y in Definition 3.6. We can do this because x and y have no order and n_1, n_2 can be arbitrary as long as $n_1 + n_2 = n$. □

Theorem 3.9. Fubini-Tonelli Theorem II (FTT-II)

Let $f \in L^1(\mathbb{R}^n)$, then even if f is not nonnegative, it still satisfies all three conditions in Definition 3.6. In particular,

$$\int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} f(x,y)\, dy \right) dx = \int_{\mathbb{R}^n} f(x,y)\, d(x,y) = \int_{\mathbb{R}^{n_2}} \left(\int_{\mathbb{R}^{n_1}} f(x,y)\, dx \right) dy$$

♡

Proof Write $f(x,y) = f_+(x,y) - (-f_-(x,y))$, where $f_+(x,y)$ and $-f_-(x,y)$ are both nonnegative measurable. Thus, $f_+ \in \mathcal{F}$ and $-f_- \in \mathcal{F}$. Since $f \in L^1(\mathbb{R}^n)$, by Definition 3.6, we have $f_+ \in L^1(\mathbb{R}^n)$ and $-f_- \in L^1(\mathbb{R}^n)$.

1. Since $-f_- \in \mathcal{F}$ and $-f_- \in L^1(\mathbb{R}^n)$, let $g_-(x) = \int_{\mathbb{R}^{n_2}} -f_-(x,y)\, dy$, and we have
$$\int_{\mathbb{R}^{n_1}} g_-(x)\, dx = \int_{\mathbb{R}^n} -f_-(x,y)\, d(x,y) < \infty$$
By Exercise 3.11, $g_-(x)$ is finite a.e. on $\mathbb{R}^{n_1}$. Let $A = \{x \in \mathbb{R}^{n_1} \mid g_-(x) = \infty\}$, then for each fixed $x \in \mathbb{R}^{n_1} \setminus A$, $g_-(x) < \infty$. This further implies for each fixed $x \in \mathbb{R}^{n_1} \setminus A$, $-f_-(x,y)$ is finite a.e. on $\mathbb{R}^{n_2}$. Let $A_x = \{y \in \mathbb{R}^{n_2} \mid -f_-(x,y) = \infty\}$, then for each fixed $x \in \mathbb{R}^{n_1} \setminus A$, $f_+(x,y) - (-f_-(x,y))$ is well-defined on $\mathbb{R}^{n_2} \setminus A_x$. Denote
$$B_1 = \{x \in \mathbb{R}^{n_1} \mid f_+(x,y) \text{ is not measurable on } \mathbb{R}^{n_2}\}$$
$$B_2 = \{x \in \mathbb{R}^{n_1} \mid -f_-(x,y) \text{ is not measurable on } \mathbb{R}^{n_2}\}$$
then $m(B_1) = m(B_2) = 0$. Let $B = B_1 \cup B_2 \cup A$, and we have $m(B) = 0$. For each fixed $x \in \mathbb{R}^{n_1} \setminus B$, since $f_+(x,y)$ and $-f_-(x,y)$ are measurable and $-f_-(x,y)$ is finite on $\mathbb{R}^{n_2} \setminus A_x$, by Exercise 2.4 & 2.5, $f_+(x,y) - (-f_-(x,y))$ is well-defined and measurable on $\mathbb{R}^{n_2} \setminus A_x$. Thus, $f(x,y)$ is measurable on $\mathbb{R}^{n_2} \setminus A_x$ for almost all $x \in \mathbb{R}^{n_1}$. Notice that $m(A_x) = 0$, so by Problem 3.3 and Exercise 2.1, $f(x,y)$ is measurable on $\mathbb{R}^{n_2}$ for almost all $x \in \mathbb{R}^{n_1}$.

2. Let $g_+(x) = \int_{\mathbb{R}^{n_2}} f_+(x,y)\ dy$ and $g(x) = \int_{\mathbb{R}^{n_2}} f(x,y)\ dy$. Since $g_-(x) < \infty$ on $\mathbb{R}^{n_1} \setminus A$, by Definition 3.4, $g(x) = g_+(x) - g_-(x)$ on $\mathbb{R}^{n_1} \setminus A$. Since $g_+(x)$ and $g_-(x)$ are both measurable on $\mathbb{R}^{n_1} \setminus A$, by Exercise 2.4 & 2.5, $g(x)$ is also measurable on $\mathbb{R}^{n_1} \setminus A$. Note that $m(A) = 0$, so by Problem 3.3 and Exercise 2.1, $g(x)$ is measurable on $\mathbb{R}^{n_1}$.
3. Since $f \in L^1(\mathbb{R}^n)$, $f_+ \in \mathcal{F}$, and $-f_- \in \mathcal{F}$, we have
$$\begin{aligned}\int_{\mathbb{R}^n} f(x,y)\ d(x,y) &= \int_{\mathbb{R}^n} f_+(x,y)\ d(x,y) - \int_{\mathbb{R}^n} -f_-(x,y)\ d(x,y) \\ &= \int_{\mathbb{R}^{n_1}} g_+(x)\ dx - \int_{\mathbb{R}^{n_1}} g_-(x)\ dx\end{aligned}$$
Also notice that $g_+ \in L^1(\mathbb{R}^{n_1})$ and $-g_- \in L^1(\mathbb{R}^{n_1})$, so by Exercise 3.12,
$$\int_{\mathbb{R}^{n_1}} g_+(x)\ dx - \int_{\mathbb{R}^{n_1}} g_-(x)\ dx = \int_{\mathbb{R}^{n_1}} [g_+(x) - g_-(x)]\ dx$$
Since $g(x) = g_+(x) - g_-(x)$ a.e. on $\mathbb{R}^{n_1}$, by Exercise 3.13,
$$\int_{\mathbb{R}^n} f(x,y)\ d(x,y) = \int_{\mathbb{R}^{n_1}} [g_+(x) - g_-(x)]\ dx = \int_{\mathbb{R}^{n_1}} g(x)\ dx$$

In conclusion, f satisfies all three conditions in Definition 3.6 even if f is not nonnegative on $\mathbb{R}^n$. Thus, by the same reason in Theorem 3.8,
$$\int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} f(x,y)\ dy\right) dx = \int_{\mathbb{R}^n} f(x,y)\ d(x,y) = \int_{\mathbb{R}^{n_2}} \left(\int_{\mathbb{R}^{n_1}} f(x,y)\ dx\right) dy$$
$\square$

Example 3.6 Suppose $E_1 \subset \mathbb{R}^{n_1}$ and $E_2 \subset \mathbb{R}^{n_2}$ are both measurable. Then $E_1 \times E_2 \subset \mathbb{R}^n$ is measurable and $m(E_1 \times E_2) = m(E_1)m(E_2)$.

Proof First we prove if we have known $E_1 \times E_2 \in \mathcal{M}$, then $m(E_1 \times E_2) = m(E_1)m(E_2)$. If $E_1 \times E_2 \in \mathcal{M}$, by Problem Set 2.1, Question 6., $I_{E_1 \times E_2}(x,y)$ is measurable on $\mathbb{R}^n$. By FTT-I,
$$\int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} I_{E_1 \times E_2}(x,y)\ dy\right) dx = \int_{\mathbb{R}^n} I_{E_1 \times E_2}(x,y)\ d(x,y) = m(E_1 \times E_2)$$
Notice that $I_{E_1 \times E_2}(x,y) = I_{E_1}(x) I_{E_2}(y)$, so we have
$$\begin{aligned}\int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} I_{E_1 \times E_2}(x,y)\ dy\right) dx &= \int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} I_{E_1}(x) I_{E_2}(y)\ dy\right) dx \\ = \int_{\mathbb{R}^{n_1}} I_{E_1}(x) \left(\int_{\mathbb{R}^{n_2}} I_{E_2}(y)\ dy\right) dx &= \int_{\mathbb{R}^{n_1}} m(E_2) I_{E_1}(x) dx \\ = m(E_2) \int_{\mathbb{R}^{n_1}} I_{E_1}(x) dx &= m(E_1)m(E_2)\end{aligned}$$
where the second and the fourth equality is by Exercise 3.2, part 3.. Thus, we have shown $m(E_1 \times E_2) = m(E_1)m(E_2)$, given that $E_1 \times E_2 \in \mathcal{M}$.

Then we prove $E_1 \times E_2 \in \mathcal{M}$. Let $A_k = (E_1 \times E_2) \cap C_k$ where $C_k = \prod_{i=1}^n [-k, k]$ for all $k \geq 1$. Since $E_1 \times E_2 = \bigcup_{k=1}^\infty A_k$, it suffices to show each $A_k \in \mathcal{M}$. Notice that $A_k = E_1^k \times E_2^k$ where $E_1^k = E_1 \cap \prod_{i=1}^{n_1} [-k, k]$ and $E_2^k = E_2 \cap \prod_{i=1}^{n_2} [-k, k]$. Thus, E_1^k and E_2^k are bounded for all $k \geq 1$. Since $E_1^k, E_2^k \in \mathcal{M}$, by Definition 1.7, for all $\epsilon > 0$, there exists open $G_1^k \subset \mathbb{R}^{n_1}$ and $G_2^k \subset \mathbb{R}^{n_2}$ s.t. $E_1^k \subset G_1^k$ and $E_2^k \subset G_2^k$ with $m(G_1^k \setminus E_1^k) < \frac{\epsilon}{100}$ and $m(G_2^k \setminus E_2^k) < \frac{\epsilon}{100}$. Also, by Problem Set 1.4, Question 2., there exists closed $F_1^k \subset \mathbb{R}^{n_1}$ and

$F_2^k \subset \mathbb{R}^{n_2}$ s.t. $F_1^k \subset E_1^k$ and $F_2^k \subset E_2^k$ with $m(E_1^k \setminus F_1^k) < \frac{\epsilon}{100}$ and $m(E_2^k \setminus F_2^k) < \frac{\epsilon}{100}$. Obviously $G_1^k \times G_2^k \supset E_1^k \times E_2^k \supset F_1^k \times F_2^k$, then we have

$$\begin{aligned} m^*(G_1^k \times G_2^k \setminus E_1^k \times E_2^k) &\le m^*(G_1^k \times G_2^k \setminus F_1^k \times F_2^k) \\ &\le m^*((G_1^k \setminus F_1^k) \times G_2^k) + m^*(G_1^k \times (G_2^k \setminus F_2^k)) \end{aligned}$$

Notice that G_1^k and $G_2^k \setminus F_2^k$ are both open, so by definition of product topology, $G_1^k \times (G_2^k \setminus F_2^k)$ is also open, hence measurable. Thus, $m(G_1^k \times (G_2^k \setminus F_2^k)) = m(G_1^k)m(G_2^k \setminus F_2^k)$. Similarly, since $(G_1^k \setminus F_1^k) \times G_2^k \in \mathcal{M}$, $m((G_1^k \setminus F_1^k) \times G_2^k) = m(G_1^k \setminus F_1^k)m(G_2^k)$. Therefore,

$$m^*(G_1^k \times G_2^k \setminus E_1^k \times E_2^k) \le m(G_1^k \setminus F_1^k)m(G_2^k) + m(G_1^k)m(G_2^k \setminus F_2^k)$$

Also, notice that $G_1^k \setminus F_1^k = (G_1^k \setminus E_1^k) \cup (E_1^k \setminus F_1^k)$ and $G_2^k \setminus F_2^k = (G_2^k \setminus E_2^k) \cup (E_2^k \setminus F_2^k)$, so we have $m(G_1^k \setminus F_1^k) < \frac{\epsilon}{50}$ and $m(G_2^k \setminus F_2^k) < \frac{\epsilon}{50}$. Since E_1^k and E_2^k are bounded, $m(E_1^k) < \infty$ and $m(E_2^k) < \infty$. This implies $m(G_1^k) < \infty$ and $m(G_2^k) < \infty$. Therefore, $m^*(G_1^k \times G_2^k \setminus E_1^k \times E_2^k) \to 0$ as $\epsilon \to 0$, and this shows $E_1^k \times E_2^k \in \mathcal{M}$. □

Example 3.7 Let $E_1 \subset \mathbb{R}^{n_1}$ and $E_2 \subset \mathbb{R}^{n_2}$ be measurable. Suppose $f \in L^1(E_1 \times E_2)$, then

$$\int_{E_1 \times E_2} f(x,y)\, d(x,y) = \int_{E_1} \left(\int_{E_2} f(x,y)\, dy \right) dx = \int_{E_2} \left(\int_{E_1} f(x,y)\, dx \right) dy$$

Proof By Example 3.6, $E_1 \times E_2 \in \mathcal{M}$. Consider $f(x,y)I_{E_1 \times E_2}(x,y)$ on $\mathbb{R}^n$, we want to show $fI_{E_1 \times E_2} \in L^1(\mathbb{R}^n)$. Since $f(x,y)I_{E_1 \times E_2}(x,y) = f(x,y)$ on $E_1 \times E_2$ and $f(x,y)$ is measurable on $E_1 \times E_2$, $f(x,y)I_{E_1 \times E_2}(x,y)$ is measurable on $E_1 \times E_2$. Since $f(x,y)I_{E_1 \times E_2}(x,y) = 0$ on $\mathbb{R}^n \setminus (E_1 \times E_2)$, by Exercise 3.10, $f(x,y)I_{E_1 \times E_2}(x,y)$ is measurable on $\mathbb{R}^n \setminus (E_1 \times E_2)$. Thus, by Exercise 2.1, $f(x,y)I_{E_1 \times E_2}(x,y)$ is measurable on $\mathbb{R}^n$. By Exercise 3.2, part 6.,

$$\begin{aligned} \int_{\mathbb{R}^n} |f(x,y)I_{E_1 \times E_2}(x,y)|\, d(x,y) &= \int_{\mathbb{R}^n} |f(x,y)|I_{E_1 \times E_2}(x,y)\, d(x,y) \\ &= \int_{E_1 \times E_2} |f(x,y)|\, d(x,y) \end{aligned}$$

Since $f \in L^1(E_1 \times E_2)$, by Exercise 3.9, we obtain the desired result $fI_{E_1 \times E_2} \in L^1(\mathbb{R}^n)$. By FTT-II, with the fact that $I_{E_1 \times E_2}(x,y) = I_{E_1}(x)I_{E_2}(y)$,

$$\int_{\mathbb{R}^n} f(x,y)I_{E_1 \times E_2}(x,y)\, d(x,y) = \int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} f(x,y)I_{E_1}(x)I_{E_2}(y)\, dy \right) dx$$

Since $f \in L^1(\mathbb{R}^n)$, by Exercise 3.19,

$$\int_{E_1 \times E_2} f(x,y)\, d(x,y) = \int_{\mathbb{R}^n} f(x,y)I_{E_1 \times E_2}(x,y)\, d(x,y)$$

By FTT-II, there exists set A with $m(A) = 0$ and $g(x) = \int_{\mathbb{R}^{n_2}} f(x,y)I_{E_1}(x)I_{E_2}(y)\, dy$ exists for $x \in \mathbb{R}^{n_1} \setminus A$. Notice that for $x \in E_1 \setminus A$, by Exercise 3.19,

$$g(x) = \int_{\mathbb{R}^{n_2}} f(x,y)I_{E_2}(y)\, dy = \int_{E_2} f(x,y)\, dy$$

Denote $E_1^c = \mathbb{R}^{n_2} \setminus E_1$, then for $x \in E_1^c \setminus A$, $g(x) = 0$. Thus, by Exercise 3.18,

$$\int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} f(x,y)I_{E_1}(x)I_{E_2}(y)\, dy \right) dx = \int_{\mathbb{R}^{n_1} \setminus A} g(x) dx$$

By Exercise 3.17,

$$\int_{\mathbb{R}^{n_1}\setminus A} g(x)dx = \int_{E_1\setminus A} g(x)dx + \int_{E_1^c\setminus A} g(x)dx = \int_{E_1\setminus A}\left(\int_{E_2} f(x,y)\,dy\right)dx$$

By the remark of Exercise 3.18, we obtain

$$\int_{E_1\setminus A}\left(\int_{E_2} f(x,y)\,dy\right)dx = \int_{E_1}\left(\int_{E_2} f(x,y)\,dy\right)dx$$

Combine all above equalities, $\int_{E_1\times E_2} f(x,y)\,d(x,y) = \int_{E_1}\left(\int_{E_2} f(x,y)\,dy\right)dx$. Similarly, we can prove the other equality $\int_{E_1\times E_2} f(x,y)\,d(x,y) = \int_{E_2}\left(\int_{E_1} f(x,y)\,dx\right)dy$. ☐

Problem Set 3.5

1. Prove Lemma 3.2, part 2..
2. Prove Lemma 3.2, part 4..
3. Let $f(x,y) \in L^1(E_1 \times E_2)$, where $x \in E_1 \subset \mathbb{R}^{n_1}$, $E_1 \in \mathcal{M}$ and $y \in E_2 \subset \mathbb{R}^{n_2}$, $E_2 \in \mathcal{M}$. Prove that $\int_{E_2} f(x,y)\,dy \in L^1(E_1)$ and $\int_{E_1} f(x,y)\,dx \in L^1(E_2)$.
4. Let $f(x)$ be nonnegative on $E \in \mathcal{M}$, $E \subset \mathbb{R}^n$. Let $A = \{(x,y) \in E\times\mathbb{R} \,|\, 0 \le y \le f(x)\}$. Prove that f is measurable on E iff $A \subset \mathbb{R}^{n+1}$ is measurable. Also prove if $f(x)$ is measurable on E, then $\int_E f(x)\,dx = m(A)$.
5. Suppose $f(x)$ is measurable on $E \subset \mathbb{R}^n$, $E \in \mathcal{M}$. For all $\lambda \ge 0$, define the **distribution function** $F(\lambda) = m(\{x \in E \,|\, |f(x)| > \lambda\})$. Prove that if $|f|^p \in L^1(E)$ where $p \ge 1$, then $\int_E |f(x)|^p\,dx = p\int_0^\infty \lambda^{p-1}F(\lambda)\,d\lambda$.

Chapter 4 L^p-space

4.1 Basic Properties of L^p-space

Definition 4.1. L^p**-norm**

Let $E \subset \mathbb{R}^n$, $E \in \mathcal{M}$, and $m(E) > 0$. For $0 < p < \infty$, define $\mathbf{L^p}$***-norm*** *of any measurable function $f(x)$ on E to be*

$$\|f\|_p = \left(\int_E |f(x)|^p \, dx \right)^{1/p}$$

Furthermore, define $\mathbf{L^\infty}$***-norm*** *of any measurable function $f(x)$ on E to be*

$$\|f\|_\infty = \inf\{C > 0 \,|\, |f(x)| \le C \text{ a.e. on } E\}$$

♣

Definition 4.2. L^p**-space**

Let $E \subset \mathbb{R}^n$, $E \in \mathcal{M}$, and $m(E) > 0$. For $0 < p \le \infty$, define $\mathbf{L^p(E)}$ *to be the set of all measurable functions on E s.t. $\|f\|_p < \infty$.*

♣

Exercise 4.1 Let $f(x)$ be measurable function on $E \in \mathcal{M}$, then $|f(x)| \le \|f\|_\infty$ a.e. on E.

Proof Take decreasing sequence c_k s.t. $c_k \to \|f\|_\infty$ as $k \to \infty$ and $|f(x)| \le c_k$ a.e. on E. Then there exists $B_k \in \mathcal{M}$ s.t. $m(B_k) = 0$ and $|f(x)| \le c_k$ on $E \setminus B_k$. Consider $E \setminus \bigcup_{k=1}^\infty B_k$, on which $|f(x)| \le c_k$ for all $k \ge 1$. Take $k \to \infty$, $|f(x)| \le \|f\|_\infty$. Since $m(\bigcup_{k=1}^\infty B_k) = 0$, $|f(x)| \le \|f\|_\infty$ a.e. on E. □

Exercise 4.2 Let $f(x)$ be measurable function on $E \in \mathcal{M}$ with $m(E) < \infty$. Prove $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

Proof If $\|f\|_\infty = 0$, then by Exercise 4.1, $|f(x)| \le 0$ a.e. on E, and so $f(x) = 0$ a.e. on E. Since $|f(x)|^p = 0$ a.e. on E, by Exercise 3.10, $\|f\|_p = 0$ for all $p > 0$. This shows $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

If $\|f\|_\infty > 0$, then for all $0 < M < \|f\|_\infty$, $m(A) > 0$, where $A = \{x \in E \,|\, |f(x)| \ge M\}$. For $0 < p < \infty$, by Exercise 3.2, part 2. & 1.,

$$\left(\int_E |f(x)|^p \, dx \right)^{1/p} \ge \left(\int_A |f(x)|^p \, dx \right)^{1/p} \ge (M^p m(A))^{1/p} = M(m(A))^{1/p}$$

Take lower limit as $p \to \infty$ on both sides, since $m(E) < \infty$, we have $\underline{\lim}_{p\to\infty} \|f\|_p \ge M$. Take $M \to \|f\|_\infty$, $\underline{\lim}_{p\to\infty} \|f\|_p \ge \|f\|_\infty$. For the other direction, by Exercise 4.1, 3.2, part 1., and 3.18 with its remark,

$$\left(\int_E |f(x)|^p \, dx \right)^{1/p} \le \left(\int_E \|f\|_\infty^p \, dx \right)^{1/p} = \|f\|_\infty (m(E))^{1/p}$$

Take upper limit as $p \to \infty$, since $m(E) < \infty$, we have $\overline{\lim}_{p\to\infty}\|f\|_p \le \|f\|_\infty$. Thus,

$$\|f\|_\infty \le \underline{\lim}_{p\to\infty}\|f\|_p \le \overline{\lim}_{p\to\infty}\|f\|_p \le \|f\|_\infty$$

This implies $\lim_{p\to\infty}\|f\|_p = \|f\|_\infty$. □

Remark The above conclusion is not true in general if $m(E) = \infty$. Consider $f(x) = 1$ on $\mathbb{R}$, then $\|f\|_\infty = 1$ but $\|f\|_p = \infty$ for all $0 < p < \infty$. However, if in addition, there exists $0 < r < \infty$ s.t. $\|f\|_r < \infty$, then the conclusion always holds even if $m(E) = \infty$ (See Problem Set 4.1, Question 7.).

Example 4.1 Let $f(x) = -\ln x$ on $E = (0,1)$. Prove $f \in L^p(E)$ but $f \notin L^\infty(E)$, and $\|f\|_p \to \infty$ as $p \to \infty$.

Proof Notice that $\lim_{x\to 0+} x^\epsilon(-\ln x) = 0$ for all $\epsilon > 0$, so there exists constant $C_\epsilon > 0$ s.t. $0 < -\ln x \le C_\epsilon x^{-\epsilon}$ for all $x \in (0,1)$. Note that $I = \int_0^1 (C_\epsilon x^{-\epsilon})^p\ dx$ is improper integral and if we take ϵ small enough s.t. $\epsilon p < 1$, then $I < \infty$. This means $C_\epsilon x^{-\epsilon} \in L^p(0,1)$ and so $f \in L^p(0,1)$. Also, it is easy to see $\|f\|_\infty = \infty$, so $f \notin L^\infty(E)$. Since $m(E) < \infty$, by Exercise 4.2, $\|f\|_p \to \infty$ as $p \to \infty$. □

Now we make an agreement as follows: if $f, g \in L^p(E)$, $0 < p \le \infty$, and $f(x) = g(x)$ a.e. on E, then we identify $f(x)$ and $g(x)$ as the same element in $L^p(E)$. For example, the Dirichlet function $I_{\mathbb{Q}}(x) = 0$ a.e. on $\mathbb{R}$, so the Dirichlet function and the constant function 0 is the same element in $L^p(\mathbb{R})$. Thus, if $f(x)$ is defined a.e. on E, then we can define $f(x)$ to be any number you like at those x's where $f(x)$ is not defined, and we can regard the new function and old function as the same element in $L^p(E)$.

✍ **Exercise 4.3** Let $E \in \mathcal{M}$. Prove for all $0 < p \le \infty$, $L^p(E)$ is a linear space, i.e., for all $f, g \in L^p(E)$, for all $c_1, c_2 \in \mathbb{R}$, we have $c_1 f + c_2 g \in L^p(E)$.

Proof Since $f, g \in L^p(E)$, $f(x)$ and $g(x)$ are finite a.e. on E, so $c_1 f(x) + c_2 g(x)$ is finite a.e. on E. Notice that it is possible that $c_1 f(x) + c_2 g(x)$ is not well-defined on a set with measure zero, but by our agreement, we can define the function value at those points to be any number we like, so $c_1 f(x) + c_2 g(x)$ is defined everywhere on E. When $p < \infty$, recall for all $a, b \in \mathbb{R}$, $|a+b|^p \le (|a| + |b|)^p \le 2^p|a|^p + 2^p|b|^p$. By Exercise 3.2, part 1., we have

$$\int_E |c_1 f(x) + c_2 g(x)|^p\ dx \le \int_E 2^p[|c_1|^p|f(x)|^p + |c_2|^p|g(x)|^p]\ dx$$

By Exericse 3.2, part 3. & 3.5, we have

$$\int_E 2^p[|c_1|^p|f(x)|^p + |c_2|^p|g(x)|^p]\ dx = 2^p|c_1|^p\|f\|_p^p + 2^p|c_2|^p\|g\|_p^p < \infty$$

This shows $\|c_1 f + c_2 g\|_p < \infty$, so $c_1 f + c_2 g \in L^p(E)$.

Now consider $p = \infty$, since $f, g \in L^p(E)$, there exists constant K_1, K_2 s.t. $|f(x)| \le K_1$ and $|g(x)| \le K_2$ a.e. on E. Thus, $|c_1 f(x) + c_2 g(x)| \le |c_1|K_1 + |c_2|K_2$ a.e. on E. This shows

$\|c_1 f + c_2 g\|_\infty \le |c_1|K_1 + |c_2|K_2 < \infty$ and $c_1 f + c_2 g \in L^\infty(E)$. $\square$

Exercise 4.4 Let $1 \le p \le \infty$, and $f(x), g(x)$ are measurable on $E \in \mathcal{M}$. Then,

$$\int_E |f(x)g(x)|\, dx \le \left(\int_E |f(x)|^p\, dx\right)^{1/p} \left(\int_E |g(x)|^q\, dx\right)^{1/q}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. This implies $\|fg\|_1 \le \|f\|_p \|g\|_q$.

Proof First consider if either $\|f\|_p$ or $\|g\|_q$ is zero, then $\|f\|_p\|g\|_q = 0$. Problem Set 3.1, Question 1. or Exercise 4.1 implies either $f(x) = 0$ a.e. or $g(x) = 0$ a.e. on E, so $f(x)g(x) = 0$ a.e. on E. Thus, $\|fg\|_1 = 0$ by Exercise 3.10, and $\|fg\|_1 \le \|f\|_p\|g\|_q$ holds.

From now on, suppose both $\|f\|_p > 0$ and $\|g\|_q > 0$. Then we consider if either $\|f\|_p$ or $\|g\|_q$ is infinity, $\|f\|_p\|g\|_q = \infty$. In this case $\|fg\|_1 \le \|f\|_p\|g\|_q$ always holds.

From now on, suppose both $\|f\|_p$ and $\|g\|_q$ are positive and finite. If $p = 1$, then $q = \infty$, and by Exercise 4.1, $|f(x)g(x)| \le \|g\|_\infty |f(x)|$ a.e. on E. By Exercise 3.2, part 1. & 3.,

$$\|fg\|_1 = \int_E |f(x)g(x)|\, dx \le \|g\|_\infty \int_E |f(x)|\, dx = \|f\|_1\|g\|_\infty$$

If $p = \infty$, then $q = 1$, and the proof is very similar.

From now on, suppose both $\|f\|_p$ and $\|g\|_q$ are positive and finite and $p, q \in (1, \infty)$. By taking logarithm on both sides and using concavity of logarithmic function, we can prove $a^{\frac{1}{p}} b^{\frac{1}{q}} \le \frac{a}{p} + \frac{b}{q}$ for all $a, b \ge 0$. Now consider

$$\begin{aligned}
\|fg\|_1 &= \|f\|_p\|g\|_q \int_E \left(\frac{|f(x)|^p}{\|f\|_p^p}\right)^{1/p} \left(\frac{|g(x)|^q}{\|g\|_q^q}\right)^{1/q} dx \\
&\le \|f\|_p\|g\|_q \int_E \left(\frac{|f(x)|^p}{p\|f\|_p^p} + \frac{|g(x)|^q}{q\|g\|_q^q}\right) dx \\
&= \left(\frac{1}{p} + \frac{1}{q}\right) \|f\|_p\|g\|_q = \|f\|_p\|g\|_q
\end{aligned}$$

where the first equality is by Exercise 3.2, part 3.; the inequality is by Exercise 3.2, part 1.; and the second equality is by Exercise 3.5 and Exercise 3.2, part 3.. $\square$

Note *The inequality in the conclusion is called Hölder's inequality.*

Example 4.2 Suppose $m(E) < \infty$, and $0 < p_1 < p_2 < \infty$. Prove $L^{p_2}(E) \subset L^{p_1}(E)$ and $\|f\|_{p_1} \le [m(E)]^{\frac{1}{p_1} - \frac{1}{p_2}} \|f\|_{p_2}$ for any measurable function $f(x)$ on E.

Proof Let $p = \frac{p_2}{p_1}$ and $q = \frac{p_2}{p_2 - p_1}$. Apply Hölder's inequality to $|f(x)|^{p_1}$ and 1, we have

$$\|f\|_{p_1}^{p_1} = \int_E |f(x)|^{p_1} \cdot 1\, dx \le \left(\int_E |f(x)|^{p_2}\, dx\right)^{\frac{p_1}{p_2}} \left(\int_E 1\, dx\right)^{\frac{p_2 - p_1}{p_2}} = \|f\|_{p_2}^{p_1} [m(E)]^{1 - \frac{p_1}{p_2}}$$

Take p_1-th square root on both sides, we obtain the desired result. $\square$

Example 4.3 Suppose $f \in L^p(0,1)$ with $1 < p \le \infty$. Let $F(x) = \int_0^x f(t)\, dt$ for all $x \in (0,1)$. Prove that $F(x) = o\left(x^{1/q}\right)$ as $x \to 0+$ where q satisfies $\frac{1}{p} + \frac{1}{q} = 1$.

Proof By Example 4.2, $f \in L^1(0,1)$, so we can apply Exercise 3.16, and $|F(x)| \le \int_0^x |f(t)|\, dt$

on $(0,1)$. By Hölder's inequality,

$$\int_0^x |f(t)|\,dt \le \left(\int_0^x |f(x)|^p\,dt\right)^{1/p}\left(\int_0^x 1\,dt\right)^{1/q} = x^{1/q}\left(\int_0^x |f(x)|^p\,dt\right)^{1/p}$$

Therefore, we have

$$\frac{F(x)}{x^{1/q}} \le \left(\int_0^x |f(x)|^p\,dt\right)^{1/p}$$

Since $f \in L^p(0,1)$, $|f|^p \in L^1(0,1)$. By Problem Set 3.4, Question 8., $\int_0^x |f(x)|^p\,dt \to 0$ as $x \to 0+$. This shows $F(x) = o\left(x^{1/q}\right)$ as $x \to 0+$. □

Exercise 4.5 Suppose $1 \le p \le \infty$, $f(x), g(x)$ are measurable on $E \in \mathcal{M}$, and $f(x)+g(x)$ is well-defined a.e. on E. Prove $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

Proof By our argeement, we can simply define $f(x)+g(x) = 0$ where $f(x)+g(x)$ is not well-defined and $\|f+g\|_p$ will be the same. When $p = 1$, since $|f(x)+g(x)| \le |f(x)|+|g(x)|$ on E, by Exercise 3.2, part 1. & 3.5,

$$\|f+g\|_1 = \int_E |f(x)+g(x)|\,dx \le \int_E |f(x)|\,dx + \int_E |g(x)|\,dx = \|f\|_1 + \|g\|_1$$

When $p = \infty$, let $A = \{c \,|\, |f(x)+g(x)| \le c \text{ a.e. on } E\}$, $A_1 = \{c_1 \,|\, |f(x)| \le c_1 \text{ a.e. on } E\}$, and $A_2 = \{c_2 \,|\, |g(x)| \le c_2 \text{ a.e. on } E\}$. To prove $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$, we only need to prove $\inf A \le \inf A_1 + \inf A_2$. For any $c_1 \in A_1$ and $c_2 \in A_2$, we have $|f(x)+g(x)| \le c_1 + c_2$ a.e. on E by triangular inequality. Thus, $\inf A \le c_1 + c_2$ for all $c_1 \in A_1$ and $c_2 \in A_2$. Take infimum on c_1 over A_1, and then on c_2 over A_2, we will obtain the desired result.

When $p \in (1,\infty)$, by triangular inequality, Exercise 3.2, part 1., & 3.5,

$$\begin{aligned}\int_E |f(x)+g(x)|^p\,dx &\le \int_E |f(x)+g(x)|^{p-1}(|f(x)|+|g(x)|)\,dx \\ &= \int_E |f(x)+g(x)|^{p-1}|f(x)|\,dx + \int_E |f(x)+g(x)|^{p-1}|g(x)|\,dx\end{aligned}$$

By Hölder's inequality,

$$\int_E |f(x)+g(x)|^{p-1}|f(x)|\,dx \le \left(\int_E |f(x)+g(x)|^p\,dx\right)^{\frac{p-1}{p}}\left(\int_E |f(x)|^p\,dx\right)^{1/p}$$

$$\int_E |f(x)+g(x)|^{p-1}|g(x)|\,dx \le \left(\int_E |f(x)+g(x)|^p\,dx\right)^{\frac{p-1}{p}}\left(\int_E |g(x)|^p\,dx\right)^{1/p}$$

Therefore, we obtain $\|f+g\|_p^p \le \|f+g\|_p^{p-1}(\|f\|_p + \|g\|_p)$. If $\|f+g\|_p = 0$, then the desired inequality trivially holds. If $\|f+g\|_p \ne 0$, we can cancel out $\|f+g\|_p^{p-1}$ on both sides, and we will obtain the desired properties. □

Note *The inequality in the conclusion is called Minkowski inequality.*

In Exercise 4.3, we have shown $L^p(E)$ is a linear space. In fact, we can further show it is a complete normed space, i.e., Banach space. However, we shall first introduce some definition about that.

Definition 4.3. Normed Space

*A **normed space** X over field $\mathbb{R}$ is a linear space in which we have a "norm" satisfying:*

1. *For all $x \in X$, $\|x\| \geq 0$.*
2. *For all $x, y \in X$, $\|x+y\| \leq \|x\| + \|y\|$.*
3. *For all $c \in \mathbb{R}$ and $x \in X$, $\|cx\| = |c|\|x\|$.*
4. *If $x \in X$ and $\|x\| = 0$, then $x = 0$.*

♣

Theorem 4.1

If $1 \leq p \leq \infty$, then $L^p(E)$ is a normed space.

♡

Proof Consider the L^p-norm defined in Definition 4.1, we need to check whether it satisfies the four conditions in Definition 4.3.

1. It is obvious that $\|f\|_p \geq 0$ for all $f \in L^p(E)$ by the Definition 4.1.
2. For all $f, g \in L^p(E)$, it is obvious that $|f|^p \in L^1(E)$ and $|g|^p \in L^1(E)$, so by Exercise 3.11, $|f(x)|^p$ and $|g(x)|^p$ are finite a.e. on E. This shows $f(x)$ and $g(x)$ are finite a.e. on E, so $f(x) + g(x)$ is well-defined a.e. on E. Thus, by Minkowski inequality, $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.
3. For $p < \infty$, since $f \in L^p(E)$, $|f|^p \in L^1(E)$, so by Exercise 3.12,
$$\left(\int_E |cf(x)|^p\, dx\right)^{1/p} = \left(\int_E |c|^p |f(x)|^p\, dx\right)^{1/p} = \left(|c|^p \int_E |f(x)|^p\, dx\right)^{1/p}$$
Thus, we have $\|cf\|_p = |c|\|f\|_p$.

   For $p = \infty$, if $c = 0$, then for all $f \in L^\infty(E)$, $cf(x) = 0$ on E, so $\|cf\|_\infty = 0$. It is obvious that $|c|\|f\|_\infty = 0$, so $\|cf\|_p = |c|\|f\|_p$ holds. Now we only consider $c \neq 0$. Let $A = \{k \,|\, |f(x)| \leq k \text{ a.e. on } E\}$ and $A_1 = \{k_1 \,|\, |cf(x)| \leq k_1 \text{ a.e. on } E\}$, then $\|cf\|_\infty = \inf A_1$ and $\|f\|_\infty = \inf A$. For all $k \in A$, $|f(x)| \leq k$ a.e. on E, so $|cf(x)| \leq |c|k$ a.e. on E. This shows $|c|k \in A_1$, and thus $|c|k \geq \|cf\|_\infty$. Since $c \neq 0$, we have $k \geq \frac{\|cf\|_\infty}{|c|}$, and by taking infimum over A on both sides, we obtain $\|f\|_\infty \geq \frac{\|cf\|_\infty}{|c|}$. This is equivalent to $|c|\|f\|_\infty \geq \|cf\|_\infty$. On the other hand, for all $k_1 \in A_1$, $|cf(x)| \leq k_1$ a.e. on E implies $|f(x)| \leq \frac{k_1}{|c|}$ a.e. on E. Thus, $\frac{k_1}{|c|} \in A$ and $\frac{k_1}{|c|} \geq \|f\|_\infty$. This is equivalent to $k_1 \geq |c|\|f\|_\infty$. By taking infimum over A_1 on both sides, $\|cf\|_\infty \geq |c|\|f\|_\infty$. Therefore, we proved $\|cf\|_\infty = |c|\|f\|_\infty$.
4. For $p < \infty$, if $\|f\|_p = 0$, by Problem Set 3.1, Question 1., $|f|^p = 0$ a.e. on E. Thus $f(x) = 0$ a.e. on E. By our agreement, $f(x)$ is just the zero element in $L^p(E)$.

   For $p = \infty$, if $\|f\|_\infty = 0$, by Exercise 4.1, $|f(x)| \leq 0$ a.e. on E, so $f(x) = 0$ a.e. on E. By our agreement, $f(x)$ is just the zero element in $L^p(E)$.

□

Definition 4.4. Cauchy Sequence

*Let X be a normed space. A sequence $\{x_k\}_{k=1}^{\infty} \subset X$ is **Cauchy** if for all $\epsilon > 0$, there exists $K_\epsilon \geq 1$ s.t. $\|x_k - x_l\| < \epsilon$, whenever $k, l \geq K_\epsilon$.* ♣

Definition 4.5. Banach Space

*A normed space in which every Cauchy sequence converges with respect to this particular norm is called a **complete normed space** or **Banach space**.* ♣

Definition 4.6. L^p-convergence

Let $f(x)$ and $f_k(x)$ be measurable on $E \in \mathcal{M}$. Suppose $f \in L^p(E)$ and $f_k \in L^p(E)$ for all $k \geq 1$. We say $f_k(x) \to f(x)$ in $L^p(E)$ if $\|f_k - f\|_p \to 0$ as $k \to \infty$. ♣

Remark From now on, if we say $f_k(x) \to f(x)$ in $L^p(E)$, then it implicitly indicates that $f \in L^p(E)$ and $f_k \in L^p(E)$ for all large enough k.

Theorem 4.2

If $1 \leq p \leq \infty$, then $L^p(E)$ is a Banach space. ♡

Proof First consider when $p = \infty$. Let $\{f_k\}_{k=1}^{\infty} \subset L^\infty(E)$ be Cauchy in $L^\infty(E)$. Define $A_{kl} = \{x \in E \,|\, |f_k(x) - f_l(x)| > \|f_k - f_l\|_\infty\}$ for all $k, l \geq 1$. By Exercise 4.1, $m(A_{kl}) = 0$. Let $A = \bigcup_{k,l=1}^{\infty} A_{kl}$, then $m(A) = 0$. Since $\{f_k\}_{k=1}^{\infty}$ is a Cauchy sequence, for all $\epsilon > 0$, there exists $K_\epsilon \geq 1$ s.t. $\|f_k - f_l\| < \epsilon$ if $k, l \geq K_\epsilon$. Thus, for all fixed $x \in E \setminus A$, $|f_k(x) - f_l(x)| < \epsilon$ if $k, l \geq K_\epsilon$. This implies for each fixed $x \in E \setminus A$, $\{f_k(x)\}_{k=1}^{\infty}$ is a Cauchy sequence. Since Cauchy sequence in $\mathbb{R}$ must converge, $f_k(x) \to f(x)$ on $E \setminus A$ and by the remark of Exercise 2.11, $f(x)$ is measurable on $E \setminus A$. Since $m(A) = 0$, we can define $f(x) = 0$ on A, and by Exercise 2.1, $f(x)$ is measurable on E. Take $l \to \infty$, $|f_k(x) - f(x)| < \epsilon$ on $E \setminus A$. By definition, we have $\|f_k - f\|_\infty \leq \epsilon$ if $k \geq K_\epsilon$. Since each $f_k \in L^\infty(E)$, by Minkowski inequality, it is easy to see $f \in L^\infty(E)$. This also shows $f_k(x) \to f(x)$ in $L^\infty(E)$, so $L^\infty(E)$ is a Banach space.

Then we consider when $p < \infty$. Let $\{f_k\}_{k=1}^{\infty}$ be Cauchy in $L^p(E)$ for all $i \geq 1$. There exists $K_i \geq 1$ s.t. if $k, l \geq K_i$, $\|f_k - f_l\|_p < \frac{1}{2^i}$. We can take $\{K_i\}_{i=1}^{\infty}$ s.t. K_i is increasing to infinity and $\|f_{K_{i+1}} - f_{K_i}\|_p < \frac{1}{2^i}$. Define $g(x) = \sum_{i=1}^{\infty} |f_{K_{i+1}}(x) - f_{K_i}(x)|$ and for all $k \geq 1$, $g_k(x) = \sum_{i=1}^{K} |f_{K_{i+1}}(x) - f_{K_i}(x)|$. Then, $g_k(x)$ is increasing in k for each fixed $x \in E$ and $g_k(x) \to g(x)$ pointwisely on E. By Minkowski inequality, $\|g_k\|_p \leq \sum_{i=1}^{k} \|f_{K_{i+1}} - f_{K_i}\|_p \leq 1$. Now since $|g_k(x)|^p \to |g(x)|^p$ pointwisely on E and $|g_k(x)|^p$ is increasing in k, by MCT-II, $\int_E |g(x)|^p \, dx = \lim_{k\to\infty} \int_E |g_k(x)|^p \, dx \leq 1$. This shows $|g|^p \in L^1(E)$ and by Exercise 3.11, $|g(x)|^p$ is finite a.e. on E, so $g(x)$ is finite a.e. on E. Thus, $\sum_{i=1}^{n-1} (f_{K_{i+1}}(x) - f_{K_i}(x))$ converges absolutely for a.e. $x \in E$. Since on $\mathbb{R}$, absolute convergence implies convergence,

$\sum_{i=1}^{n-1}(f_{K_{i+1}}(x) - f_{K_i}(x))$ converges for a.e. $x \in E$. However,

$$\sum_{i=1}^{n-1}(f_{K_{i+1}}(x) - f_{K_i}(x)) = f_{K_n}(x) - f_{K_1}(x)$$

so $f_{K_i}(x)$ converges to some $f(x)$ a.e. on E. By a similar argument to the $p = \infty$ case, $f(x)$ is measurable. Since $\{f_k(x)\}_{k=1}^{\infty}$ is Cauchy in $L^p(E)$, for all $\epsilon > 0$, there exists $K_\epsilon \geq 1$ s.t. if $k, l \geq K_\epsilon$, $\|f_k - f_l\|_p < \infty$. Take $l = K_i$ for large enough i s.t. $K_i \geq K_\epsilon$, then $\int_E |f_k(x) - f_{K_i}(x)|^p \, dx < \epsilon^p$ for all large i. By Fatou's lemma,

$$\int_E \varliminf_{i\to\infty} |f_k(x) - f_{K_i}(x)|^p \, dx \leq \varliminf_{i\to\infty} \int_E |f_k(x) - f_{K_i}(x)|^p \, dx \leq \epsilon^p$$

Thus, $\int_E |f_k(x) - f(x)|^p \, dx \leq \epsilon^p$ and $\|f_k - f\|_p \leq \epsilon$ for all $k \geq K_\epsilon$. Since each $f_k \in L^p(E)$, by Minkowski inequality, it is easy to see $f \in L^p(E)$. This also shows $f_k(x) \to f(x)$ in $L^p(E)$, so $L^p(E)$ is a Banach space. □

Recall in the previous chapter, we have shown that L^1-convergence implies convergence in measure. Now we generalize it to L^p-convergence implies convergence in measure.

Theorem 4.3

Let $f(x)$ and $f_k(x)$ be measurable on $E \in \mathcal{M}$. If $f_k(x) \to f(x)$ in $L^p(E)$, then $f_k(x) \to f(x)$ in measure on E.

Proof By Markov's inequality on $|f_k(x) - f(x)|^p$, for all $\alpha \in (0, \infty)$, we have

$$\alpha m\left(\{x \in E \,|\, |f_k(x) - f(x)|^p > \alpha\}\right) \leq \int_E |f_k(x) - f(x)|^p \, dx$$

For all $\sigma > 0$, take α s.t. $\alpha^{1/p} = \sigma$, then we have

$$m\left(\{x \in E \,|\, |f_k(x) - f(x)|^p > \alpha\}\right) \leq \frac{1}{\sigma^p}\|f_k - f\|_p^p \to 0$$

Therefore, $f_k(x) \to f(x)$ in measure on E. □

Problem Set 4.1

1. Let $0 < p < 1$ and $q = \frac{p}{p-1}$. Assume that if $g = 0$ on E then $\|g\|_{L^q(E)} = 0$.
   (a). Proved for f, g measurable on $E \in \mathcal{M}$ and $m(E) > 0$, we have the reversed Hölder's inequality, i.e., $\|fg\|_{L^1(E)} \geq \|f\|_{L^p(E)}\|g\|_{L^q(E)}$.
   (b). Prove reversed Minkowski inequality, i.e., for measurable f, g s.t. $f \geq 0$, $g \geq 0$ on E, we have $\|f\|_{L^p(E)} + \|g\|_{L^p(E)} \leq \|f + g\|_{L^p(E)}$.
   (c). Construct f and g s.t. $\|f\|_{L^p(E)} + \|g\|_{L^p(E)} < \|f + g\|_{L^p(E)}$.
2. Let X be a normed space. Suppose $x_\infty \in X$ and $x_k \in X$ for all $k \geq 1$. Prove that if $\|x_k - x_\infty\| \to 0$ as $k \to \infty$, then $\|x_k\| \to \|x_\infty\|$. In $L^1(-1, 1)$, construct a counterexample s.t. $\|f_k\|_{L^1} \to \|f_\infty\|_{L^1}$ but $f_k \not\to f_\infty$ in L^1.
3. Let $E \subset \mathbb{R}^m$, $F \subset \mathbb{R}^n$, and $f(x, y)$ be measurable on $E \times F$, where $x \in E$, $y \in F$. For $1 \leq p < \infty$, if $\int_F \|f(x, y)\|_{L_x^p(E)} \, dy < \infty$, prove

(a). For almost every fixed $x \in E$, $f(x,y) \in L^1_y(F)$.

(b). $\int_F f(x,y)\, dy$ is a measurable function of x on E and $\int_F f(x,y)\, dy \in L^p_x(E)$.

(c). $\left\| \int_F f(x,y)\, dy \right\|_{L^p_x(E)} \le \int_F \|f(x,y)\|_{L^p_x(E)}\, dy$.

4. Let $1 < p < \infty$. For all $f \in L^p(0,\infty)$, define $Tf = \frac{1}{x}\int_0^x f(y)\, dy$ for $x \in (0,\infty)$. Prove that $\|Tf\|_{L^p(0,\infty)} \le \frac{p}{p-1}\|f\|_{L^p(0,\infty)}$.
5. Let $f(x)$ be measurable on $\mathbb{R}^n$.

(a). Prove that $f(x-y)$ as a function of $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ is measurable.

(b). Prove that for all $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, $f * g \in L^p(\mathbb{R}^n)$ where $f * g = \int_{\mathbb{R}^n} f(x-y)g(y)\, dy$.

(c). Prove $\|f * g\|_{L^p(\mathbb{R}^n)} \le \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)}$.

6. Let f be continuous on the interval $(0,1)$. Prove that $\|f\|_{L^\infty(0,1)} = \sup_{x\in(0,1)} |f(x)|$.
7. Let f be measurable on E and there exists $r > 0$ s.t. $f \in L^r(E)$. Prove that $\lim_{p\to\infty}\|f\|_{L^p(E)} = \|f\|_{L^\infty(E)}$.
8. Let $f \in L^2(0,1)$ and $\int_0^1 f(x)x^n\, dx = 0, \forall\, n \in \mathbb{N}$. Prove $f(x) = 0$ a.e. on $(0,1)$.
9. Let f be positive and measurable on $(0,1)$. Prove that $1 \le \left(\int_0^1 f(x)\, dx\right)\left(\int_0^1 \frac{1}{f(x)}\, dx\right)$.
10. Let $f_k(x)$ be measurable on $(0,1)$ for all $k \ge 1$. Suppose $f_k \to f$ a.e. on $(0,1)$ and for some $r \in (0,\infty)$, $\int_0^1 |f_k(x)|^r\, dx \le M$ for constant M and for all $k \ge 1$. Prove that for all $0 < p < r$, $\int_0^1 |f_k(x) - f(x)|^p\, dx \to 0$ as $k \to \infty$.

4.2 Dense Subsets of L^p-space

In this section we are going to explore several density theorems for L^p-space. The main idea of these theorems is to use a sequence of "good" functions to approximate a general function in L^p-space.

Theorem 4.4

Suppose $1 \le p \le \infty$, then for all $f \in L^p(E)$ with $E \in \mathcal{M}$, there exists a sequence of measurable simple function $f_k(x)$ on E s.t. $f_k(x) \to f(x)$ in $L^p(E)$.

Proof We first consider the case when $p < \infty$. Recall by simple approximation theorem, there exists a sequence of measurable simple function $\{f_k(x)\}_{k=1}^\infty$ on E s.t. $f_k(x) \to f(x)$ pointwisely on E with $f_k(x)$ finite on E for all $k \ge 1$. Furthermore, if we scrutinize the proof of simple approximation theorem, we can see $|f_k(x)| \le |f(x)|$ on E. Notice that $|f_k(x) - f(x)|^p \to 0$ pointwisely on E, and $|f_k(x) - f(x)|^p \le 2^p(|f(x)|^p + |f_k(x)|^p) \le 2^{p+1}|f(x)|^p$. Since f is in $L^p(E)$, $|f|^p \in L^1(E)$. By Exercise 4.3, $L^1(E)$ is a linear space, so $2^{p+1}|f|^p \in L^1(E)$. By DCT, $\lim_{k\to\infty}\int_E |f_k(x) - f(x)|^p\, dx = \int_E \lim_{k\to\infty}|f_k(x) - f(x)|^p\, dx = 0$.

Then we consider the case when $p = \infty$. By Exercise 4.1, $|f(x)| \le \|f\|_\infty < \infty$ a.e. on E. Thus, we can find a measurable set A s.t. $m(A) = 0$ and $|f(x)| \le \|f\|_\infty$ for all $x \in E \setminus A$. By simple approximation theorem, there exists a sequence of measurable simple function $\{f_k(x)\}_{k=1}^{\infty}$ and $f_k(x) \to f(x)$ uniformly on $E \setminus A$. For all $\epsilon > 0$, there exists $K_\epsilon \geq 1$ s.t. $|f_k(x) - f(x)| < \epsilon$ for all $x \in E \setminus A$ and $k \geq K_\epsilon$. This implies $\|f_k - f\|_\infty \leq \epsilon$ for all $k \geq K_\epsilon$, which shows $f_k(x) \to f(x)$ in $L^\infty(E)$. $\square$

Remark This theorem shows that the set of measurable simple functions in L^p is a dense subset of L^p for all $1 \leq p \leq \infty$.

Theorem 4.5

Suppose $1 \leq p < \infty$, then for all $f \in L^p(E)$ with $E \in \mathcal{M}$, there exists a sequence of measurable simple function $f_k(x)$ on E s.t. $f_k(x) \to f(x)$ in $L^p(E)$ and $f_k(x)$ has bounded support, i.e., $\{x \in E \mid f_k(x) \neq 0\}$ is bounded, for all $k \geq 1$. ♡

Proof For $l \geq 1$, let $I_l(x) = I_{B_l(0)}(x)$, where $B_l(0)$ is the ball centered at the origin with radius l. Notice that for each fixed k, $I_l(x)f_k(x) \to f_k(x)$ pointwisely as $l \to \infty$. Also, $|I_l(x)f_k(x)|^p \leq |f_k(x)|^p \in L^1(E)$, so by DCT, as $l \to \infty$,

$$\int_E |f_k(x)|^p |I_l(x) - 1|^p \, dx = \int_E |[I_l(x)f_k(x)]^p - [f_k(x)]^p| \, dx \to 0$$

This shows for each fixed k, there exists $l_k \geq 1$ s.t. $\|I_{l_k} f_k - f_k\|_p \leq \frac{1}{k}$. By Theorem 4.4, there exists a sequence of measurable simple functions $\{g_k(x)\}_{k=1}^{\infty}$ s.t. $g_k(x) \to f(x)$ in $L^p(E)$. Let $f_k(x) = I_{l_k}(x)g_k(x)$, then $f_k(x)$ is measurable simple function with bounded support. Thus,

$$\|f_k(x) - f(x)\|_p \leq \|f_k(x) - g_k(x)\|_p + \|g_k(x) - f(x)\|_p \leq \frac{1}{k} + \|g_k(x) - f(x)\|_p \to 0$$

as $k \to \infty$. $\square$

Remark This theorem shows that the set of measurable simple functions with bounded support in L^p is a dense subset of L^p for all $1 \leq p < \infty$. Notice that this is not always true for $L^\infty(E)$. To find a counter-example, one can consider $f(x) = 1$ on $\mathbb{R}$.

Theorem 4.6

Suppose $1 \leq p < \infty$, then for all $f \in L^p(E)$ with $E \in \mathcal{M}$, there exists a bounded continuous function $g(x)$ on $\mathbb{R}^n$ s.t. $\|f - g\|_p < \epsilon$. There also exists a sequence of bounded continuous functions $\{g_k(x)\}_{k=1}^{\infty}$ on $\mathbb{R}^n$ s.t. $g_k \in L^p(E)$ for all $k \geq 1$ and $g_k(x) \to f(x)$ in $L^p(E)$. ♡

Proof We are going to prove this theorem in two steps. We first prove that for all $\epsilon > 0$, there exists bounded measurable function $h(x)$ s.t. $h \in L^p(E)$ and $\|f - h\|_p < \epsilon$. Then we apply Lusin's theorem to $h(x)$ to obtain the desired bounded continuous function.

Step 1: For all $k \geq 1$, define $f_k(x)$ on E by

$$f_k(x) = \begin{cases} f(x) & \text{if } |f(x)| < k \\ k & \text{if } f(x) \geq k \\ -k & \text{if } f(x) \leq -k \end{cases}$$

In this case, $|f_k(x)| \le |f(x)|$ for all $x \in E$ and $f_k(x) \to f(x)$ a.e. on E. Notice that

$$|f_k(x) - f(x)|^p \le (|f_k(x)| + |f(x)|)^p \le 2^p |f(x)|^p \in L^1(E)$$

By DCT, $|f_k(x) - f(x)|^p \to 0$ in $L^1(E)$, so there exists k_0 s.t. $\|f_{k_0} - f\|_p < \epsilon$. Since $f_{k_0}(x)$ is bounded measurable, we take $h(x) = f_{k_0}(x)$.

Step 2: By Lusin's theorem, there exists closed $F \subset E$ s.t. $m(E \setminus F) < \epsilon$ and $h\big|_F(x)$ is continuous on F. By Tietze extension theorem (a famous theorem in general topology), there exists continuous function $g(x)$ on $\mathbb{R}^n$ s.t. $g\big|_F(x) = h\big|_F(x)$ and $g(x)$ preserves boundedness of $f(x)$. Thus, we have

$$\|f - g\|_p \le \|f - h\|_p + \|h - g\|_p < \epsilon + [(2M)^p m(E \setminus F)]^{1/p}$$

This proves the first part of the theorem. The second part is trivial by simply taking $\epsilon = \frac{1}{n}$. □

Remark This theorem shows that the set of bounded continuous functions in L^p is a dense subset of L^p for all $1 \le p < \infty$. Notice that this is not always true for $L^\infty(E)$. To find a counter-example, one can consider $f(x) = I_{(0,1]}(x) - I_{[-1,0]}(x)$ on $E = [-1, 1]$. If the theorem is true, then there exists $k_0 \ge 1$ s.t. $\|f - g_{k_0}\|_\infty < \frac{1}{100}$, so $|f(x) - g_{k_0}(x)| < \frac{1}{100}$ for all $x \in E \setminus B$ with $m(B) = 0$. Since $g_{k_0}(x)$ is continuous, by intermediate value property, there exists $x_0 \in [-1, 1]$ s.t. $g_{k_0}(x_0) = 0$. Since $m(B) = 0$, there exists $x_n \to x_0$ s.t. $x_n \notin B$ for all n. Notice that if $x_0 \in [-1, 0]$, we can pick $x_n < x_0$ for all n; if $x_0 \in (0, 1]$, we pick $x_n > x_0$ for all n. In this case, $|f(x_n) - g_{k_0}(x_n)| < \frac{1}{100}$ for all $n \ge 1$. Take $n \to \infty$, by one-side continuity of f and continuity of g_{k_0}, we have $|f(x_0) - g_{k_0}(x_0)| < \frac{1}{100}$ which is a contradiction because $|f(x_0) - g_{k_0}(x_0)| = |f(x_0) - 0| = 1$.

Theorem 4.7

The set of all polynomial functions on $[a, b]$ is dense in $L^p([a, b])$ with a, b finite and $1 \le p < \infty$. ♡

Proof First, polynomial functions on $[a, b]$ are always in $L^p([a, b])$. For all $f \in L^p([a, b])$, for all $\epsilon > 0$, there exists bounded continuous function g defined on $\mathbb{R}$ s.t. $\|f - g\|_p < \epsilon$. Since g is continuous on $[a, b]$, by Weierstrass Approximation theorem, there exists polynomial $h(x)$ s.t. $\max_{[a,b]} |g(x) - h(x)| < \epsilon$. Consider

$$\|f - h\|_p \le \|f - g\|_p + \|g - h\|_p < \epsilon + (\epsilon^p (b - a))^{1/p} = K\epsilon$$

where K is a positive constant. □

Remark Notice that this is not always true for $L^\infty([a, b])$. To see this, consider polynomials on bounded interval $[a, b]$ as a special type of bounded continuous functions and use Theorem 4.6.

Definition 4.7. Step Function

Step function *is a function that can be written as a finite linear combination of indicator functions of disjoint intervals.* ♣

Theorem 4.8

The set of all step functions in $L^p([a,b])$ is dense in $L^p([a,b])$ with a, b finite and $p \in [1, \infty)$. ♡

Proof By Theorem 4.4, for all $\epsilon > 0$, there exists a simple measurable function $g(x)$ s.t. $\|f - g\|_p < \epsilon$. Denote $g(x) = \sum_{i=1}^k C_i I_{E_i}(x)$, where $E_i \in \mathcal{M}$ and $E_i \subset (a,b)$ for all $i = 1, \ldots, k$. It suffices to show $I_{E_i}(x)$ can be approximated by step function. By Theorem 1.1, since $m(E_i) \le b - a$, for all $\epsilon > 0$, there exists finitely many closed intervals $I_1, \ldots, I_J$ s.t. they are almost disjoint, $I_j \subset (a,b)$ for all j, and if $U = \bigcup_{j=1}^J I_j$, we have $m(E_i \triangle U) < \epsilon$. If two intervals are almost disjoint but not disjoint, then we denote the union of them as a new closed interval and replace the orginal two by this new closed interval. In this way, we can assume all I_j's are disjoint. Let $S(x) = \sum_{j=1}^J I_{I_j}(x)$, and we can observe that $|S(x) - I_{E_i}(x)| = I_{E_i \triangle U}(x)$. This shows $\|S - I_{E_i}\|_p = [m(E_i \triangle U)]^{1/p} < \epsilon^{1/p}$, and thus, $I_{E_i}(x)$ can be approximated by step function $S(x)$. □

Remark Notice that this is not always true for $L^\infty([a,b])$. See Problem Set 4.2, Question 1..

Problem Set 4.2

1. Prove that step functions are not dense in $L^\infty([0,1])$.
2. Let $f(x)$ be measurable and bounded on $\mathbb{R}$ and periodic with period $T > 0$. Consider $g \in L^1(0,a)$, where $0 < a < \infty$. Prove that as $\epsilon \to 0+$,
$$\int_0^a f(x/\epsilon) g(x)\, dx \to \langle f \rangle \int_0^a g(x)\, dx, \quad \langle f \rangle = \frac{1}{T}\int_0^T f(y)\, dy$$
3. Consider Fourier transform:
$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, dx$$
Prove that if $f \in L^1(\mathbb{R})$, then $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.
4. In Step 2 of the proof of Theorem 4.6, we use Tietze extension theorem. In fact, we only need to use a special version of it, that is, for every bounded continuous real-valued function $g(x)$ on a closed set $F \subset \mathbb{R}^n$, there exists a bounded continuous real-valued function $G(x)$ on $\mathbb{R}^n$ s.t. $G\big|_F(x) = g(x)$. If $|g(x)| \le M$ on F, $|G(x)| \le M$ on $\mathbb{R}^n$. To prove this special case of Tietze extension theorem, show that
   (a). there exists a continuous function $h : \mathbb{R}^n \mapsto \mathbb{R}$ s.t.
   - $|h(x)| \le \frac{1}{3}M$ for all $x \in F$,
   - $|h(x)| < \frac{1}{3}M$ for all $x \in F^c$,
   - $|g(x) - h(x)| \le \frac{2}{3}M$ for all $x \in F$.

   (b). there exists bounded continuous function $G(x)$ on $\mathbb{R}^n$ s.t. $G\big|_F(x) = g(x)$ on F and $|G(x)| \le M$ on $\mathbb{R}^n$.

4.3 Applications of Density Theorems in L^p-space

In this section we are going to apply the density theorems studied in the last section to verify some more advanced but essential properties of L^p-space. In short, we will first show the generalized Riemann-Lebesgue lemma. After that, we will discuss the continuity and separability of L^p-space.

Theorem 4.9. Generalized Riemann-Lebesgue Lemma

Suppose $\{g_n(x)\}_{n=1}^{\infty}$ are measurable and uniformly bounded on bounded interval $[a, b]$, i.e., there exists constant $M > 0$ s.t. $|g_n(x)| \leq M$ for all $x \in [a, b]$ and $n \geq 1$. Assume for all $c \in [a, b]$, $\int_a^c g_n(x)\ dx \to 0$ as $n \to \infty$, then $\int_a^b f(x)g_n(x)\ dx \to 0$ for all $f \in L^1([a, b])$.

Proof Step 1: Suppose f is a step function, then we can write $f(x) = \sum_{i=1}^k c_i I_{I_i}(x)$, where interval $I_i \subset [a.b]$. Denote $c_i \leq d_i$ as the two end points of interval I_i, then we have

$$\int_a^b f(x)g_n(x)\ dx = \sum_{i=1}^k c_i \int_a^b I_{I_i}(x)g_n(x)\ dx = \sum_{i=1}^k c_i \int_{c_i}^{d_i} g_n(x)\ dx$$

Since for each i, $\int_{c_i}^{d_i} g_n(x)\ dx = \int_a^{d_i} g_n(x)\ dx - \int_a^{c_i} g_n(x)\ dx$. Both terms on the right hand side converge to zero as $n \to \infty$ because $c_i, d_i \in [a, b]$. This shows $\int_a^b f(x)g_n(x)\ dx \to 0$.

Step 2: For any $f \in L^1(a, b)$, by Theorem 4.8, for all $\epsilon > 0$, there exists step function g s.t. $\|f - g\|_1 < \epsilon$. Consider

$$\begin{aligned}
\left|\int_a^b f(x)g_n(x)\ dx\right| &\leq \left|\int_a^b (f(x) - g(x))g_n(x)\ dx\right| + \left|\int_a^b g(x)g_n(x)\ dx\right| \\
&\leq M \int_a^b |f(x) - g(x)|\ dx + \left|\int_a^b g(x)g_n(x)\ dx\right| \\
&< M\epsilon + \left|\int_a^b g(x)g_n(x)\ dx\right|
\end{aligned}$$

Take $\limsup_{n\to\infty}$ on both sides, since $g(x)$ is step function, by Step 1,

$$\limsup_{n\to\infty} \left|\int_a^b f(x)g_n(x)\ dx\right| \leq M\epsilon$$

Since this is true for all $\epsilon > 0$, by taking $\epsilon \to 0$, we obtain the desired result. □

Theorem 4.10. Continuity in L^p-space

Let $1 \leq p < \infty$. For all $f \in L^p(\mathbb{R}^n)$, $\|f(x + h) - f(x)\|_{L^p_x(\mathbb{R}^n)} \to 0$ as $h \to 0$.

Proof Recall Theorem 4.6, for all $\epsilon > 0$, there exists bounded continuous function $g(x)$ s.t. $\|f - g\|_p < \epsilon$. Define function $\phi(r)$ for $r \geq 0$ to be

$$\phi(r) = \begin{cases} 1 & \text{if } r \in [0, 1] \\ -r + 2 & \text{if } r \in (1, 2] \\ 0 & \text{if } r > 2 \end{cases}$$

Let $g_k(x) = g(x)\phi\left(\frac{|x|}{k}\right)$ for all $k \geq 1$, then

$$g_k(x) = \begin{cases} g(x) & \text{if } |x| \leq k \\ 0 & \text{if } |x| \geq 2k \\ c_x^k \in (0, g(x)) & \text{if } k < |x| < 2k \end{cases}$$

Thus, $g_k(x)$ is uniformly continuous on $\bar{B}_{2k}(0)$, where $\bar{B}_r(x)$ is the closed ball centered at x with radius r. Abbreviate $\|\bullet\|_{L_x^p(\mathbb{R}^n)}$ as $\|\bullet\|_p$. Consider

$$\|f(x+h) - f(x)\|_p \leq \underbrace{\|f(x+h) - g(x+h)\|_p}_{\text{I}} + \underbrace{\|g(x+h) - g(x)\|_p}_{\text{II}} + \underbrace{\|g(x) - f(x)\|_p}_{\text{III}}$$

Notice that III $< \epsilon$ by construction. By applying FTT-II $n-1$ times (properly choosing objective function to apply),

$$\begin{aligned}
(\text{I})^p &= \int_{\mathbb{R}^n} |f(x+h) - g(x+h)|^p \, dx \\
&= \underbrace{\int \cdots \int_{-\infty}^{\infty}}_{n \text{ copies}} |f(x_1 + h_1, \cdots, x_n + h_n) - g(x_1 + h_1, \cdots, x_n + h_n)|^p \, dx_1 \cdots dx_n \\
&= \underbrace{\int \cdots \int_{-\infty}^{\infty}}_{n \text{ copies}} |f(x_1, \cdots, x_n) - g(x_1, \cdots, x_n)|^p \, dx_1 \cdots dx_n \\
&= \int_{\mathbb{R}^n} |f(x) - g(x)|^p \, dx = (\text{III})^p
\end{aligned}$$

where the third equality is by appying change of variables technique in Problem Set 3.3, Question 3.. Thus, I $< \epsilon$. Now we only need to focus on part II, where

$$\text{II} = \underbrace{\|g(x+h) - g_k(x+h)\|_p}_{a} + \underbrace{\|g_k(x+h) - g_k(x)\|_p}_{b} + \underbrace{\|g_k(x) - g(x)\|_p}_{c}$$

Similarly, we can prove $a = c$. Since g and g_k are all bounded continuous function, and $g_k(x) \to g(x)$ pointwisely, by DCT, $c \to 0$. Thus, there exists K s.t. $\|g_K - g\|_p < \epsilon$. Fix this K, and consider when $|h| < 1$,

$$b^p = \int_{B_{2K+1}(0)} |g_K(x+h) - g_K(x)|^p \, dx$$

because $g_K(x) = g_K(x+h) = 0$ when $|x| \geq 2K+1$ by definition of g_K. Up to now we have $\|f(x+h) - f(x)\|_p \leq 4\epsilon + b$. As $h \to 0$, since $g_K(x)$ is continuous, $g_K(x+h) - g_K(x) \to 0$. Also note that $g_K(x+h) - g_K(x)$ is bounded, so by DCT, $b \to 0$ as $h \to 0$. Thus, by taking $\overline{\lim}_{h\to 0}$ on both sides, we obtain $\overline{\lim}_{h\to 0}\|f(x+h) - f(x)\|_p \leq 4\epsilon$. Take $\epsilon \to 0$, we have shown that $\|f(x+h) - f(x)\|_p \to 0$ as $p \to 0$. □

Example 4.4 Let $E \subset \mathbb{R}^n$, $E \in \mathcal{M}$ with $m(E) < \infty$. Prove $\lim_{h\to 0} m((E+h) \cap E) = m(E)$.

Proof Note that it suffices to show as $h \to 0$,

$$\int_{\mathbb{R}^n} I_{(E+h)\cap E}(x) \, dx \to \int_{\mathbb{R}^n} I_E(x) \, dx < \infty$$

Since $I_{(E+h)\cap E}(x) = I_{E+h}(x)I_E(x)$, consider

$$\left|\int_{\mathbb{R}^n} I_E(x)(I_{E+h}(x) - I_E(x))\,dx\right| \le \int_{\mathbb{R}^n} |I_{E+h}(x) - I_E(x)|\,dx$$
$$= \int_{\mathbb{R}^n} |I_E(x-h) - I_E(x)|\,dx$$

Since $m(E) < \infty$, apply continuity of L^1-norm with $f(x) = I_E(x) \in L^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |I_E(x-h) - I_E(x)|\,dx = \|I_E(x-h) - I_E(x)\|_1 \to 0$$

□

Example 4.5 Let $E \subset \mathbb{R}^n$, $E \in \mathcal{M}$ with $m(E) > 0$. Prove $E - E \supset B_\delta(0)$ for $\delta > 0$, where $B_\delta(0)$ is the open ball centered at the origin with radius δ.

Proof Recall the first paragraph of the proof of Lemma 1.1, it suffices to show the desired result holds for the case when $m(E) < \infty$. In this case, by Example 4.4, $m((E+h)\cap E) \to m(E) > 0$ as $h \to 0$. Thus, there exists $\delta > 0$ s.t. $m((E+h)\cap E) > 0$ when $|h| < \delta$. Thus, there exists $x \in E$, $y + h \in E + h$ s.t. $x = y + h$, and hence $x - y = h$. This shows $h \in E - E$ for all $|h| < \delta$. Thus, $B_\delta(0) \subset E - E$. □

Remark This is exactly the Steinhauss Theorem we proved in Chapter 1 (see Lemma 1.1). In Chapter 1, we provided an elementary but rather tedious proof. However, with continuity property of L^p-space, we can prove it within a few lines.

At the end of this section, we are going to introduce a topological property, called "Separability", of a topological space (here we restricted to normed space). This property may be widely used in your graduate study.

Definition 4.8. Separability of Normed Space

*Normed space X is **separable** if it has a countable dense subset.* ♣

Example 4.6 Let $X = \mathbb{R}$, then X is separable because $\mathbb{Q}$ is a countable dense subset of it.

Theorem 4.11

For $1 \le p < \infty$, $L^p(\mathbb{R})$ is separable. ♡

Proof Let D be the set of all functions in the form of $p(x)I_{B_r(0)}(x)$ on $x \in \mathbb{R}^n$, where $p(x)$ is the polynomial with rational coefficient and $r \in \mathbb{Q}^+$. In this case, it is easy to see D is countable. Claim: D is dense in $L^p(\mathbb{R}^n)$. Take arbitrary function $f \in L^p(\mathbb{R}^n)$. For all $\epsilon > 0$, by Theorem 4.6, there exists bounded continuous function g s.t. $\|f - g\|_p < \frac{\epsilon}{100}$. Consider

$$\|g - gI_{B_r(0)}\|_p = \int_{\mathbb{R}^n} |g(x)|^p(1 - I_{B_r(0)}(x))\,dx$$

We want to show $\|g - gI_{B_r(0)}\|_p \to 0$ as $r \to \infty$. This is done by applying DCT because

$gI_{B_r(0)}(x) \to g(x)$ pointwisely and g is bounded. Thus, there exists large $R \in \mathbb{Q}$ so that $\|g - gI_{B_R(0)}\|_p < \frac{\epsilon}{100}$. Fix this R and consider $g(x)$ on $B_R(0)$. By Weierstrass approximation theorem, there exists polynomial $p(x)$ s.t. $|p(x) - g(x)| < \frac{\epsilon}{100c}$ for all $x \in B_R(0)$, where constant c is the volume of $B_R(0)$. WLOG, we can assume $p(x)$ has only rational coefficients because the set of polynomials with rational coefficients on $\bar{B}_R(0)$ is also a dense subset of polynomials with real coefficients on $\bar{B}_R(0)$ (because rational number is dense in real number and $\bar{B}_R(0)$ is compact). Thus, $\|gI_{B_R(0)} - pI_{B_R(0)}\|_p \leq \frac{\epsilon}{100}$. In conclusion,

$$\|f - pI_{B_R(0)}\|_p = \|f - g\|_p + \|g - gI_{B_R(0)}\|_p + \|gI_{B_R(0)} - pI_{B_R(0)}\|_p \leq \frac{3\epsilon}{100} < \epsilon$$

This shows exactly D is dense in $L^p(\mathbb{R}^n)$. □

Remark For $1 \leq p < \infty$, $L^p(E)$ is separable for any $E \subset \mathbb{R}^n$, $E \in \mathcal{M}$. To prove this, let D' denote the set of all functions in the form of $p(x)I_{B_r(0)}(x)I_E(x)$ on $\mathbb{R}^n$ and check D' is dense in $L^p(E)$.

Theorem 4.12

For $p = \infty$, $L^p(E)$ is not separable for $E \subset \mathbb{R}^n$ and $E \in \mathcal{M}$. ♡

Proof Define $f(r) = m(E \cap B_r(0))$ for $r \geq 0$. Then $f(0) = 0$ and $f(r) \to m(E)$ as $r \to \infty$ (by continuity of measure). Also, $f(r)$ is increasing on $r \in [0, \infty)$. Furthermore, we claim that $f(r)$ is continuous on $[0, \infty)$. To prove it, consider any $0 \leq r < t < \infty$,

$$\begin{aligned} 0 \leq f(t) - f(r) &= m(E \cap B_t(0)) - m(E \cap B_r(0)) \\ &= m(E \cap (B_t(0) \setminus B_r(0))) \leq m(B_t(0) \setminus B_r(0)) \to 0 \end{aligned}$$

as $|r - t| \to 0$. Thus, $f(r)$ is continuous. Define

$$A = \{\text{closed nondegenerate interval } I \text{ s.t. } f\big|_I \text{ is constant, and } I \text{ is maximal}\}$$

Then for $I, J \in A$, if $I \neq J$, $I \cap J = \varnothing$. Since $f(\bigcup_{I \in A} I) \subset [0, m(E)]$ is countable and $f([0, \infty)) \supset [0, m(E))$ is uncountable, $S = [0, \infty) \setminus \bigcup_{I \in A} I$ is uncountable. This is because f maps countable set to countable set. Note that for $s < t$ with $s, t \in S$, $f(s) < f(t)$ because if $f(s) = f(t)$, then f is a constant on (s, t), and $(s, t) \in A$. Now, for all $s \in S$, define $I_s(x) = I_{E \cap B_s(0)}(x)$, then $\|I_s - I_t\|_\infty = 1$ for $t > s$ because $m(B_t(0) \setminus B_s(0)) > 0$. Suppose $L^\infty(E)$ is separable, then there exists dense countable set $\{g_k\}_{k=1}^\infty$. For all $s \in S$, pick $k(s) \geq 1$ s.t. $\|I_s - g_{k(s)}\|_\infty < \frac{1}{4}$.

Claim: $k(s)$ is injective mapping. If $k(s) = k(t)$ but $s \neq t$, then $\|I_s - I_t\|_\infty = 1$. However, this is impossible because

$$\|I_s - I_t\|_\infty \leq \|I_s - g_{k(s)}\|_\infty + \|g_{k(s)} - I_t\|_\infty = \|I_s - g_{k(s)}\|_\infty + \|g_{k(t)} - I_t\|_\infty < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

This shows $k(s)$ is injective. Notice that $k(s)$ maps uncountable set S to countable set $\mathbb{N}$, but injective mapping cannot map a set to another set with smaller cardinality, so we obtain a contradiction. Thus, $L^\infty(E)$ is not separable. □

Problem Set 4.3

1. Recall the heat equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) & x \in \mathbb{R},\ t > 0 \\ u(x,0) = \phi(x) & x \in \mathbb{R} \end{cases}$$

whose solution is given by

$$u(x,t) = \int_{-\infty}^{\infty} \Gamma(x-y,t)\phi(y)\,dy$$

where $\Gamma(x,t)$ is the fundamental solution of heat equation given by

$$\Gamma(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \quad x \in \mathbb{R}, t > 0$$

which is the solution of heat equation with $\phi(x)$ equal to delta function $\delta(x)$.

(a). Prove for any fixed $y \in \mathbb{R}$,

$$\frac{\partial}{\partial t}\Gamma(x-y,t) = \frac{\partial^2}{\partial x^2}\Gamma(x-y,t), \quad \forall\, x \in \mathbb{R},\ \forall\, t > 0$$

(b). Suppose $\phi \in L^1(\mathbb{R})$ from now on, and prove $u(x,t)$ satisfies the equation $u_t(x,t) = u_{xx}(x,t)$ for $x \in \mathbb{R}$, $t > 0$.

(c). Prove $\|u(\cdot,t) - \phi(\cdot)\|_{L^1(\mathbb{R})} \to 0$ as $t \to 0+$.

(d). Prove that $|u(x,t)| \leq \frac{1}{\sqrt{4\pi t}}\|\phi\|_{L^1(\mathbb{R})}$, for all $x \in \mathbb{R}$, all $t > 0$. Give physical intepretation of this result.

2. Answer the following questions:

(a). For all measurable subset $A \subset [0, 2\pi]$, prove that

$$\lim_{t\to\infty} \int_A \cos(tx)\,dx = 0$$

(b). Let $t_k \to \infty$ as $k \to \infty$. Define $E = \{x \in [0,2\pi] \mid \sin(t_k x) \text{ converges as } k \to \infty\}$. Prove $m(E) = 0$.

3. Suppose $f \in L^1(0,1)$. Let $g(x) = \int_x^1 \frac{f(t)}{t}\,dt$, $0 < x \leq 1$. Prove that $g \in L^1(0,1)$, $\lim_{x\to 0+} xg(x) = 0$, and $\int_0^1 g(x)\,dx = \int_0^1 f(t)\,dt$.

4. Let $f \in L^1(\mathbb{R}^n)$, $g \in L^\infty(\mathbb{R}^n)$. Prove that

(a). $(f * g)(x)$ is uniformly continuous in x on $\mathbb{R}^n$.

(b). If $g \in L^1(\mathbb{R}^n)$, then $(f * g)(x) \to 0$ as $|x| \to \infty$.

Chapter 5 Lebesgue Differentiation

5.1 Differentiability of Monotone Functions

In this section, we are going to focus on the differentiablity of monotone functions. Our ultimate goal is to introduce two main theorems related to the differentiablity of monotone functions, namely, the Lebesgue's theorem for the differentiability of monotone functions and Fubini's theorem on differentiation. However, before going into that, we need to introduce some new concepts and useful tools.

Definition 5.1. Dini's Derivatives

Suppose $f(x)$ is real-valued on $(x_0-\delta, x_0+\delta)$ for $\delta > 0$, then the four types of ***Dini's derivatives*** *are given by*

$$D^+f(x_0) = \overline{\lim_{x\to x_0+}} \frac{f(x)-f(x_0)}{x-x_0}, \quad D^-f(x_0) = \overline{\lim_{x\to x_0-}} \frac{f(x)-f(x_0)}{x-x_0}$$
$$D_+f(x_0) = \underline{\lim_{x\to x_0+}} \frac{f(x)-f(x_0)}{x-x_0}, \quad D_-f(x_0) = \underline{\lim_{x\to x_0-}} \frac{f(x)-f(x_0)}{x-x_0}$$

♣

Example 5.1 Let $f(x) = |x|$ on $(-1,1)$. We can compute $D^+f(0) = 1 = D_+f(0)$ and $D^-f(0) = D_-f(0) = -1$.

✍ **Exercise 5.1** Consider four types of Dini's derivatives, and show that $D^+f(x_0) \geq D_+f(x_0)$ and $D^-f(x_0) \geq D_-f(x_0)$. Also, $D^+f(x_0) = D_+f(x_0) = D^-f(x_0) = D_-f(x_0) < \infty$ if and only if f is differentiable at x_0.

Example 5.2 Suppose f is continuous on bounded interval $[a,b]$ and $D^+f(x) \geq 0$ for all $x \in (a,b)$, then f is increasing on $[a,b]$. The same conclusion holds if $D^-f(x) \geq 0$ on (a,b).

Proof Special case: $D^+f(x) > 0$ on (a,b). Suppose there exists $a < x_1 < x_2 < b$ s.t. $f(x_2) < f(x_1)$, fix $\alpha \in (f(x_2), f(x_1))$. Let $E_\alpha = \{x \in (x_1,x_2) \mid f(x) = \alpha\}$. By intermediate value theorem, $E_\alpha \neq \varnothing$. Since f is continuous, E_α is closed and bounded, hence compact. Thus, there exists $c \in E_\alpha$ s.t. $c \geq x$ for all $x \in E_\alpha$. Note that $c \in (x_1, x_2)$, so $f(x) \leq f(c)$ for $x \in (c, x_2)$. Suppose not, then there exists x_0 s.t. $f(x_0) > f(c)$ and $x_0 \in (c, x_2)$. Note that $f(c) > f(x_2)$, so by intermediate value theorem, there exists $d \in (x_0, x_2)$ s.t. $f(d) = \alpha$. Then $d \in E_\alpha$ and $d > c$ leads to a contradiction. Thus, we have $D^+f(c) = \overline{\lim}_{x\to c+} \frac{f(x)-f(c)}{x-c} \leq 0$. This contradicts $D^+f(c) > 0$ and so we can conclude that $f(x)$ must be increasing on (a,b). Since $f(x)$ is continuous on $[a,b]$, $f(x)$ is also increasing on $[a,b]$.
General case: $D^+f(x) \geq 0$ on (a,b). For any $\epsilon > 0$, define $f_\epsilon(x) = f(x) + \epsilon x$. It is easy to see $D^+f_\epsilon(x) = D^+f(x) + \epsilon > 0$ given $D^+f(x) \geq 0$. Thus, by special case $f_\epsilon(x)$ is increasing

on $[a,b]$, i.e., $f(x_1)+\epsilon x_1 \le f(x_2)+\epsilon x_2$ for all $a \le x_1 \le x_2 \le b$. Take $\epsilon \to 0$ on both sides, $f(x_1) \le f(x_2)$ for all $a \le x_1 \le x_2 \le b$, which means $f(x)$ is increasing. $\square$

Definition 5.2. Vitali Covering

*Let $E \subset \mathbb{R}$ and Γ be a set of intervals I in $\mathbb{R}$. If for all $x \in E$ and all $\epsilon > 0$, we can find $I \in \Gamma$ and $x \in I$ with $0 < |I| < \epsilon$, then Γ is called a **Vitali covering** of E.* ♣

Example 5.3 Let $\mathbb{Q} = \{r_n\}_{n=1}^{\infty}$, $\Gamma = \{[r_n - \frac{1}{k}, r_n + \frac{1}{k}] \,|\, n,k = 1,2,\ldots\}$. Then Γ is a Vitali covering of $\mathbb{Q}$.

Theorem 5.1. Vitali Covering Theorem

Let $E \subset \mathbb{R}$ with $m^*(E) < \infty$ and Γ is a Vitali covering of E. Then for all $\epsilon > 0$, there exists a finite disjoint collection $\{I_n\}_{n=1}^N$ of intervals in Γ s.t. $m^*(E \setminus \bigcup_{n=1}^N I_n) < \epsilon$. ♡

Proof Notice that if we prove the desired result for the case when all intervals $I \in \Gamma$ are closed, we can easily prove the general case. If Γ is a Vitali covering of E, we can define $\bar{\Gamma} = \{\bar{I} \,|\, I \in \Gamma\}$ and $\bar{\Gamma}$ is also a Vitali covering of E because $|\bar{I}| = |I|$. Then by closed interval case, for all $\epsilon > 0$, there exists $\{\bar{I}_n\}_{n=1}^N$ s.t. $m^*(E \setminus \bigcup_{n=1}^N \bar{I}_n) < \epsilon$. Thus, $\{I_n\}_{n=1}^N$ satisfies $m^*(E \setminus \bigcup_{n=1}^N I_n) < \epsilon$. This proves the general case. Thus, WLOG, we can assume all $I \in \Gamma$ are closed.

Since $m(E) < \infty$, there exists open set G s.t. $G \supset E$ and $m(G) < \infty$. Define $\Gamma_1 = \{I \in \Gamma \,|\, I \subset G\}$, then $\Gamma_1 \subset \Gamma$ is also a Vitali covering of E. To see this, for all $x \in E$, consider for all $\epsilon > 0$, there exists neighborhood $N_\epsilon(x)$ of x with radius ϵ s.t. $N_\epsilon(x) \subset G$. Since Γ is a Vitali covering, there exists $I \in \Gamma$ s.t. $|I| < \frac{\epsilon}{4}$ and $x \in I$. Notice that $I \subset N_\epsilon(x)$, so $I \in \Gamma_1$. This shows for all $\epsilon > 0$ and $x \in E$, we can find $I \in \Gamma_1$ s.t. $x \in I$ and $0 < |I| < \epsilon$. Thus, Γ_1 is indeed a Vitali covering of E and it suffices to choose the desired $\{I_n\}_{n=1}^N$ from Γ_1.

We choose $\{I_n\}_{n=1}^N$ inductively. Choose $I_1 \in \Gamma_1$ arbitrarily. For $n \ge 2$, let $\Gamma_{n+1} = \{I \,|\, I \in \Gamma_1,\ I \cap I_k = \varnothing \text{ for } k = 1,\ldots,n\}$. If $\Gamma_{n+1} = \varnothing$ (called "finite termination"), then denote the current index n as N and claim that $\{I_n\}_{n=1}^N$ satisfies the desired property. Actually, in this case $E \subset \bigcup_{n=1}^N I_n$, so $m^*(E \setminus \bigcup_{n=1}^N I_n) = 0$. This is because if there exists $x \in E \setminus \bigcup_{n=1}^N I_n$, then we can find $\delta > 0$ s.t. $N_\delta(x) \subset G \setminus \bigcup_{n=1}^N I_n$ because I_n's are closed and $G \setminus \bigcup_{n=1}^N I_n$ is open. Then there exists $I \in \Gamma_1$ s.t. $|I| < \frac{\delta}{4}$ and $I \subset N_\delta(x)$. This shows $I \cap I_n = \varnothing$ for $n = 1,\ldots,N$, i.e., $I \in \Gamma_{n+1}$, which is a contradiction. If $\Gamma_{n+1} \ne \varnothing$, then let $k_n = \sup_{I \in \Gamma_{n+1}} |I|$, and $k_n > 0$. There exists $I_{n+1} \in \Gamma_{n+1}$ with $|I_{n+1}| > \frac{1}{2} k_n$, and we can continue to choose I_{n+2} by the same procedure.

If Γ_n is noempty for all $n \ge 1$, we will obtain a sequence of disjoint closed intervals $\{I_n\}_{n=1}^{\infty}$. Since $I_n \subset G$, $\sum_{n=1}^{\infty} |I_n| = m(\bigcup_{n=1}^{\infty} I_n) \le m(G) < \infty$. Thus, for all $\epsilon > 0$, we can find N s.t. $\sum_{n=N+1}^{\infty} |I_n| < \frac{\epsilon}{5}$. We claim $\{I_1,\ldots,I_N\}$ is the desired collection, i.e., $m^*(E \setminus \bigcup_{n=1}^N I_n) < \epsilon$.

Let $x \in E \backslash \bigcup_{n=1}^{N} I_n$, then by the same argument as the finite termination case, there exists $I \in \Gamma_1$ s.t. $x \in I$ and $I \cap I_n = \varnothing$ for all $n = 1, \ldots, N$. For all $n \geq 1$, we have $|I| \leq k_n < 2|I_{n+1}|$. Since $|I_n| \to 0$ as $n \to \infty$, $k_n \to 0$. We can always find the smallest $N_1 > N$ s.t. $|I| > k_{N_1}$. This shows $|I| \leq k_{N_1-1} < 2|I_{N_1}|$. Note that $|I| > k_{N_1}$ but $|I| \leq k_{N_1-1}$ implies that $I \cap I_{N_1} \neq \varnothing$. Since $x \in I$, the distance of x to the midpoint of I_n is at most $|I| + \frac{1}{2}|I_{N_1}| \leq \frac{5}{2}|I_{N_1}|$. Let J_{N_1} be the interval with the same midpoint of I_{N_1} but $|J_{N_1}| = 5|I_{N_1}|$, then $x \in J_{N_1}$. Thus, $E \setminus \bigcup_{n=1}^{N} I_n \subset \bigcup_{N+1}^{\infty} J_n$ and $m^*(E \setminus \bigcup_{n=1}^{N} I_n) \leq \sum_{n=N+1}^{\infty} |J_n| = 5 \sum_{n=N+1}^{\infty} |I_n| < \epsilon$. $\square$

Remark The theorem may not be true if $m^*(E) = \infty$. Consider

$$\Gamma = \{[x - \frac{1}{n}, x + \frac{1}{n}] \,|\, \forall x \in \mathbb{R}, \forall n \in \mathbb{N}^+\}$$

is a Vitali covering of $\mathbb{R}$. However, for any finitely many intervals $\{[x - \frac{1}{n}, x + \frac{1}{n}]\}_{n=1}^{N}$, $m(\mathbb{R} \setminus \bigcup_{n=1}^{N} [x - \frac{1}{n}, x + \frac{1}{n}]) = \infty$.

Remark From the proof of this theorem, one can easily show there exists at most countable disjoint intervals $\{I_n\}_{n=1}^{\infty} \subset \Gamma$ ($\{I_n\}_{n=1}^{N}$ for finite termination case) s.t. $m^*(E \setminus \bigcup_{n=1}^{\infty} I_n) = 0$ ($m^*(E \setminus \bigcup_{n=1}^{N} I_n) = 0$ for finite termination case).

Remark Interestingly, the above remark is still true even if $m^*(E) = \infty$. We can define $E_k = \{x \in E \,|\, k < |x| < k+1\}$ for all $k \geq 0$, then $E = (\bigcup_{k=1}^{\infty} E_k) \cup Z$ with $m(Z) = 0$. Define $\Gamma_k = \{I \in \Gamma \,|\, k < |x| < k+1 \text{ for all } x \in I\}$. We claim that Γ_k is a Vitalli covering of E_k for all $k \geq 0$. It can be verified by using exactly the same method as in the second paragraph of proof of the theorem. Apply the theorem, there exists $\{I_n^k\}_{n=1}^{\infty}$ s.t. $m^*(E_k \setminus \bigcup_{n=1}^{\infty} I_n^k) = 0$ for all $k \geq 0$. Notice that $I_n^k \cap I_{n'}^{k'} = \varnothing$ if $(k, n) \neq (k', n')$. Thus,

$$E \setminus \left(\bigcup_{k=0}^{\infty} \bigcup_{n=1}^{\infty} I_n^k \right) \subset \left[\bigcup_{k=0}^{\infty} E_k \setminus \left(\bigcup_{k=0}^{\infty} \bigcup_{n=1}^{\infty} I_n^k \right) \right] \cup Z \subset \bigcup_{k=0}^{\infty} \left(E_k \setminus \bigcup_{n=1}^{\infty} I_n^k \right) \cup Z$$

By taking outer measure on both sides together with monotonicity and σ-subadditivity, we obtain $m^*(E \setminus \bigcup_{k=0}^{\infty} \bigcup_{n=1}^{\infty} I_n^k) \leq \sum_{k=0}^{\infty} m^*(E_k \setminus \bigcup_{n=1}^{\infty} I_n^k) = 0$.

Before we state and prove the main theorem in this section, there is one more crucial lemma that will be very helpful for the proof of our main theorem.

Lemma 5.1

Let $F : [a, b] \mapsto \mathbb{R}$ be an increasing function defined on bounded interval $[a, b]$. For two real numbers $r < R$, the set $E = \{x \in (a, b) \,|\, D_-F(x) < r < R < D^+F(x)\}$ has measure zero.

Proof Let $m^*(E) = s$. For any $\epsilon > 0$, there exists open set $O \supset E$ s.t. $m(O) < s + \epsilon$. Let $x \in E$. Then $D_-F(x) < r$ implies for all $\delta > 0$, there exists $0 < h < \delta$ s.t. $\frac{F(x) - F(x-h)}{h} < r$. Collect all of such intervals $[x - h, x] \subset O$, we obtain a Vitali covering of E. By Vitali covering theorem, there exists disjoint intervals $I_1, \ldots, I_N$, where $I_k = [x_k - h_k, x_k]$ s.t. $m^*(E \setminus \bigcup_{k=1}^{N} I_k) < \epsilon$. Denote $I_k^o = (x_k - h_k, x_k)$, then $m^*(E \setminus \bigcup_{k=1}^{N} I_k^o) < \epsilon$. Define

$A = E \cap (\bigcup_{k=1}^N I_k^o)$, then $m^*(A) > s - \epsilon$. Moreover, we have

$$\sum_{k=1}^N (F(x_k) - F(x_k - h_k)) < r \sum_{k=1}^N h_k < rm(O) < r(s+\epsilon)$$

Let $y \in A$. Then $D^+F(y) > R$ implies there exists arbitrarily small $k > 0$ s.t. $[y, y+k] \subset I_k$ for some k and $\frac{F(y+k)-F(y)}{k} > R$. The collection of such intervals is a Vitali covering of A, so by Vitali covering theorem again, there exists disjoint $J_1, \dots, J_M$ with $J_j = [y_j, y_j + k_j]$ s.t. $m^*(A \setminus \bigcup_{j=1}^M J_j) < \epsilon$. It further implies $m^*(A \cap (\bigcup_{j=1}^M J_j)) > s - 2\epsilon$. Moreover,

$$\sum_{j=1}^M (F(y_j + k_j) - F(y_j)) > R \sum_{j=1}^M k_j > R(s - 2\epsilon)$$

Also, each J_j is contained in some I_n, so for each fixed n, by increasing property of F,

$$\sum_{j: J_j \subset I_n} (F(y_j + k_j) - F(y_j)) \le F(x_n) - F(x_n - h_n)$$

Sum both sides over $n = 1, \dots, N$,

$$\begin{aligned} \sum_{n=1}^N (F(x_n) - F(x_n - h_n)) &\ge \sum_{n=1}^N \sum_{j: J_j \subset I_n} (F(y_j + k_j) - F(y_j)) \\ &= \sum_{j=1}^M (F(y_j + k_j) - F(y_j)) > R(s - 2\epsilon) \end{aligned}$$

Thus, we have $r(s+\epsilon) > R(s - 2\epsilon)$ for all $\epsilon > 0$. Take $\epsilon \to 0$, we obtain $rs \ge Rs$. Since $r < R$, $s = 0$ and we are done. $\square$

Theorem 5.2. Lebesgue's Theorem

Suppose real-valued function $f(x)$ is increasing on $[a, b]$. Then $f'(x)$ exists a.e. in (a, b). Moreover, $f'(x)$ is measurable, nonnegaitve, and the Lebesgue integral of $f'(x)$ satisfies

$$\int_a^b f'(x)\, dx \le f(b) - f(a)$$

♡

Proof Notice that

$$\{x \in (a,b) \,|\, D_-f(x) < D^+f(x)\} = \bigcup_{r, R \in \mathbb{Q}} \{x \in (a,b) \,|\, D_-f(x) < r < R < D^+f(x)\}$$

By Lemma 5.1, $m(\{x \in (a,b) \,|\, D_-f(x) < r < R < D^+f(x)\}) = 0$ for all $r, R \in \mathbb{Q}$, so $m(\{x \in (a,b) \,|\, D_-f(x) < D^+f(x)\}) = 0$, i.e., $D_-f(x) \ge D^+f(x)$ a.e. on (a,b). Now consider function $-f(-x)$, it is also increasing. Thus, we can apply the same argument on $g(x) = -f(-x)$ on $(-b, -a)$ and obtain $D_-g(x) \ge D^+g(x)$ a.e. on (a,b). Notice that

$$\begin{aligned} D_-g(x) &= \varliminf_{y \to x-} \frac{g(y) - g(x)}{y - x} = \varliminf_{y \to x-} \frac{f(-y) - f(-x)}{-y - (-x)} \\ &= \varliminf_{(-y) \to (-x)+} \frac{f(-y) - f(-x)}{-y - (-x)} = \varliminf_{z \to w+} \frac{f(z) - f(w)}{z - w} = D_+f(w) \end{aligned}$$

Similarly, we can obtain $D^+g(x) = D^-f(w)$, so $D_+f(w) \ge D^-f(w)$ for all $w = -x$ where $x \in (-b, -a)$. Thus, $D_+f(x) \ge D^-f(x)$ for $x \in (a,b)$. Note that $D^-f(x) \ge D_-f(x)$ and

$D^+f(x) \geq D_+f(x)$ is always true by Exercise 5.1. Therefore, for almost all $x \in (a, b)$,

$$D_+f(x) \geq D^-f(x) \geq D_-f(x) \geq D^+f(x) \geq D_+f(x)$$

which implies $D_+f(x) = D^-f(x) = D_-f(x) = D^+f(x)$ a.e. on (a, b). Then we can conclude $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists (possibly infinity) a.e. on (a, b). Define $f(x) = f(b)$ for all $x \geq b$. Let $g_n(x) = n(f(x + \frac{1}{n}) - f(x))$, then $g_n(x)$ is measurable, nonnegative on $[a, b]$ and $g_n(x) \to g(x)$ a.e. on (a, b). This shows $f'(x) \geq 0$ is measurable. By Fatou's lemma,

$$\begin{aligned}
\int_a^b f'(x)\, dx &\leq \varliminf_{n\to\infty} \int_a^b g_n(x)\, dx = \varliminf_{n\to\infty} n \int_a^b \left[f\left(x + \frac{1}{n}\right) - f(x) \right] dx \\
&\overset{(1)}{=} \varliminf_{n\to\infty} n \left(\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f(x)\, dx - \int_a^b f(x)\, dx \right) \\
&= \varliminf_{n\to\infty} n \left(\int_b^{b+\frac{1}{n}} f(x)\, dx - \int_a^{a+\frac{1}{n}} f(x)\, dx \right) \\
&\leq \varliminf_{n\to\infty} n \left(\frac{f(b)}{n} - \frac{f(a)}{n} \right) = f(b) - f(a)
\end{aligned}$$

Since $f(a)$ and $f(b)$ are both real value and $f(x)$ is increasing on $[a, b]$, $f(x)$ is bounded on $[a, b]$. Since $[a, b]$ is also bounded interval, $f \in L^1(a, b)$. Thus, (1) follows from change of variable of integrable function (see Problem Set 3.3, Question 3.). The above inequality shows $f'(x) \in L^1(a, b)$, so $f'(x)$ is finite a.e. on (a, b). Thus, $f(x)$ is differentiable a.e. on (a, b). □

Theorem 5.3. Fubini's Theorem on Differentiation

Suppose $f_n(x)$ is increasing on $[a, b]$ for all $n \geq 1$ and $S(x) \triangleq \sum_{n=1}^{\infty} f_n(x)$ is convergent for all $x \in [a, b]$. Then $S(x)$ is differentiable a.e. on (a, b), $\sum_{n=1}^{\infty} f'_n(x)$ is convergent a.e. on (a, b), and $S'(x) = \sum_{n=1}^{\infty} f'_n(x)$ a.e. on (a, b). ♡

Proof Let $S_N(x) = \sum_{n=1}^{N} f_n(x)$ and $R_N(x) = \sum_{n=N+1}^{\infty} f_n(x)$ for all $N \geq 1$. Then $S(x)$, $S_N(x)$, and $R_N(x)$ are all increasing on $[a, b]$ and hence differentiable a.e. on (a, b) by Lebesgue's Theorem (Theorem 5.2). Thus, there exists $A \subset (a, b)$ with $m((a, b) \setminus A) = 0$ and $S(x)$, $S_N(x)$, and $R_N(x)$ are all differentiable at every $x \in A$ for all $N \geq 1$. From Lebesgue's Theorem, we also know $R'_N(x) \geq 0$ on A. Also, $S'(x) = S'_N(x) + R'_N(x) \geq S'_N(x) = \sum_{n=1}^{N} f'_n(x) \geq 0$ on A, where the last equality is because each $f_n(x) = S_n(x) - S_{n-1}(x)$ (Define $S_0(x) = 0$) is differentiable on A. Also, $f'_n(x) \geq 0$ for all $n \geq 1$ on A. Since $S'(x)$ is finite and $\sum_{n=1}^{N} f'_n(x) \leq S'(x)$ for all $N \geq 1$, take $N \to \infty$, $\sum_{n=1}^{N} f'_n(x)$ converges to $\sum_{n=1}^{\infty} f'_n(x) < \infty$ for each fixed $x \in A$. Thus, $\sum_{n=1}^{\infty} f'_n(x)$ is convergent a.e. on (a, b).

Now it remains to show $S'(x) = \sum_{n=1}^{\infty} f'_n(x)$ a.e. on (a, b). It suffices to show $R'_N(x) \to 0$ as $N \to \infty$ a.e. on (a, b). Notice that $R'_N(x) = R'_{N+1}(x) + f'_{N+1}(x) \geq R'_{N+1}(x)$, so $R'_N(x)$ is decreasing in N for any fixed $x \in A$. Thus, $R'_N(x)$ is convergent because it is bounded below by zero. Now we only need to show there exists a subsequence $R'_{N_i}(x) \to 0$ as $i \to \infty$ a.e. on (a, b). Since $\sum_{n=1}^{\infty} f_n(b)$ and $\sum_{n=1}^{\infty} f_n(b)$ converges, $R_N(b) \to 0$ and $R_N(a) \to 0$ as $N \to \infty$. This also shows $R_N(b) - R_N(a) \to 0$ as $N \to \infty$. Take subsequence $N_i \to \infty$ as

$i \to \infty$ s.t. $0 \le R_{N_i}(b) - R_{N_i}(a) < \frac{1}{2^i}$. Since $f_n(x)$ is an increasing function for all $n \ge 1$, $0 \le R_{N_i}(x) - R_{N_i}(a) < \frac{1}{2^i}$ for all $x \in [a,b]$. This implies $R(x) \triangleq \sum_{i=1}^{\infty}(R_{N_i}(x) - R_{N_i}(a))$ converges.

It is easy to see $R(x)$ is increasing on $[a,b]$, so by Lebesgue's Theorem, $R(x)$ is differentiable a.e. on (a,b) and $R'(x) \ge 0$. There exists $B \subset (a,b)$ s.t. $m((a,b) \setminus B) = 0$ and $R(x)$ is differentiable at every point in B. Also, define $U_M(x) = \sum_{i=1}^{M}(R_{N_i}(x) - R_{N_i}(a))$ and $V_M(x) = \sum_{i=M+1}^{\infty}(R_{N_i}(x) - R_{N_i}(a))$, then $U_M(x)$ is differentiable at all $x \in A$ for all $M \ge 1$ and hence $V_M(x) = R(x) - U_M(x)$ is differentiable at all $x \in A \cap B$ for all $M \ge 1$. Thus, on $A \cap B$, $R'(x) = U'_M(x) + V'_M(x) \ge U'_M(x) = \sum_{i=1}^{M} R'_{N_i}(x) \ge 0$. Take $M \to \infty$, we have $R'(x) \ge \sum_{i=1}^{\infty} R'_{N_i}(x)$. Since $R'(x)$ is finite, $\sum_{i=1}^{\infty} R'_{N_i}(x) < \infty$, and thus $R'_{N_i}(x) \to 0$ for all $x \in A \cap B$. Notice that $m((a,b) \setminus (A \cap B)) = 0$, so $R'_{N_i}(x) \to 0$ a.e. on (a,b). $\square$

Problem Set 5.1

1. Let $f(x)$ be increasing on $[a,b]$. Prove that the set of discontinuous points of f is at most countable.
2. Let $f(x) = x \sin\frac{1}{x}$ for $x \ne 0$ and $f(x) = 0$ for $x = 0$. Find Dini's derivative $D^{\pm}f(0)$ and $D_{\pm}f(0)$.
3. Let $f(x)$ be real-valued on (a,b). Define $E = \{x \in (a,b) \,|\, D^+f(x) < D_-f(x)\}$. Prove that E is at most countable.
4. Let $f(x)$ be increasing on (a,b). Let $E \subset (a,b)$ s.t. $E \in \mathcal{M}$ and for all $\epsilon > 0$, there exists open $G \subset (a,b)$, $G \supset E$ s.t. $\sum_i (f(b_i) - f(a_i)) < \epsilon$, where $G = \bigcup_i (a_i, b_i)$. Prove that $f'(x) = 0$ for a.e. $x \in E$.
5. Suppose $f(x)$ is continuous on I. Prove that it is impossible that $D^+f(x) > c > D_-f(x)$ for all $x \in I$, where c is a constant and I is an interval.
6. Find a function $f(x)$ that is strictly increasing on $\mathbb{R}$, discontinuous at and only at every $q \in \mathbb{Q}$, and $f'(x) = 0$ a.e. on $\mathbb{R}$.

5.2 Function of Bounded Variations

Definition 5.3. Total Variation

*Suppose $f(x)$ is defined on $[a,b]$ and it is real-valued. Let Δ be a partition of $[a,b]$, i.e., $\Delta = \{a = x_0, x_1, \ldots, x_{n-1}, x_n = b\}$. Define $v_\Delta = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$ and $V_a^b(f) = \sup\{v_\Delta \,|\, \Delta$ is a partition of $[a,b]\}$. We call $V_a^b(f)$ the **total variation** of f over $[a,b]$.*

Recall our definition of positive part and negative part of any real numbers, i.e., for all $t \in \mathbb{R}$, $t^+ = \max\{0,t\} \ge 0$ and $t^- = \min\{0,t\} \le 0$. Also, $t = t^+ + t^-$ and $|t| = t^+ - t^-$.

Thus, we can define positive variation and negative variation respectively as follows:

Definition 5.4. Positive & Negative Variation

*Suppose $f(x)$ is defined on $[a, b]$ and it is real-valued. Let Δ be a partition of $[a, b]$. Define $p_\Delta = \sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))^+$ and $P_a^b(f) = \sup\{p_\Delta \mid \Delta$ is a partition of $[a, b]\}$. Then we call $P_a^b(f)$ the **positive variation** of f over $[a, b]$. Similarly, we can define $n_\Delta = -\sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))^-$ and $N_a^b(f) = \sup\{n_\Delta \mid \Delta$ is a partition of $[a, b]\}$. We call $N_a^b(f)$ the **negative variation** of f over $[a, b]$.* ♣

Definition 5.5. Functions of Bounded Variation

*Suppose $f(x)$ is defined on $[a, b]$ and it is real-valued. If the total variation of f is finite, i.e., $V_a^b(f) < \infty$, then we say f has **bounded variation** on $[a, b]$ and denote it as $f \in \mathrm{BV}([a, b])$.* ♣

Example 5.4 Suppose f is monotone on $[a, b]$, then $V_a^b(f) = |f(b) - f(a)|$.

Example 5.5 Suppose f is continuous on $[a, b]$ and differentiable on (a, b) with $|f'(x)| \le M$ for some constant $M > 0$, then $f \in \mathrm{BV}([a, b])$.

Proof Take any partition Δ of $[a, b]$, we have $v_\Delta = \sum_{i=1}^{n}|f(x_i) - f(x_{i-1})|$. By mean value theorem on $[x_{i-1}, x_i]$, there exists $c_i \in (x_{i-1}, x_i)$ s.t. $f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1})$. Thus, we have $v_\Delta = \sum_{i=1}^{n}|f'(c_i)||x_i - x_{i-1}| \le M\sum_{i=1}^{n}|x_i - x_{i-1}| = M(b - a)$. This shows for all Δ, $v_\Delta \le M(b - a)$, so by taking supremum over all Δ on both sides, we have $V_a^b(f) \le M(b - a) < \infty$. Therefore, $f \in \mathrm{BV}([a, b])$. □

Example 5.6 Let $f(x) = x^\alpha \sin\frac{1}{x}$ for $x \in (0, 1]$ and $f(0) = 0$, where $\alpha > 0$ is a constant. It is easy to see $f(x)$ is continuous on $[0, 1]$. Discuss on $[0, 1]$, for which value of α, we have $f \in \mathrm{BV}([0, 1])$.

Proof Case 1: $\alpha \ge 2$. Compute $|f'(x)| = |\alpha x^{\alpha-1}\sin\frac{1}{x} + x^{\alpha-2}\cos\frac{1}{x}| \le \alpha + 1$. Since $f(x)$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$ with $|f'(x)| \le \alpha + 1$ for all $x \in (0, 1)$, by Example 5.5, $f \in \mathrm{BV}([0, 1])$.

Case 2: $\alpha \in (0, 1]$. For all $m \ge 2$, define partition $\Delta_m = \{0, \frac{2}{(2m+1)\pi}, \frac{2}{(2m-1)\pi}, \ldots, \frac{2}{3\pi}, 1\}$. Note that $|f(\frac{2}{(2m+1)\pi}) - f(\frac{2}{(2m-1)\pi})| = \frac{2^\alpha}{(2m+1)^\alpha\pi^\alpha} + \frac{2^\alpha}{(2m-1)^\alpha\pi^\alpha}$ for all $m \ge 2$. Thus,

$$v_{\Delta_m} = \sum_{k=0}^{m}|f(x_{k+1}) - f(x_k)| \ge \frac{2^\alpha}{\pi^\alpha}\sum_{k=1}^{m}\frac{1}{(2k+1)^\alpha}$$

Since $\alpha \in (0, 1]$, the series diverges by comparing it with harmonic series. Thus, $v_{\Delta_m} \to \infty$ as $m \to \infty$, so $V_0^1(f) = \infty$ and f is not of bounded variation.

Case 3: $\alpha \in (1, 2)$. In this case, the first term of $f'(x)$, namely, $\alpha x^{\alpha-1}\sin\frac{1}{x}$ is bounded on $[0, 1]$. We focus on the second term, i.e., $x^{\alpha-2}\cos\frac{1}{x}$. When $x \to 0$, this term can be arbitrarily large. However, we can check the improper integral of it is absolutely convergent, i.e., $(\mathcal{I})\int_0^1 |x^{\alpha-2}\cos\frac{1}{x}|\ dx \le (\mathcal{I})\int_0^1 x^{\alpha-2}\ dx = \lim_{n\to\infty}(\mathcal{R})\int_{1/n}^1 x^{\alpha-2}\ dx = \frac{1}{\alpha-1} < \infty$, where the last equality is by using Fundamental Theorem of Calculus for Riemann integral. Knowing this, it is not hard to show $(\mathcal{I})\int_0^1 |f'(x)|\ dx < \infty$. Notice that for any partition $\Delta = \{0 = x_0, x_1, \ldots, x_n = 1\}$, for all $i \ge 2$, $f'(x)$ is bounded continuous on $[x_{i-1}, x_i]$, hence Riemann integrable, so Fundamental Theorem of Calculus for Riemann integral implies $|f(x_i) - f(x_{i-1})| = (\mathcal{R})|\int_{x_{i-1}}^{x_i} f'(t)\ dt| \le (\mathcal{R})\int_{x_{i-1}}^{x_i} |f'(t)|\ dt$. In addition, $f(x_1) - f(x_0) = \lim_{n\to\infty}[f(x_1) - f(x_0 + \frac{1}{n})] = \lim_{n\to\infty}(\mathcal{R})\int_{1/n}^{x_1} f'(t)\ dt$. Therefore, we have $v_\Delta \le \lim_{n\to\infty}(\mathcal{R})\int_{1/n}^1 |f'(t)|\ dt = (\mathcal{I})\int_0^1 |f'(t)|\ dt$. Take supremum over all Δ, and we obtain $V_0^1(f) \le (\mathcal{I})\int_0^1 |f'(t)|\ dt < \infty$, so $f \in \mathrm{BV}([0,1])$. □

✍ **Exercise 5.2** If $f \in \mathrm{BV}([a,b])$, then f is bounded on $[a,b]$.

Proof For all $x \in [a,b]$, let $\Delta_x = \{a, x, b\}$. Since $v_{\Delta_x} = |f(x) - f(a)| + |f(b) - f(x)| \le V_a^b(f)$ for all $x \in [a,b]$, by triangular inequality, $|f(x)| - |f(a)| + |f(x)| - |f(b)| \le V_a^b(f)$. This impiles that $|f(x)| \le \frac{1}{2}(V_a^b(f) + |f(a)| + |f(b)|) \triangleq M$. Since $f \in \mathrm{BV}([a,b])$, M is a finite number, and this shows f is bounded by M on $[a,b]$. □

✍ **Exercise 5.3** If $f, g \in \mathrm{BV}([a,b])$, then $c_1 f + c_2 g \in \mathrm{BV}([a,b])$ for any constants c_1, c_2 and $f \cdot g \in \mathrm{BV}([a,b])$. Moreover, if $|g(x)| \ge c$ for constant $c > 0$ on $[a,b]$, then $\frac{f}{g} \in \mathrm{BV}([a,b])$.

Proof First we prove $c_1 f + c_2 g \in \mathrm{BV}([a,b])$. For any partition Δ, consider

$$\begin{aligned} v_\Delta(c_1 f + c_2 g) &= \sum_{i=1}^n |c_1 f(x_i) + c_2 g(x_i) - c_1 f(x_{i-1}) - c_2 g(x_{i-1})| \\ &\le \sum_{i=1}^n |c_1||f(x_i) - f(x_{i-1})| + \sum_{i=1}^n |c_2||g(x_i) - g(x_{i-1})| \\ &= |c_1| v_\Delta(f) + |c_2| v_\Delta(g) \le |c_1| V_a^b(f) + |c_2| V_a^b(g) \end{aligned}$$

Take supremum over all partition Δ, we have $V_a^b(c_1 f + c_2 g) \le |c_1| V_a^b(f) + |c_2| V_a^b(g) < \infty$, and this shows $c_1 f + c_2 g \in \mathrm{BV}([a,b])$.

Next we prove $f \cdot g \in \mathrm{BV}([a,b])$. For any partition Δ, consider

$$\begin{aligned} v_\Delta(fg) &= \sum_{i=1}^n |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})| \\ &= \sum_{i=1}^n |f(x_i)g(x_i) - f(x_i)g(x_{i-1}) + f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_{i-1})| \\ &\le \sum_{i=1}^n |f(x_i)||g(x_i) - g(x_{i-1})| + \sum_{i=1}^n |g(x_{i-1})||f(x_i) - f(x_{i-1})| \end{aligned}$$

By Exercise 5.2, $|f(x)| \le M$ and $|g(x)| \le N$ on $[a,b]$ for some constant $M, N > 0$.

$$v_\Delta(fg) \le M \sum_{i=1}^n |g(x_i) - g(x_{i-1})| + N \sum_{i=1}^n |f(x_i) - f(x_{i-1})| \le M V_a^b(g) + N V_a^b(f)$$

Take supremum over all partition Δ, we have $V_a^b(fg) \le M V_a^b(g) + N V_a^b(f) < \infty$, and this shows $fg \in \mathrm{BV}([a,b])$.

Finally, we prove $\frac{f(x)}{g(x)} \in \mathrm{BV}([a,b])$ when $g(x)$ is bounded away from zero. By product case, it suffices to show $\frac{1}{g} \in \mathrm{BV}([a,b])$. For any partition Δ, consider

$$v_\Delta\left(\frac{1}{g}\right) = \sum_{i=1}^{n}\left|\frac{1}{g(x_i)} - \frac{1}{g(x_{i-1})}\right| = \sum_{i=1}^{n}\frac{|g(x_i)-g(x_{i-1})|}{|g(x_i)g(x_{i-1})|} \leq \frac{1}{c^2}v_\Delta(g) \leq \frac{V_a^b(g)}{c^2} < \infty$$

Thus, $\frac{1}{g} \in \mathrm{BV}([a,b])$ and it further implies $\frac{f}{g} \in \mathrm{BV}([a,b])$ by the previous conclusion. $\square$

Exercise 5.4 If $f \in \mathrm{BV}([a,b])$, $V_a^b(f) = P_a^b(f) + N_a^b(f)$ and $f(b) - f(a) = P_a^b(f) - N_a^b(f)$.

Proof By definition, for any partition Δ, we have

$$\begin{aligned} p_\Delta - n_\Delta &= \sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))^+ + \sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))^- \\ &= \sum_{i=1}^{n}(f(x_i) - f(x_{i-1})) = f(b) - f(a) \end{aligned}$$

Thus, $p_\Delta = n_\Delta + f(b) - f(a)$. Take supremum over Δ on both sides, it is easy to see $P_a^b(f) = N_a^b(f) + f(b) - f(a)$. Since it is easy to see $n_\Delta \leq v_\Delta$, $N_a^b(f) \leq V_a^b(f) < \infty$. Thus, by subtracting $N_a^b(f)$ on both sides, we obtain $P_a^b(f) - N_a^b(f) = f(b) - f(a)$. To prove the other equality, consider $v_\Delta = p_\Delta + n_\Delta = 2n_\Delta + f(b) - f(a)$. Take supremum over all Δ on both sides, we have $V_a^b(f) = 2N_a^b(f) + f(b) - f(a)$. Since we have proved $P_a^b(f) - N_a^b(f) = f(b) - f(a)$, by eliminating $f(b) - f(a)$, we have $V_a^b(f) = P_a^b(f) + N_a^b(f)$. $\square$

Exercise 5.5 The functions $V_a^x(f)$, $P_a^x(f)$ and $N_a^x(f)$ are all increasing in x on $[a,b]$.

Proof Consider $a \leq x_1 < x_2 \leq b$, for any partition Δ_{x_1} of $[a, x_1]$, there is a partition Δ_{x_2} of $[a, x_2]$ satisfying $\Delta_{x_2} = \Delta_{x_1} \cup \{x_2\}$. Notice that

$$v_{\Delta_{x_1}} \leq v_{\Delta_{x_1}} + |f(x_2) - f(x_1)| = v_{\Delta_{x_2}} \leq V_a^{x_2}(f)$$

so by taking supremum over all Δ_{x_1}, we have $V_a^{x_1}(f) \leq V_a^{x_2}(f)$. Thus, $V_a^x(f)$ is increasing on $[a,b]$. The other two can be proved in exactly the same way, so we omit the proof. $\square$

Theorem 5.4. Jordan Decomposition for BV Function

A function $f \in \mathrm{BV}([a,b])$ if and only if $f = g - h$ where g and h are real-valued increasing functions on $[a,b]$. ♡

Proof First we prove the "If" part. Since g, h are increasing on $[a,b]$, by Example 5.4, $g, h \in \mathrm{BV}([a,b])$. By Exercise 5.3, $f = g - h \in \mathrm{BV}([a,b])$. Then we prove the "only if" part. Since $f \in \mathrm{BV}([a,b])$, by Exercise 5.5, for all $x \in [a,b]$, $f \in \mathrm{BV}([a,x])$. Thus, by Exercise 5.4, $f(x) = f(a) + P_a^x(f) - N_a^x(f)$. By Exercise 5.5, $P_a^x(f)$ and $N_a^x(f)$ are increasing on $[a,b]$. Let $g(x) = f(a) + P_a^x(f)$ and $h(x) = N_a^x(f)$, then $f(x) = g(x) - h(x)$ where g, h are increasing functions on $[a,b]$. Since $P_a^x(f)$ is increasing, $0 \leq P_a^x(f) \leq P_a^b(f)$. Note that $N_a^b(f) \geq 0$, together with $V_a^b(f) = P_a^b(f) + N_a^b(f)$, we have $P_a^b(f) \leq V_a^b(f) < \infty$, so $g(x)$ is real-valued on $[a, b]$. Similarly, we can prove $h(x)$ is real-valued on $[a, b]$. $\square$

Corollary 5.1

If $f \in \mathrm{BV}([a, b])$, then f is differentiable a.e. on $[a, b]$. ♡

Proof The proof follows directly from Jordan decomposition for function of bounded variations and Lebesgue's theorem for the differentiability of monotone functions. $\square$

Theorem 5.5

Suppose $f \in L^1(a, b)$. Define indefinite integral $F(x) = \int_a^x f(t)\, dt$ for $x \in [a, b]$. Then

(i) F is continuous on $[a, b]$.

(ii) $F \in \mathrm{BV}([a, b])$ and $V_a^b(F) = \int_a^b |f(x)|\, dx$. ♡

Proof We prove the above two parts separately:

(i) Fix $x_0 \in [a, b]$. Let $A = [x_0, x]$ or $[x, x_0]$, then $m(A) \to 0$ as $x \to x_0$. Since $f \in L^1(a, b)$, $|F(x) - F(x_0)| = |\int_A f(t)\, dts| \to 0$ as $m(A) \to 0$ by Problem Set 3.4, Question 8.. Thus, we obtain $|F(x) - F(x_0)| \to 0$ as $x \to x_0$. This shows $F(x)$ is continuous at x_0.

(ii) For any partition $\Delta = \{a = x_0, \ldots, x_{n-1}, x_n = b\}$, compute

$$v_\Delta(F) = \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \le \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(x)|\, dx = \int_a^b |f(x)|\, dx < \infty$$

where the first inequality is due to Exercise 3.16. Take supremum over all Δ, we have $V_a^b(F) \le \int_a^b |f(x)|\, dx$. It remains to show $V_a^b(F) \ge \int_a^b |f(x)|\, dx$.

Let $E^+ = \{x \in (a, b) \,|\, f(x) > 0\}$ and $E^- = \{x \in (a, b) \,|\, f(x) < 0\}$. We define $I(x) = I_{E^+}(x) - I_{E^-}(x)$, then $I(x)f(x) = |f(x)|$ on (a, b). By Theorem 4.8, there exists sequence of step functions $S_n^+(x) \to I_{E^+}(x)$ in $L^1(a, b)$ as $n \to \infty$. According to the proof of Theorem 4.8, we can assume $S_n^+(x)$ only take value 1 or 0 on (a, b). Similarly, there exists sequence of step functions $S_n^-(x) \to I_{E^-}(x)$ in $L^1(a, b)$, where $S_n^-(x) = 0$ or 1 on (a, b). Let $S_n(x) = S_n^+(x) - S_n^-(x)$. Then $S_n(x)$ is a step function and $S_n(x) \in \{1, 0, -1\}$ on (a, b). Furthermore, it is easy to show by Minkowski inequality that $S_n(x) \to I(x)$ in $L^1(a, b)$. Thus, by Theorem 3.7, $S_n(x) \to I(x)$ in measure and so there exists a subsequence $S_{n_p} \to I(x)$ a.e. on (a, b). Since $f \in L^1(a, b)$, by Exercise 3.11, $f(x)$ is finite a.e. on (a, b). This shows for almost all fixed $x \in (a, b)$, $S_{n_p}(x)f(x) \to I(x)f(x)$ as $p \to \infty$, so $S_{n_p}(x)f(x) \to I(x)f(x)$ a.e. on (a, b). Note that $|S_{n_p}(x)f(x)| \le |f(x)|$ where $f \in L^1(a, b)$, so by DCT, $\int_a^b S_{n_p}(x)f(x)\, dx \to \int_a^b I(x)f(x)\, dx = \int_a^b |f(x)|\, dx$. Also, we can assume $S_n(x) = \sum_{j=1}^{k_n} c_{n,j} I_{R_{n,j}}(x)$ where $R_{n,j}$'s are disjoint intervals and $c_{n,j} \in \{0, \pm 1\}$ for each fixed n. Notice that

$$\int_a^b S_n(x)f(x)\, dx = \sum_{j=1}^{k_n} \int_a^b c_{n,j} I_{R_{n,j}}(x)f(x)\, dx \le \sum_{j=1}^{k_n} \left| \int_{R_{n,j}} f(x)\, dx \right|$$

Define partition Δ of $[a, b]$ as the collection of two end points of each interval $R_{n,j}$, then by definition of $F(x)$, for each n,

$$\int_a^b S_n(x)f(x)\,dx = \sum_{j=1}^{k_n} |F(R_{n,j}^r) - F(R_{n,j}^l)| \le v_\Delta(F) \le V_a^b(F)$$

Thus by considering $n = n_p$ and taking $p \to \infty$, $\int_a^b |f(x)|\,dx \le V_a^b(F)$. Combined with the previous result, $V_a^b(F) = \int_a^b |f(x)|\,dx$ and $F \in \mathrm{BV}([a,b])$ because $f \in L^1(a,b)$.

□

Problem Set 5.2

1. Let $\Delta_0 = \{a = x_0, x_1, x_2, x_3, b = x_4\}$. Then if a continuous function $f(x)$ defined on $[a,b]$ is increasing on $[a,x_1]$ and $[x_2,x_3]$, decreasing on $[x_1,x_2]$ and $[x_3,b]$, then $V_a^b(f) = v_{\Delta_0}$.
2. Observe that $v_\Delta \le v_{\Delta_1}$ if Δ_1 is a finer partition of $[a,b]$ than Δ. Use this observation to prove if f is real-valued on $[a,b]$ and $c \in (a,b)$, then $V_a^b(f) = V_a^c(f) + V_c^b(f)$.
3. Find $V_0^{2\pi}(\sin 2x)$ by using Question 2. in this Problem Set.
4. Let $f_k(x) \in \mathrm{BV}([a,b])$ for all $k \ge 1$. Suppose $V_a^b(f_k) \le M$ for all $k \ge 1$, and $f_k \to f$ pointwise on $[a,b]$ as $k \to \infty$. Prove $f \in \mathrm{BV}([a,b])$ and $V_a^b(f) \le M$.
5. Denote $\gamma : [0,1] \mapsto \mathbb{C}$ by $\gamma(t) = x(t) + iy(t)$, where $x(t)$ and $y(t)$ are real-valued continuous functions on $[0,1]$. A curve γ is rectifiable if $V_0^1(\gamma) < \infty$. In this case, the length of γ is defined to be $V_0^1(\gamma)$. Prove that if $x(t)$ and $y(t)$ are continuously differentiable on $[0,1]$, then $V_0^1(\gamma) = \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$.
6. Suppose $f \in \mathrm{BV}([0,1])$. Define $F(x) = \frac{1}{x}\int_0^x f(t)\,dt$ for $x \in (0,1]$ and $F(0) = 2020$. Prove that $F \in \mathrm{BV}([0,1])$ and $\lim_{x\to 0+} F(x)$ exists as a finite number.
7. Let $f(x)$ be real-valued on $[a,b]$, satisfying that for all $\epsilon > 0$, $V_{a+\epsilon}^b(f) \le M$, where M is a constant. Prove that $f \in \mathrm{BV}([a,b])$.

5.3 Fundamental Theorem of Calculus and Absolutely Continuous Function

In this section, we are going to derive a sufficient and necessary condition for Fundamental Theorem of Calculus (FTC) for Lebesgue integrable function. This is a huge extension for the basic FTC for Riemann integral that you should learn in any basic calculus course. FTC for Lebesgue integral consists of two parts. The first part is relatively easier and only requires the following lemma as an extra prerequisite:

Lemma 5.2

If $f \in L^1(a,b)$ and $\int_a^x f(t)\,dt = 0$ for all $x \in [a,b]$, then $f(x) = 0$ a.e. on (a,b).

Proof Let I be an interval s.t. $I \subset [a,b]$, then $\int_I f(x)\,dx = 0$ by using the assumption. Recall Problem 1.1, for any open $G \subset [a,b]$, we can write $G = \bigcup_{k=1}^\infty I_k$, where I_k's are disjoint open intervals. Thus, by Problem Set 3.4, Question 4., $\int_G f(x)\ dx = \sum_{k=1}^{\infty} \int_{I_k} f(x)\ dx = 0$. Let $E^+ = \{x \in (a,b) \mid f(x) > 0\}$ and $E^- = \{x \in (a,b) \mid f(x) < 0\}$.

We want to show $m(E^{\pm}) = 0$. Suppose $m(E^+) > 0$. Since for all $\delta > 0$, there exists closed $F \subset E^+$ s.t. $m(E^+ \setminus F) < \delta$. Take $\delta = \frac{m(E^+)}{100}$, we have $m(E^+) - m(F) < \delta$, so $m(F) > \frac{99}{100} m(E^+) > 0$. However, since $(a,b) \setminus F$ is open,

$$0 = \int_a^b f(x)\ dx = \int_F f(x)\ dx + \int_{(a,b)\setminus F} f(x)\ dx = \int_F f(x)\ dx$$

This shows $\int_F f(x)\ dx = 0$. Notice that $f(x) > 0$ on F with $m(F) > 0$, by using Problem Set 3.1, Question 2., we obtain a contradiction. Therefore, $m(E^+) = 0$. Similarly, we can show $m(E^-) = 0$. This shows that $f(x) = 0$ a.e. on (a,b). □

Theorem 5.6. Fundamental Theorem of Calculus I (FTC-I)

Suppose $f \in L^1(a,b)$ and define $F(x) \triangleq \int_a^x f(t)\ dt$, then $F'(x)$ exists and $F'(x) = f(x)$ a.e. on (a,b). ♡

Proof Special case: Assume f is bounded, i.e., $|f(x)| \le C$ for all $x \in [a,b]$. WLOG, define $f(x) = f(b)$ for all $x > b$. Define $F_n(x) = n(F(x + \frac{1}{n}) - F(x))$ for all $x \in [a,b]$, then $F_n(x) = n \int_x^{x+\frac{1}{n}} f(t)\ dt$. By Theorem 5.5, each $F_n(x)$ and $F(x)$ are continuous with bounded variation on $[a,b]$. By Corollary 5.1, $F(x)$ is differentiable a.e. on (a,b), so $F'(x)$ exists a.e. on (a,b). By definition, $F_n(x) \to F'(x)$ a.e. on (a,b). Note that $|F_n(x)| \le C$ on $[a,b]$ for all $n \ge 1$, so for each $c \in [a,b]$, by DCT,

$$\begin{aligned}
\int_a^c F'(x)\ dx &= \lim_{n\to\infty} \int_a^c F_n(x)\ dx = \lim_{n\to\infty} n \int_a^c \left[F\left(x + \frac{1}{n}\right) - F(x)\right]\ dx \\
&\overset{(\star)}{=} \lim_{n\to\infty} n \left[\int_{a+\frac{1}{n}}^{c+\frac{1}{n}} F(x)\ dx - \int_a^c F(x)\ dx\right] \\
&= \lim_{n\to\infty} n \left[\int_c^{c+\frac{1}{n}} F(x)\ dx - \int_a^{a+\frac{1}{n}} F(x)\ dx\right] \\
&= F(c) - F(a) = F(c) = \int_a^c f(x)\ dx
\end{aligned}$$

where $(\star)$ is due to change of variable for Riemann-integral (we can regard it as Riemann-integral because $F(x)$ is continuous and $[a,b]$ is bounded).

We tend to conclude that $\int_a^c (F'(x) - f(x))\ dx = 0$ for all $c \in [a,b]$ by using Exercise 3.12 but it requires us to show $F' \in L^1(a,b)$. Since $F \in \mathrm{BV}([a,b])$, by Jordan Decomposition, $F = g - h$ where g and h are increasing on $[a,b]$ and $F' = g' - h'$ a.e. on (a,b). By Lebesgue's Theorem, $\int_a^b g'(x)\ dx \le g(b) - g(a) < \infty$, so $g' \in L^1(a,b)$. Similarly, $h' \in L^1(a,b)$, so $F' \in L^1(a,b)$. Therefore, we can conclude $\int_a^c (F'(x) - f(x))\ dx = 0$ for all $c \in [a,b]$. By Lemma 5.2 we just proved, $F'(x) = f(x)$ a.e. on (a,b).

General case: Assume $f \in L^1(a,b)$ only. Consider $f^+ \in L^1(a,b)$ and $-f^- \in L^1(a,b)$, we can write $F(x) = \int_a^x f^+(t)\ dt - \int_a^x (-f^-(t))\ dt$. Thus, if we can prove the desired result for nonnegative $f \in L^1(a,b)$, then we can prove the most general case easily. Now assume $f \in L^1(a,b)$ is nonnegative on $[a,b]$. Define

$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) \le n \\ n & \text{if } f(x) > n \end{cases}$$

then $f_n(x) \le f(x)$ is nonnegative, measurable and bounded. Since $f(x) \ge 0$, $F(x)$ is increasing on $[a,b]$. By Lebesgue's Theorem, for all $c \in [a,b]$, $\int_a^c F'(x)\ dx \le F(c) - F(a) = \int_a^c f(x)\ dx$. Similar to the argument in special case, we have $\int_a^c (F'(x) - f(x))\ dx \le 0$ for all $c \in [a,b]$. Now it suffices to show $F'(x) - f(x) \ge 0$ a.e. on (a,b). Let $F_n(x) = \int_a^x f_n(t)\ dt$, then by special case, $F_n'(x)$ exists and $F_n'(x) = f_n(x)$ a.e. on (a,b). Finally, notice that $F(x) - F_n(x)$ is increasing on $[a,b]$ because $F(x) - F_n(x) = \int_a^x (f(t) - f_n(t))\ dt$ and the integrand is nonnegative. Thus, by Lebesgue's Theorem, it is differentiable and its derivative is nonnegative a.e. on (a,b), i.e., $(F(x) - F_n(x))' = F'(x) - F_n'(x) = F'(x) - f_n(x) \ge 0$ a.e. on (a,b). Since $f_n(x) \to f(x)$ a.e. on (a,b), by taking $n \to \infty$, $F'(x) - f(x) \ge 0$ a.e. on (a,b). This shows $\int_a^c (F'(x) - f(x))\ dx \ge 0$. Combined with the previous result, we actually have $\int_a^c (F'(x) - f(x))\ dx = 0$ for all $c \in [a,b]$. By Lemma 5.2, $F'(x) = f(x)$ a.e. on (a,b). □

Having FTC-I, we can prove the 1-dimensional version of the well-known Lebesgue's Differentiation Theorem easily. You will learn the general version of it in Harmonic Analysis.

Theorem 5.7. Lebesgue's Differentiation Theorem

Suppose $f \in L^1(a,b)$, where (a,b) may be unbounded, e.g. $\mathbb{R}$. We have

1. *For almost all $x \in (a,b)$, $\lim_{h\to 0^+} \frac{1}{|B_h(x)|} \int_{B_h(x)} f(y)\ dy = f(x)$.*
2. *Furthermore, $\lim_{h\to 0^+} \frac{1}{|B_h(x)|} \int_{B_h(x)} |f(y) - f(x)|\ dy = 0$ a.e. on (a,b).*

where $B_h(x)$ is the open ball centered at x with radius h. ♡

Proof

1. Define $E_k = (a,b) \cap \{x \in \mathbb{R} \mid k \le x < k+1\}$ for all $k \in \mathbb{Z}$. Notice that each E_k is a bounded interval, so we denote the two end points of E_k as a_k, b_k ($a_k \le b_k$). If $a_k = b_k$, then ignore such E_k. Let $F_k(x) = \int_{a_k}^x f(t)\ dt$, since $F_k \in L^1(a_k, b_k)$, by FTC-I on (a_k, b_k), for almost all $x \in (a_k, b_k)$,
$$\begin{aligned} \lim_{h\to 0^+} \frac{1}{|B_h(x)|} \int_{B_h(x)} f(y)\ dy &= \lim_{h\to 0^+} \frac{F_k(x+h) - F_k(x-h)}{2h} \\ &= \frac{1}{2} \lim_{h\to 0^+} \left[\frac{F_k(x+h) - F_k(x)}{h} + \frac{F_k(x) - F_k(x-h)}{h} \right] \\ &= \frac{1}{2}(F_k'(x) + F_k'(x)) = F_k'(x) = f(x) \end{aligned}$$
Since the above result holds a.e. on each E_k and $(a,b) = \bigcup_{k=-\infty}^{\infty} E_k$, the desired result holds a.e. on (a,b).
2. For all fixed $r \in \mathbb{Q}$, consider the function $|f(x) - r|$, it is in $L^1(a,b)$. Apply part 1 to

$|f(x) - r|$, for almost all $x \in (a, b)$,

$$\lim_{h \to 0^+} \frac{1}{|B_h(x)|} \int_{B_h(x)} |f(y) - r| \, dy = |f(x) - r| \tag{$\star$}$$

Thus, there exists $E_r \subset (a, b)$ s.t. $m(E_r) = 0$ and $(\star)$ holds for all $x \in (a, b) \setminus E_r$. Let $E = \bigcup_{r \in \mathbb{Q}} E_r$, then it is easy to show $m(E) = 0$ and for all $x \in (a, b) \setminus E$ and $r \in \mathbb{Q}$, $(\star)$ holds. Now fix $x \in (a, b) \setminus E$, for all $\epsilon > 0$, pick $r_x \in \mathbb{Q}$ s.t. $|f(x) - r_x| < \epsilon$, and we have

$$\begin{aligned} \frac{1}{|B_h(x)|} \int_{B_h(x)} |f(y) - f(x)| \, dy &\le \frac{1}{|B_h(x)|} \int_{B_h(x)} (|f(y) - r_x| + |r_x - f(x)|) \, dy \\ &\le \frac{1}{|B_h(x)|} \int_{B_h(x)} |f(y) - r_x| \, dy + \epsilon \end{aligned}$$

Take $\overline{\lim}_{h \to 0^+}$ on both sides,

$$\overline{\lim_{h \to 0^+}} \frac{1}{|B_h(x)|} \int_{B_h(x)} |f(y) - f(x)| \, dy \le |f(x) - r_x| + \epsilon < 2\epsilon$$

Take $\epsilon \to 0$, we obtain the desired result for all $x \in (a, b) \setminus E$, hence a.e. on (a, b).

□

The following theorem is a special case of 1-dimensional Lebesgue's Differentiation Theorem. It tells us that for any measurable set E, almost all points in E is has "density" 1, and the set of points on the boundary of E, which has density less than 1, can be ignored.

Theorem 5.8. Lebesgue's Density Theorem

Suppose $E \subset \mathbb{R}$, $E \in \mathcal{M}$. Then

$$\lim_{h \to 0^+} \frac{m(E \cap B_h(x))}{m(B_h(x))} = \begin{cases} 1 & \text{for almost all } x \in E \\ 0 & \text{for almost all } x \in E^c \end{cases}$$

where $B_h(x)$ is the open ball centered at x with radius h.

Proof For any $x \in \mathbb{R}$, we can write

$$\frac{m(E \cap B_h(x))}{m(B_h(x))} = \frac{1}{|B_h(x)|} \int_{B_h(x)} I_E(y) \, dy$$

Similar to the proof of Lebesgue's Differentiation Theorem, we can define $E_k = E \cap [k, k+1)$ for all $k \in \mathbb{Z}$. Notice that $I_{E_k} \in L^1(k, k+1)$, so we can apply Lebesgue's Differentiation Theorem to conclude

$$\lim_{h \to 0^+} \frac{m(E \cap B_h(x))}{m(B_h(x))} = I_E(x) \tag{$\star$}$$

for almost all $x \in [k, k+1)$. Since $\mathbb{R} = \bigcup_{k=-\infty}^{\infty} [k, k+1)$, $(\star)$ holds for almost all $x \in \mathbb{R}$, and this proves the desired results. □

Next we are going to derive the second part of FTC. However, before that, we will first introduce an essential concept, that is, absolutely continuous function and derive some properties of it. These results would be helpful for us to prove the second part of FTC.

Definition 5.6

*Let $f(x)$ be real-valued on $[a,b]$. The function f is **absolutely continuous** on $[a,b]$ if for all $\epsilon > 0$, there exists $\delta > 0$ s.t. for any finite collection of disjoint open intervals $(x_1,y_1),\ldots,(x_n,y_n)$ contained in (a,b) satisfying $\sum_{i=1}^n (y_i - x_i) < \delta$, we have $\sum_{i=1}^n |f(y_i) - f(x_i)| < \epsilon$. In addition, denote $f \in \mathrm{AC}([a,b])$ if and only if f is **absolutely continuous** on $[a,b]$.* ♣

Problem 5.1 If $f \in \mathrm{AC}([a,b])$, then f is uniformly continuous on $[a,b]$.

Problem 5.2 If f is Lipschitz continuous on $[a,b]$, then $f \in \mathrm{AC}([a,b])$.

Exercise 5.6 If $f \in \mathrm{AC}([a,b])$, and $f'(x) = 0$ a.e. on $[a,b]$, then $f(x) = c$ on $[a,b]$ for some constant c.

Proof We prove by contradiction, and it suffices to show if $f'(x) = 0$ a.e. on $[a,b]$ and f is not a constant, then there exists $\epsilon_0 > 0$ s.t. for all $\delta > 0$, there exists a finite collection of disjoint open intervals $(x_1,y_1),\ldots,(x_n,y_n)$ containted in (a,b) satisfying $\sum_{i=1}^n (y_i - x_i) < \delta$ and $\sum_{i=1}^n |f(y_i) - f(x_i)| \geq \epsilon_0$.

Pick $c \in (a,b]$ s.t. $f(c) \neq f(a)$. Let $E_c = \{x \in (a,c) \mid f'(x) = 0\}$, then $m(E_c) = c - a$. Fix $r > 0$, for all $x \in E_c$, since $f'(x) = 0$, there exists small interval $[x, x+h_x] \subset (a,c)$ s.t. $|f(x+h_x) - f(x)| < rh_x$, Now consider $\Gamma = \{[x,x+h] \mid x \in E_c,\ 0 < h \leq h_x\}$, we can easily see it is a Vitali covering of E_c. By Vitali Covering Theorem, for all $\delta > 0$, there exists a finite collection of disjoint intervals $\{[x_i, x_i + h_i]\}_{i=1}^m \subset \Gamma$ s.t. $m(E_c \setminus \bigcup_{i=1}^m [x_i, x_i+h_i]) < \delta$. This implies $m((a,c) \setminus \bigcup_{i=1}^m [x_i, x_i + h_i]) < \delta$. Let $x_0 = a$ and $x_{m+1} = c$, then WLOG, assume $x_0 < x_1 < x_1 + h_1 < \cdots < x_m < x_m + h_m < x_{m+1}$. Let $h_0 = 0$, then

$$\begin{aligned}
|f(c) - f(a)| &\leq \sum_{i=0}^m |f(x_{i+1}) - f(x_i + h_i)| + \sum_{i=1}^m |f(x_i + h_i) - f(x_i)| \\
&\leq \sum_{i=0}^m |f(x_{i+1}) - f(x_i + h_i)| + r\sum_{i=1}^m h_i \\
&\leq \sum_{i=0}^m |f(x_{i+1}) - f(x_i + h_i)| + r(b-a)
\end{aligned}$$

Take $\epsilon_0 = \frac{1}{2}|f(c) - f(a)|$ and $r = \frac{\epsilon_0}{b-a}$, we have $\sum_{i=0}^m |f(x_{i+1}) - f(x_i + h_i)| \geq \epsilon_0$. Now let $y_{i+1} = x_{i+1}$ and $z_{i+1} = x_i + h_i$ for all $i = 0,\ldots,m$. It is easy to see $\{(z_i, y_i)\}_{i=1}^{m+1}$ is a finite collection of disjoint open intervals satisfying $\sum_{i=1}^{m+1} (y_i - z_i) = m((a,c) \setminus \bigcup_{i=1}^m [x_i, x_i + h_i]) < \delta$ and $\sum_{i=1}^{m+1} |f(y_i) - f(z_i)| \geq \epsilon_0$, so the desired result holds. □

Remark This may be the most intuitive reason for introducing a new concept of absolutely continuous function. Recall in Lebesgue's Theorem, we have $\int_a^x f'(t)\,dt \leq f(b) - f(a)$. If the equality holds, then we obtain FTC. However, the strict inequality may holds if $f'(x) = 0$ a.e.

on (a, b) cannot imply $f(x)$ is a constant. Let $f(x)$ be Cantor function, then $f'(x) = 0$ a.e. on $(0, 1)$ but $0 = \int_a^x f'(t)\, dt < f(b) - f(a) = 1$. Thus, FTC does not hold for Cantor function. On the contrary, if we impose absolutely continuity, then such a strange case will be eliminated (we will see this later).

Exercise 5.7 Let $f \in L^1(a, b)$ and $F(x) = \int_a^x f(t)\, dt$, then $F \in \mathrm{AC}([a, b])$.

Proof Recall Problem Set 3.4, Question 8., for all $\epsilon > 0$, there exists $\delta > 0$ s.t. for any subset $e \subset (a, b)$ if $m(e) < \delta$, $\int_e |f(x)|\, dx < \epsilon$. Now consider any finite collection of disjoint open intervals $\{(x_i, y_i)\}_{i=1}^n$ contained in (a, b) satisfying $\sum_{i=1}^n (y_i - x_i) < \delta$, let $e = \bigcup_{i=1}^n (x_i, y_i)$, then since $m(e) = \sum_{i=1}^n (y_i - x_i) < \delta$, $\int_{\bigcup_{i=1}^n (x_i, y_i)} |f(x)|\, dx < \epsilon$. Notice that

$$\sum_{i=1}^n |F(y_i) - F(x_i)| = \sum_{i=1}^n \left| \int_{x_i}^{y_i} f(t)\, dt \right| \leq \sum_{i=1}^n \int_{x_i}^{y_i} |f(t)|\, dt = \int_{\bigcup_{i=1}^n (x_i, y_i)} |f(x)|\, dx < \epsilon$$

Thus, $F \in \mathrm{AC}([a, b])$. □

Exercise 5.8 If $f \in \mathrm{AC}([a, b])$ and $g \in \mathrm{AC}([a, b])$, then $c_1 f + c_2 g \in \mathrm{AC}([a, b])$, where c_1, c_2 are two constants. Furthermore, $fg \in \mathrm{AC}([a, b])$.

Proof By assumption, for all $\epsilon > 0$, there exists $\delta > 0$ s.t. for any finite collection of disjoint open intervals $\{(x_i, y_i)\}_{i=1}^n$ contained in (a, b), if $\sum_{i=1}^n (y_i - x_i) < \delta$, then $\sum_{i=1}^n |f(y_i) - f(x_i)| < \epsilon$ and $\sum_{i=1}^n |g(y_i) - g(x_i)| < \epsilon$. Let $h = c_1 f + c_2 g$, by triangular inequality,

$$\sum_{i=1}^n |h(y_i) - h(x_i)| \leq |c_1| \sum_{i=1}^n |f(y_i) - f(x_i)| + |c_2| \sum_{i=1}^n |g(y_i) - g(x_i)| < (|c_1| + |c_2|)\epsilon$$

This implies that $h \in \mathrm{AC}([a, b])$. Furthermore, by Problem 5.1, f and g are uniformly continuous on $[a, b]$, hence they are bounded on $[a, b]$, i.e., $|f(x)| \leq M$ and $|g(x)| \leq N$ on $[a, b]$ for some constant $M, N > 0$. Let $\phi = fg$, then

$$\begin{aligned}\sum_{i=1}^n |\phi(y_i) - \phi(x_i)| &\leq \sum_{i=1}^n |f(y_i)g(y_i) - f(y_i)g(x_i) + f(y_i)g(x_i) - f(x_i)g(x_i)| \\ &\leq \sum_{i=1}^n |f(y_i)||g(y_i) - g(x_i)| + \sum_{i=1}^n |g(x_i)||f(y_i) - f(x_i)| \\ &\leq M \sum_{i=1}^n |g(y_i) - g(x_i)| + N \sum_{i=1}^n |f(y_i) - f(x_i)| < (M + N)\epsilon\end{aligned}$$

Thus, $fg \in \mathrm{AC}([a, b])$. □

Exercise 5.9 If $f \in \mathrm{AC}([a, b])$, then $f \in \mathrm{BV}([a, b])$.

Proof By assumption, there exists $\delta > 0$ s.t. for any finite collection of disjoint open intervals $\{(x_i, y_i)\}_{i=1}^n$ contained in (a, b), if $\sum_{i=1}^n (y_i - x_i) < \delta$, then $\sum_{i=1}^n |f(y_i) - f(x_i)| < 1$. Choose $N \geq 1$ s.t. $\frac{b-a}{N} < \delta$. Define a partition of $[a, b]$ by $\Delta_0 = \{a = x_0, x_1, \ldots, x_N = b\}$. Let Δ be any partition of $[a, b]$, and define $\Delta_1 = \Delta_0 \cup \Delta = \{z_0, z_1, \ldots, z_K\}$. By Problem Set 5.2, Question 2., since Δ_1 is finer than Δ, we have $v_\Delta \le v_{\Delta_1}$. However,

$$v_{\Delta_1} = \sum_{k=1}^{K} |f(z_k) - f(z_{k-1})| = \sum_{i=1}^{N} \sum_{(z_{k-1},z_k)\subset(x_{i-1},x_i)} |f(z_k) - f(z_{k-1})| < N$$

This shows for any partition Δ, $v_\Delta < N$. Thus, by taking the supremum over all partition Δ on both sides, $V_a^b(f) \le N$, which shows $f \in \mathrm{BV}([a,b])$. $\square$

Theorem 5.9. Fundamental Theorem of Calculus II (FTC-II)

If $f \in \mathrm{AC}([a,b])$, then $f'(x)$ exists a.e. on (a,b) and $f' \in L^1(a,b)$. Furthermore, $f(x) - f(a) = \int_a^x f'(t)\, dt$ for all $x \in [a,b]$.

Proof By Exercise 5.9, $f \in \mathrm{BV}([a,b])$, so by Corollary 5.1, $f'(x)$ exists a.e. on (a,b). Moreover, by Jordan Decomposition, $f = g - h$ for some increasing function g and h. By Lebesgue's Theorem, $0 \le \int_a^b g'(x)\, dx \le g(b) - g(a)$, so $g' \in L^1(a,b)$. Similarly, $h' \in L^1(a,b)$, thus $f' \in L^1(a,b)$. Define $\tilde{f}(x) \triangleq f(a) + \int_a^x f'(t)\, dt$, we want to show $\tilde{f}(x) = f(x)$ on $[a,b]$. By FTC-I, $\tilde{f}'(x) = f'(x)$ a.e. on (a,b), so $(\tilde{f} - f)'(x) = 0$ a.e. on $[a,b]$. By Exercise 5.7, $\tilde{f} \in \mathrm{AC}([a,c])$. By Exercise 5.8, $\tilde{f} - f \in \mathrm{AC}([a,b])$. By Exercise 5.6, $\tilde{f} - f$ is a constant on $[a,b]$. However, it is obvious that $\tilde{f}(a) - f(a) = 0$, so $\tilde{f}(x) = f(x)$ on $[a,b]$. $\square$

Remark The converse of FTC-II is also true, i.e., if $f(x) - f(a) = \int_a^x g(t)\ dt$ for some $g \in L^1(a,b)$, then $f \in \mathrm{AC}([a,b])$ and $f'(x) = g(x)$ a.e. on (a,b).

A direct application of FTC-II is called "integration by parts", a famous and classical technique that is widely used in Riemann integration.

Theorem 5.10. Integration by Parts

Suppose $f, g \in \mathrm{AC}([a,b])$, then $\int_a^b f'(x)g(x)\ dx = f(x)g(x)\big|_a^b - \int_a^b f(x)g'(x)\ dx$.

Proof Since $f, g \in \mathrm{AC}([a,b])$, by Exercise 5.8, $fg \in \mathrm{AC}([a,b])$. By FTC-II, $f(x)$, $g(x)$, and $f(x)g(x)$ are all differentiable a.e. on (a,b). By FTC-II, $\int_a^b (f(x)g(x))'\ dx = f(x)g(x)\big|_a^b$. By product rule for differentiation, $(f(x)g(x))' = f'(x)g(x) + g'(x)f(x)$ a.e. on (a,b), so $\int_a^b (f(x)g(x))'\ dx = \int_a^b f'(x)g(x)\ dx + \int_a^b g'(x)f(x)\ dx$. To be rigorous, we need to argue $f'(x)g(x)$ and $g'(x)f(x)$ are in $L^1(a,b)$. This is because FTC-II guarantees $f', g' \in L^1(a,b)$ and Exercise 5.1 guarantees f, g are bounded on $[a,b]$. $\square$

Our final task for this section is to connect measurablity preserving property with absolute continuity. This is because in general, verifying a function is absolutely continuous is quite hard. After connecting them, we can derive some user-friendly criteria for checking absolute continuity.

Exercise 5.10 If $f \in \mathrm{AC}([a,b])$ with $E \subset [a,b]$ and $m(E) = 0$, then $m(f(E)) = 0$.

Proof By assumption, for all $\epsilon > 0$, there exists $\delta > 0$ s.t. if any finite collection of open intervals $\{(x_i, y_i)\}_{i=1}^n$ contained in (a, b) satisfies $\sum_{i=1}^n (y_i - x_i) < \delta$, then $\sum_{i=1}^n |f(y_i) - f(x_i)| < \epsilon$. Since $m(E) = 0$, there exists open set $G \subset (a, b)$ s.t. $G \supset E$ and $m(G) < \delta$. By Problem 1.1, write $G = \bigcup_{k=1}^{\infty}(a_k, b_k)$ where $\{(a_k, b_k)\}_{k=1}^{\infty}$ is a collection of disjoint open intervals. Thus, $f(E) \subset f(G) \subset \bigcup_{k=1}^{\infty} f([a_k, b_k])$. Let $m_k = \min_{[a_k,b_k]} f(x)$ and $M_k = \max_{[a_k,b_k]} f(x)$. Also let $c_k = \arg\min_{[a_k,b_k]} f(x)$ and $d_k = \arg\max_{[a_k,b_k]} f(x)$. Therefore,

$$m^*(f(E)) \leq \sum_{k=1}^{\infty}(M_k - m_k) = \sum_{k=1}^{\infty}(f(d_k) - f(c_k))$$

For any fixed N, $\sum_{k=1}^{N}(d_k - c_k) \leq \sum_{k=1}^{\infty}(b_k - a_k) = m(G) < \delta$, so by using the definition of absolute continuity, $\sum_{k=1}^{N}(f(d_k) - f(c_k)) < \epsilon$. Take $N \to \infty$, $\sum_{k=1}^{\infty}(f(d_k) - f(c_k)) < \epsilon$. Take $\epsilon \to 0$, we have $m^*(f(E)) = 0$. □

Theorem 5.11

*Suppose $f(x)$ is continuous on $[a, b]$. Then f is **measurability preserving**, i.e., $E \in \mathcal{M}$ implies $f(E) \in \mathcal{M}$, if and only if $m(E) = 0$ implies $m(f(E)) = 0$.* ♡

Proof For "if" part, use the same argument as the second paragraph in the proof of Theorem 1.5. For "only if" part, suppose there exists $E \subset [a, b]$ s.t. $m(E) = 0$ but $m(f(E)) > 0$. Then by the remark right after Theorem 1.3, there exists $S \subset f(E)$ with $S \notin \mathcal{M}$. However, $f^{-1}(S) \cap E \in \mathcal{M}$ because $m(E) = 0$ implies $m(f^{-1}(S) \cap E) = 0$. By assumption, f is measurability preserving, so $f(f^{-1}(S) \cap E) = S$ is measurable. This is a contradiction, so the desired property is proved. □

Corollary 5.2

If $f \in \mathrm{AC}([a, b])$, then f is measurability preserving. ♡

Now we display our main theorem for connecting absolute continuity and measurability preserving. However, since its proof is too complicated, we will not prove it immediately; instead, we shall verify it progressively by first introducing two useful lemmas.

Theorem 5.12

Suppose $f(x)$ is continuous on $[a, b]$ and $f'(x)$ exists a.e. on (a, b). Furthermore, $f' \in L^1(a, b)$ and for any subset $E \subset [a, b]$, $m(E) = 0$ implies $m(f(E)) = 0$. Then $f \in \mathrm{AC}([a.b])$ and FTC-II holds for all $x \in [a, b]$. ♡

Lemma 5.3

Suppose $f(x)$ is measurable on $[a, b]$, $E \subset (a, b)$ and $E \in \mathcal{M}$. Furthermore, $f'(x)$ exists and $|f'(x)| \leq c$ for some constant c on $[a, b]$. Then $m^*(f(E)) \leq cm(E)$. ♡

Proof For all $\epsilon > 0$, there exists open G s.t. $E \subset G \subset (a,b)$ and $m(G) < m(E) + \epsilon$. By assumption, for all $x \in E$, $|f'(x)| < c + \epsilon$. Thus, there exists small enough $h_x > 0$ s.t. $|f(y) - f(x)| < (c+\epsilon)|y-x|$ for all $y \neq x$ and $y \in [x - h_x, x + h_x] \subset G$. This implies $|f(y) - f(x)| < (c+\epsilon)h$ for all $y \in [x-h, x+h]$ with any $0 < h \leq h_x$. This further implies $f(y) \in (f(x) - (c+\epsilon)h, f(x) + (c+\epsilon)h)$, so $f([x-h, x+h]) \subset (f(x)-(c+\epsilon)h, f(x)+(c+\epsilon)h)$ for any $0 < h \leq h_x$. Let $\Gamma = \{[f(x) - (c+\epsilon)h, f(x) + (c+\epsilon)h] \,|\, x \in E,\ 0 < h \leq h_x\}$. It is easy to see Γ is a Vitali covering of $f(E)$. By the second remark following Theorem 5.1, there exists a countable collection of disjoint intervals $\{[f(x_i) - (c+\epsilon)h_i, f(x_i) + (c+\epsilon)h_i]\}_{i=1}^{\infty} \subset \Gamma$ s.t. $m^*(f(E) \setminus \bigcup_{i=1}^{\infty}[f(x_i) - (c+\epsilon)h_i, f(x_i) + (c+\epsilon)h_i]) = 0$. Therefore,

$$\begin{aligned} m^*(f(E)) &\leq \sum_{i=1}^{\infty} m([f(x_i) - (c+\epsilon)h_i, f(x_i) + (c+\epsilon)h_i]) \\ &= \sum_{i=1}^{\infty} 2(c+\epsilon)h_i = (c+\epsilon)\sum_{i=1}^{\infty} m([x_i - h_i, x_i + h_i]) \end{aligned}$$

Note that $\{[x_i - h_i, x_i + h_i]\}_{i=1}^{\infty}$ should be a collection of disjoint intervals. If not, then $f([x_i - h_i, x_i + h_i]) \cap f([x_j - h_j, x_j + h_j]) \neq \varnothing$. However, this is a contradiction because $f([x_i - h_i, x_i + h_i]) \subset (f(x_i) - (c+\epsilon)h_i, f(x_i) + (c+\epsilon)h_i)$ for any $i \in \mathbb{N}^+$ and we pick $\{[f(x_i) - (c+\epsilon)h_i, f(x_i) + (c+\epsilon)h_i]\}_{i=1}^{\infty}$ to be a collection of disjoint intervals. Thus,

$$m^*(f(E)) \leq (c+\epsilon)\sum_{i=1}^{\infty} m([x_i - h_i, x_i + h_i]) = (c+\epsilon) m\left(\bigcup_{i=1}^{\infty}[x_i - h_i, x_i + h_i]\right)$$

This shows $m^*(f(E)) \leq (c+\epsilon)m(G) < (c+\epsilon)(m(E)+\epsilon)$ for all $\epsilon > 0$. By taking $\epsilon \to 0$ and using $m(E) < \infty$, $m^*(f(E)) \leq c\,m(E)$. $\square$

Lemma 5.4

Suppose $f(x)$ is measurable on $[a,b]$ and $f'(x)$ exists for all $x \in E \subset (a,b)$ with $E \in \mathcal{M}$. Then $m^*(f(E)) \leq \int_E |f'(x)|\, dx$. ♡

Proof Let $0 = y_0 < y_1 < \cdots < y_n < \cdots$ with $y_n \to \infty$ as $n \to \infty$. Assume $y_{n+1} - y_n < \delta$ for all $n \geq 0$. Also, define $E_n = \{x \in E \,|\, y_{n-1} \leq |f'(x)| < y_n\}$ for all $n \geq 1$. Notice that $E = \bigcup_{n=1}^{\infty} E_n$ implies $f(E) = \bigcup_{n=1}^{\infty} f(E_n)$, so $m^*(f(E)) \leq \sum_{n=1}^{\infty} m^*(f(E_n))$. By Lemma 5.3, $m^*(f(E_n)) \leq y_n m(E_n)$. Thus, $m^*(f(E)) \leq \sum_{n=1}^{\infty} y_n m(E_n)$ for any $\delta > 0$. Recall in Example 3.2, we have $\lim_{\delta \to 0} \sum_{i=1}^{\infty} y_n m(E_n) = \int_E |f'(x)|\, dx$. Therefore, by taking $\delta \to 0$ on both sides, $m^*(f(E)) \leq \int_E |f'(x)|\, dx$. $\square$

After so much preparation, we can finally prove our main theorem on absolute continuity and measurability preserving. In addition, several corollaries following the main theorem will give some user-friendly versions of it.

Proof [Theorem 5.12] Consider any finite collection of disjoint open intervals $\{(x_i, y_i)\}_{i=1}^{n}$ contained in (a,b), define $E_k = \{x \in [x_k, y_k] \,|\, f'(x) \text{ exists as a finite number}\}$. Since $f'(x)$

exists a.e. on (a, b), $m([x_k, y_k] \setminus E_k) = 0$. By assumption, $m(f([x_k, y_k] \setminus E_k)) = 0$, so

$$m^*(f(E_k)) \leq m^*(f([x_k, y_k])) \leq m^*(f(E_k)) + m^*(f([x_k, y_k] \setminus E_k)) = m^*(f(E_k))$$

Thus, $m^*(f(E_k)) = m^*(f([x_k, y_k]))$. Now by intermediate value property of continuous function, $|f(y_k) - f(x_k)| \leq m^*(f([x_k, y_k]))$. By Lemma 5.4, $m^*(f(E_k)) \leq \int_{E_k} |f'(x)|\, dx$. Therefore, $|f(y_k) - f(x_k)| \leq \int_{E_k} |f'(x)|\, dx$ for $k = 1, \ldots, n$. By summing over k on both sides and since E_k's are almost disjoint,

$$\sum_{k=1}^{n} |f(y_k) - f(x_k)| \leq \sum_{k=1}^{n} \int_{E_k} |f'(x)|\, dx = \int_{\bigcup_{k=1}^{n} E_k} |f'(x)|\, dx \leq \int_{\bigcup_{k=1}^{n} [x_k, y_k]} |f'(x)|\, dx$$

Since $f' \in L^1(a, b)$, by Problem Set 3.4, Question 8., for all $\epsilon > 0$, there exists $\delta > 0$ s.t. for any subset $e \subset [a, b]$ and $e \in \mathcal{M}$, if $m(e) < \delta$, then $\int_e |f'(x)|\, dx < \epsilon$. Let $e = \bigcup_{k=1}^{n} [x_k, y_k]$, then if $\sum_{k=1}^{n} (y_k - x_k) < \delta$, $\sum_{k=1}^{n} |f(y_k) - f(x_k)| \leq \int_{\bigcup_{k=1}^{n} [x_k, y_k]} |f'(x)|\, dx < \epsilon$. ☐

Corollary 5.3

Suppose $f(x)$ is continuous on $[a, b]$ and $f'(x)$ exists a.e. on (a, b) with $f' \in L^1(a, b)$. Then $f \in \mathrm{AC}([a, b])$ if and only if $m(f(E)) = 0$ for any subset $E \subset (a, b)$ with $m(E) = 0$. ♡

Proof The "if" and "only if" parts follow from Theorem 5.12 and Exercise 5.10. ☐

Corollary 5.4

Suppose $f(x)$ is continuous on $[a, b]$ and $f \in \mathrm{BV}([a, b])$. Moreover, $m(f(E)) = 0$ for any subset $E \subset (a, b)$ with $m(E) = 0$. Then $f \in \mathrm{AC}([a, b])$. ♡

Proof If $f \in \mathrm{BV}([a, b])$, by Corollary 5.1, $f'(x)$ exists a.e. on (a, b). To show $f' \in L^1(a, b)$, we can use exactly the same argument as in the second paragraph in the proof of FTC-I. Then Theorem 5.12 gives the desired result. ☐

Corollary 5.5

Suppose $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) with $f' \in L^1(a, b)$. Then $f \in \mathrm{AC}([a, b])$ and FTC-II holds for all $x \in [a, b]$. ♡

Proof Since $f(x)$ is differentiable on $E \subset (a, b)$, by Lemma 5.4, $m^*(f(E)) \leq \int_E |f'(x)|\, dx$. If $m(E) = 0$, then $\int_E |f'(x)|\, dx = 0$, so $m^*(f(E)) = 0$. Therefore, we can apply Theorem 5.12 to conclude the desired result. ☐

Example 5.7 Let $f(x) = x^\alpha \sin \frac{1}{x}$ for $x \in (0, 1]$ and $f(0) = 0$. Then $f \in \mathrm{AC}([0, 1])$ if $\alpha > 1$.

Proof It is easy to see $f(x)$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$. To see $f' \in L^1(0, 1)$, consider $f'(x) = \alpha x^{\alpha-1} \sin \frac{1}{x} - x^{\alpha-2} \cos \frac{1}{x}$. Notice that $|x^{\alpha-1} \sin \frac{1}{x}| \leq 1$ on $[0, 1]$, so $x^{\alpha-1} \sin \frac{1}{x} \in L^1(0, 1)$. Also, $|x^{\alpha-2} \cos \frac{1}{x}| \leq x^{\alpha-2} \in L^1(0, 1)$. Thus, $f' \in L^1(0, 1)$. By Corollary 5.5, $f \in \mathrm{AC}([0, 1])$. ☐

Note *In the following problem set, all (a,b) or $[a,b]$ are assumed to be bounded intervals.*

Problem Set 5.3

1. Let $f(x)$ be continuous and increasing on $[a,b]$. Prove $f \in \mathrm{AC}([a,b])$ if and only if for all $\epsilon > 0$, there exists $\delta > 0$ s.t. whenever $E \subset (a,b)$, $E \in \mathcal{M}$, $m(E) < \delta$, we have $m^*(f(E)) < \epsilon$.
2. Let $f \in L^1(a,b)$ and $\int_a^b x^n f(x)\,dx = 0$ for all $n \geq 0$. Prove that $f(x) = 0$ a.e. on $[a,b]$.
3. Let f be increasing on $[a,b]$, satisfying $\int_a^b f'(x)\,dx = f(b) - f(a)$. Prove that f is absolutely continuous on $[a,b]$.
4. Suppose f is differentiable on $\mathbb{R}$ and $f, f' \in L^1(\mathbb{R})$. Prove that $\int_{\mathbb{R}} f'(x)\,dx = 0$.
5. Let $f_k(x)$ be increasing and absolutely continuous on $[a,b]$ for all $k \geq 1$. Suppose $\sum_{k=1}^{\infty} f_k(x)$ converges pointwise on $[a,b]$. Prove that $\sum_{k=1}^{\infty} f_k(x)$ is absolutely continuous on $[a,b]$.
6. Let $E \in \mathcal{M}$ be a subset of $[0,1]$ s.t. $\exists$ constant $\alpha > 0$ satisfying $m(E \cap [a,b]) \geq \alpha(b-a)$ for all $0 \leq a < b \leq 1$. Prove that $m(E) = 1$.
7. Let f be continuous on $[a,b]$ and differentiable at every $x \in (a,b) \setminus S$, where S is at most countable. Suppose $f'(x) \in L^1(a,b)$. Prove that
$$f(x) = f(a) + \int_a^x f'(t)\,dt, \quad \forall\, x \in [a,b] \tag{1}$$
8. Suppose $f \in \mathrm{AC}([a,b])$ and $f(0) = 0$. Prove that
$$\int_0^1 |f(x)f'(x)|\,dx \leq \frac{1}{\sqrt{2}} \int_0^1 (f'(x))^2\,dx$$
9. Let $\{g_k\}_{k=1}^{\infty} \subset \mathrm{AC}([a,b])$. Assume
   - $|g_k'(x)| \leq F(x)$ a.e. on (a,b) for all $k \geq 1$, where $F \in L^1(a,b)$.
   - there exists $c \in [a,b]$ s.t. $\lim_{k\to\infty} g_k(c)$ exists as a finite number.
   - $\lim_{k\to\infty} g_k'(x)$ exists and equal to some finite $f(x)$ a.e. on (a,b).

   Prove

   (a). $\lim_{k\to\infty} g_k(x)$ exists and equal to some finite $g(x)$ for every $x \in [a,b]$.

   (b). Show $g \in \mathrm{AC}([a,b])$ and $g' = f$ a.e. on (a,b).
10. Let $f \in \mathrm{BV}([a,b])$. Define $v(x) = V_a^x(f)$. Prove that $f \in \mathrm{AC}([a,b])$ if and only if $v \in \mathrm{AC}([a,b])$.

5.4 Change of Variables

In this section, we are going to derive another useful technique that has already been widely used in the Riemann integral, that is, the change of variables technique, in the context of Lebesgue integral. After that, we will also introduce some user-friendly versions of it and illustrate how to use it by some concrete examples.

Throughout this section, we are going to use the notation and assumption below. Let $G \subset \mathbb{R}^n$ and $\phi : G \mapsto \mathbb{R}^n$, i.e., $x = (x_1, \ldots, x_n)$ and $\phi(x) = (\phi_1(x), \ldots, \phi_n(x))$. Suppose ϕ is $\mathcal{C}^1$-smooth (continuously differentiable) and injective. Denote its Jacobian matrix as

$$d\phi(x) = \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_n}{\partial x_1} & \cdots & \frac{\partial \phi_n}{\partial x_n} \end{bmatrix} = \frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)}$$

We assume $d\phi(x)$ is nonsingular at every $x \in G$, i.e., $\det(d\phi(x)) \neq 0$ for all $x \in G$. By Inverse Function Theorem, $D \triangleq \phi(G)$ is open and $\phi^{-1} : D \mapsto G$ is also $\mathcal{C}^1$-smooth.

We first recall the baby version (for Riemman integral) of change of variables technique, which should be learnt in any calculus or mathematical analysis course. In addition to the above assumption, we also assume

- $d\phi(x)$ is bounded on G;
- $m(\partial G) = 0 = m(\partial D)$ where ∂S denotes the boundary of S for any set S;
- $f : D \mapsto \mathbb{R}$ is continuous and bounded;
- G is bounded

Then we conclude that $(\mathcal{R}) \int_D f(y)\, dy = (\mathcal{R}) \int_G f(\phi(x)) \det(d\phi(x))\, dx$ where $y = \phi(x)$.

From the baby version of change of variables technique, we can see that the situations for using such technique are usually:

1. $f(y)$ is too complicated but $f(\phi(x)) \det(d\phi(x))$ has a nicer form;
2. D is too compliacted but G has a nicer form.

Next we are going to introduce the statement of grown-up version (for Lebesgue integral) of change of variables technique. However, we will not give a proof of it until we derive an important lemma.

Theorem 5.13. Change of Variables

Assume the conditions in the second paragraph of this section hold. In addition, suppose $f(y) \in L^1(D)$. Then $f(\phi(x)) \det(d\phi(x)) \in L^1(G)$ and

$$(\mathcal{L}) \int_D f(y)\, dy = (\mathcal{L}) \int_G f(\phi(x)) \det(d\phi(x))\, dx$$

♡

Lemma 5.5

For any $E \subset D$ with $E \in \mathcal{M}$, we have $\phi^{-1}(E) \in \mathcal{M}$. If $E \subset D$ with $m(E) = 0$, then $m(\phi^{-1}(E)) = 0$.

♡

Proof For all $k \geq 1$, let $D_k = \{x \in D \,|\, \|x\|_2 < k, \operatorname{dist}(x, \partial D) > \frac{1}{k}\}$ where the distance function $\operatorname{dist}(x, \partial D) \triangleq \inf_{z \in \partial D} \|z - x\|_2$. It is a standard exercise to check D_k is open.

Also, the closure $\overline{D_k} \subset D$ is compact, and $D = \bigcup_{k=1}^{\infty} D_k$. Then $E = \bigcup_{k=1}^{\infty}(D_k \cap E)$ and $\phi^{-1}(E) = \bigcup_{k=1}^{\infty} \phi^{-1}(D_k \cap E)$. For all $x \in \overline{D_k} \subset D$, there exists closed ball $\overline{B_x} \subset D$ with x as the center of it. Thus, $\{B_x\}_{x \in \overline{D_k}}$ is an open cover of $\overline{D_k}$. Since $\overline{D_k}$ is compact, there exists a finite collection $\{B_{x_i}\}_{i=1}^{m} \subset \{B_x\}_{x \in \overline{D_k}}$ s.t. $\bigcup_{i=1}^{m} B_{x_i} \supset \overline{D_k}$. Then, $E \cap D_k \subset \bigcup_{i=1}^{m}(B_{x_i} \cap E)$ implies $\phi^{-1}(E \cap D_k) = \bigcup_{i=1}^{m} \phi^{-1}(B_{x_i} \cap E \cap D_k)$. Consider $\phi^{-1} : \overline{B_{x_i}} \mapsto \mathbb{R}^n$ is $\mathcal{C}^1$-smooth, ϕ^{-1} is Lipschitz continuous on compact set $\overline{B_{x_i}}$. By (slightly modifying the domain of function) Theorem 1.5, $\phi^{-1}(B_{x_i} \cap E \cap D_k) \in \mathcal{M}$. Therefore, $\phi^{-1}(E \cap D_k) \in \mathcal{M}$ and $\phi^{-1}(E) \in \mathcal{M}$. Since ϕ^{-1} is continuous and measurability preserving, by using the same argument (except that the domain is different) as in Theorem 5.11, for any $E \subset D$ with $m(E) = 0$, $m(\phi^{-1}(E)) = 0$.

□

Now we are going to prove the Change of Variables theorem. To make this complicated proof easier to follow, we divide the whole proof into five steps.

Proof [Theorem 5.13] Step 1: For simplicity, denote $J(x) = d\phi(x)$ and $\det(d\phi(x)) = |J(x)|$. For any rectangle I (can be open, closed, or half-open half-closed) so that its closure (a closed rectangle) $\bar{I} \subset D$,

$$m(I) = (\mathcal{L}) \int_I 1 \; dy = (\mathcal{R}) \int_I 1 \; dy = (\mathcal{R}) \int_{\phi^{-1}(I)} |J(x)| \; dx = (\mathcal{L}) \int_{\phi^{-1}(I)} |J(x)| \; dx$$

where the third equality is by change of variables for Riemann integration. If $\{I_i\}_{i=1}^{\infty}$ are rectangles, almost disjoint and their closures are contained in D, then

$$m\left(\bigcup_{i=1}^{\infty} I_i\right) = \sum_{i=1}^{\infty} m(I_i) = \sum_{i=1}^{\infty} (\mathcal{L}) \int_{\phi^{-1}(I_i)} |J(x)| \; dx = \sum_{i=1}^{\infty} (\mathcal{L}) \int_{\phi^{-1}(I_i^o)} |J(x)| \; dx$$

where I_i^o denotes the interior of rectangle I_i. The last equality is because $\phi^{-1}(I_i) = \phi^{-1}(I_i^o) \cup Z$ with $Z \subset \phi^{-1}(\partial I_i)$, and by Lemma 5.5, $m(\partial I_i) = 0$ implies $m(\phi^{-1}(\partial I_i)) = 0$ and $m(Z) = 0$. Since ϕ^{-1} is injective and maps set with zero measure to set with zero measure, that I_i's are almost disjoint implies $\phi^{-1}(I_i)$'s are almost disjoint. Thus,

$$m\left(\bigcup_{i=1}^{\infty} I_i\right) = (\mathcal{L}) \int_{\bigcup_{i=1}^{\infty} \phi^{-1}(I_i^o)} |J(x)| \; dx = (\mathcal{L}) \int_{\bigcup_{i=1}^{\infty} \phi^{-1}(I_i)} |J(x)| \; dx$$

Consider any open set $\Omega \subset D$, by Exercise 1.3, $\Omega = \bigcup_{k=1}^{\infty} c_k$ where c_k's are almost disjoint cubes. Therefore,

$$m(\Omega) = (\mathcal{L}) \int_{\bigcup_{k=1}^{\infty} \phi^{-1}(c_k)} |J(x)| \; dx = (\mathcal{L}) \int_{\phi^{-1}(\bigcup_{k=1}^{\infty} c_k)} |J(x)| \; dx$$

In conclusion, for any open set $\Omega \subset D$, $m(\Omega) = (\mathcal{L}) \int_{\phi^{-1}(\Omega)} |J(x)| \; dx$. From now on, all integrals (except for specially declared) are Lebesgue integral, so we will drop the symbol $(\mathcal{R})$ and $(\mathcal{L})$ in front of the integral sign.

Step 2: Prove for any $E \subset D$ with $E \in \mathcal{M}$ and $m(E) < \infty$, $m(E) = \int_{\phi^{-1}(E)} |J(x)| \; dx$. By Theorem 1.1, there exists a G_δ set $H = \bigcap_{k=1}^{\infty} D_k$ so that $H \supset E$, $D_k \subset D$ open, $m(D_1) < m(E) + 1$, and $m(H \setminus E) = 0$. Claim that $m(H) = \int_{\phi^{-1}(H)} |J(x)| \; dx$. Define

$E_k = \bigcap_{i=1}^k D_i$ for all $k \geq 1$, then E_k decreases to H as $k \to \infty$. Since $\phi^{-1}(E_k) \in \mathcal{M}$ and $\phi^{-1}(H) \in \mathcal{M}$, $\phi^{-1}(E_k)$ decreases to $\phi^{-1}(H)$. Since E_k is open, by Step 1,

$$m(E_k) = \int_{\phi^{-1}(E_k)} |J(x)|\ dx = \int_{\mathbb{R}^n} |J(x)| I_{\phi^{-1}(E_k)}(x)\ dx$$

Take limit as $k \to \infty$ on both sides, by Continuity of Lebesgue Measure,

$$m(H) = \lim_{k\to\infty} \int_{\mathbb{R}^n} |J(x)| I_{\phi^{-1}(E_k)}(x)\ dx$$

Since $|J(x)| I_{\phi^{-1}(E_k)}(x) \leq |J(x)| I_{\phi^{-1}(E_1)}(x)$ and

$$\int_{\mathbb{R}^n} |J(x)| I_{\phi^{-1}(E_1)}(x)\ dx = \int_{\phi^{-1}(E_1)} |J(x)|\ dx = m(E_1) < \infty$$

By DCT, $m(H) = \int_{\mathbb{R}^n} |J(x)| I_{\phi^{-1}(H)}(x)\ dx = \int_{\phi^{-1}(H)} |J(x)|\ dx$, which proves the claim. Now we can conclude that

$$\begin{aligned} m(E) = m(H) &= \int_{\phi^{-1}(H)} |J(x)|\ dx \\ &= \int_{\phi^{-1}(E)} |J(x)|\ dx + \int_{\phi^{-1}(H\setminus E)} |J(x)|\ dx = \int_{\phi^{-1}(E)} |J(x)|\ dx \end{aligned}$$

where the last equality is because $m(H \setminus E) = 0$ with Lemma 5.5 implies $m(\phi^{-1}(H \setminus E)) = 0$.

Step 3: Suppose $f(y) \in L^1(D)$ is simple function. Write $f(y) = \sum_{k=1}^m c_k I_{E_k}(x)$ where E_k's are measurable (with $0 < m(E) < \infty$), disjoint, and c_k's are nonzero. Observe

$$\int_D f(y)\ dy = \sum_{k=1}^m c_k m(E_k) = c_k \sum_{k=1}^m \int_{\phi^{-1}(E_k)} |J(x)|\ dx = \int_G |J(x)| \sum_{k=1}^m c_k I_{\phi^{-1}(E_k)}(x)\ dx$$

where the second equality is because of Step 2. Notice that $I_{\phi^{-1}(E_k)}(x) = I_{E_k}(\phi(x))$, so

$$\int_D f(y)\ dy = \int_G |J(x)| \sum_{k=1}^m c_k I_{E_k}(\phi(x))\ dx = \int_G |J(x)| f(\phi(x))\ dx$$

Step 4: Suppose $f(x) \geq 0$ with $f \in L^1(D)$. By simple approximation theorem, there exists measurale simple function $f_k \geq 0$ s.t. $f_k(x)$ increases to $f(y)$ on D as $k \to \infty$. Thus, $f_k(\phi(x))$ increases to $f(\phi(x))$ on G. By Step 3, $\int_D f_k(y)\ dy = \int_G f(\phi(x))|J(x)|\ dx$ for all $k \geq 1$. Take $k \to \infty$ on both sides, by MCT, $\int_D f(y)\ dy = \int_G f(\phi(x))|J(x)|\ dx$.

Step 5: Suppose $f \in L^1(D)$, then $f(y) = f^+(y) - (-f^-(y))$ with $f^+(y) \geq 0$ and $f^-(y) \leq 0$ on D. By Step 4, we have

$$\begin{aligned} \int_D f(y)\ dy &= \int_D f^+(y)\ dy - \int_D -f^-(y)\ dy \\ &= \int_G f^+(\phi(x))|J(x)|\ dx - \int_G -f^-(\phi(x))|J(x)|\ dx \\ &= \int_G f(\phi(x))|J(x)|\ dx \end{aligned}$$

Thus, the Change of Variables theorem is proved. □

Remark Since $f(y) \in L^1(D)$, we implicitly assume $f(y)$ is measurable on D. To be rigorous, we need to prove $f(\phi(x))$ is measurable on G, i.e., for all $t \in \mathbb{R}$, $\{x \in G \mid f(\phi(x)) > t\} \in \mathcal{M}$. Notice that $x \in \phi^{-1}\{y \in D \mid f(y) > t\}$ if and only if $\phi(x) \in \{y \in D \mid f(y) > t\}$. Thus,

$\{x \in G \,|\, f(\phi(x)) > t\} = \phi^{-1}\{y \in D \,|\, f(y) > t\}$. Since $f(y)$ is measurable on D, for all $t \in \mathbb{R}$, $\{y \in D \,|\, f(y) > t\} \in \mathcal{M}$, so by Lemma 5.5, $\phi^{-1}\{y \in D \,|\, f(y) > t\} \in \mathcal{M}$ and $\{x \in G \,|\, f(\phi(x)) > t\} \in \mathcal{M}$.

Next we derive some user-friendly versions of the Change of Variables theorem. For all of the following Corollaries, we always assume the same conditions on sets G, D, function ϕ as in Theorem 5.13. After that, we also give some examples to illustrate how to invoke those user-friendly versions of theorem to solve practical problems.

Corollary 5.6

Suppose $E \subset D$, $E \in \mathcal{M}$ and $f(y) \in L^1(E)$, then

$$\int_E f(y)\, dy = \int_{\phi^{-1}(E)} f(\phi(x))|J(x)|\, dx$$

♡

Proof Since $f(y) \in L^1(E)$, $f(y)I_E(y) \in L^1(D)$. By Theorem 5.13, $f(\phi(x))I_E(\phi(x))|J(x)|$ is in $L^1(G)$. Notice that $I_E(\phi(x)) = I_{\phi^{-1}(E)}(x)$, so by Theorem 5.13 again, we have

$$\int_D f(y)I_E(y)\, dy = \int_G f(\phi(x))I_{\phi^{-1}(E)}(x)|J(x)|\, dx$$

Since $E \subset D$ implies $\phi^{-1}(E) \subset \phi^{-1}(D) = G$, the desired result follows immediately. □

Corollary 5.7

If $f(\phi(x))|J(x)| \in L^1(G)$, then $f(y) \in L^1(D)$ and

$$\int_D f(y)\, dy = \int_G f(\phi(x))|J(x)|\, dx$$

♡

Proof Let $g(x) = f(\phi(x))|J(x)|$ for $x \in G$. By assumption, $g(x) \in L^1(G)$. Consider $\phi^{-1} : D \mapsto G$ defined by $x = \phi^{-1}(y)$ for $y \in D$, it is bijective and $\mathcal{C}^1$-smooth (by Inverse Function Theorem). Thus, applying Theorem 5.13 to g and ϕ^{-1}, we can conclude $g(\phi^{-1}(y)) \det(d\phi^{-1}(y)) \in L^1(D)$ and $\int_G g(x)\, dx = \int_D g(\phi^{-1}(y)) \det(d\phi^{-1}(y))\, dy$. Note that $g(\phi^{-1}(y)) \det(d\phi^{-1}(y)) = f(y) \det(d\phi(x)) \det(d\phi^{-1}(y)) = f(y)$ because by Inverse Function Theorem, $d\phi^{-1}(y) = (d\phi(x))^{-1}$. Therefore, $f(y) \in L^1(D)$ and

$$\int_G f(\phi(x))|J(x)|\, dx = \int_G g(x)\, dx = \int_D g(\phi^{-1}(y)) \det(d\phi^{-1}(y))\, dy = \int_D f(y)\, dy$$

which gives the desired result. □

Corollary 5.8

If $f(y) \geq 0$ and measurable on D, then $f(\phi(x))|J(x)|$ is measurable on G and

$$\int_D f(y)\, dy = \int_G f(\phi(x))|J(x)|\, dx$$

♡

Proof For all $k \geq 1$, define

$$f_k(y) = \begin{cases} f(y) & \text{if } f(y) \leq k \\ k & \text{if } f(y) > k \end{cases}$$

Let $D_k = D \cap B_k(0)$ and $G_k = \phi^{-1}(D_k)$, then $f_k(y)$ is bounded on D_k with D_k bounded. By the remark right after the proof of Theorem 5.13, $f(\phi(x))|J(x)|$ is measurable. Thus, $f_k \in L^1(D_k)$. By Theorem 5.13, we conclude that $\int_{D_k} f_k(y)\ dy = \int_{G_k} f_k(\phi(x))|J(x)|\ dx$. This implies $\int_{\mathbb{R}^n} f_k(y) I_{D_k}(y)\ dy = \int_{\mathbb{R}^n} f_k(\phi(x)) I_{G_k}(x)|J(x)|\ dx$. Notice that as $k \to \infty$, for any fixed $x \in G$, $f_k(\phi(x))$ and $I_{G_k}(x)$ are nonnegative increasing to $f(\phi(x))$ and $I_G(x)$; for any fixed $y \in D$, $f_k(y)$ and $I_{D_k}(y)$ are nonnegative increasing to $f(y)$ and $I_D(y)$. Therefore, by MCT, $\int_{\mathbb{R}^n} f(y) I_D(y)\ dy = \int_{\mathbb{R}^n} f(\phi(x)) I_G(x)|J(x)|\ dx$, which gives the desired result. □

Example 5.8 Let A be $n \times n$ real matrix with $\det(A) \neq 0$. Then for all $E \subset \mathbb{R}^n$ with $E \in \mathcal{M}$, $A(E) = \{y \in \mathbb{R}^n \,|\, Ax = y,\ x \in E\}$ is measurable and $m(A(E)) = m(E)|\det(A)|$.

Proof Let $G = \mathbb{R}^n$, $\phi(x) = A^{-1}x$, $D = \mathbb{R}^n$ and $f(y) = I_E(y)$. Since $E \in \mathcal{M}$, $f(y)$ is measurable. By Corollary 5.8, $f(\phi(x))|J(x)| = I_E(A^{-1}x)|\det(A^{-1})| = I_E(A^{-1}x)|\det(A)|^{-1}$ is measurable on $\mathbb{R}^n$. Furthermore, since $I_E(A^{-1}x) = I_{A(E)}(x)$,

$$\int_{\mathbb{R}^n} I_E(y)\ dy = \int_{\mathbb{R}^n} I_E(A^{-1}x)|\det(A)|^{-1}\ dx = \int_{A(E)} |\det(A)|^{-1}\ dx = \frac{m(A(E))}{|\det(A)|}$$

Also, that $I_E(A^{-1}x)|\det(A)|^{-1}$ is measurable implies $I_E(A^{-1}x)$ is measurable, because $|\det(A)|^{-1}$ is a finite constant. Then $I_{A(E)}(x)$ is measurable, so $A(E)$ is measurable. □

Example 5.9 Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathbb{R}^n$ be linearly independent. Define parallelepiped $P \subset \mathbb{R}^n$ spanned by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ as $P = \{\sum_{i=1}^n x_i \boldsymbol{a}_i \,|\, x_i \in [0,1],\ \forall\, i = 1, \ldots, n\}$. Prove that P is measurable and $m(P) = |\det(A)|$ where the i-th column of A is $\boldsymbol{a}_i$.

Proof Let $E = \{x \in \mathbb{R}^n \,|\, x_i \in [0,1],\ \forall\, i = 1, \ldots, n\}$, then E is a closed cube in $\mathbb{R}^n$, hence measurable. Notice that $P = A(E)$ with $\det(A) \neq 0$, so by Exercise 5.8, $P \in \mathcal{M}$ and $m(P) = |\det(A)| m(E)$. It is trivial that $m(E) = |E| = 1$, so $m(P) = |\det(A)|$. □

Example 5.10 Let A be a 3×3 positive definite real matrix (symmetric) with eigenvalues $\lambda_i = i$ for $i = 1, 2, 3$. Define $H(x) = x^{\mathrm{T}} A x$ on $x \in \mathbb{R}^3$. Compute $\iiint_{H(y)<1} e^{\sqrt{H(y)}}\ dy$.

Proof Since A is symmetric positive definite, by eigenvalue decomposition, there exists orthogonal matrix Q s.t. $A = Q \Lambda Q^{\mathrm{T}}$ where $\Lambda = \operatorname{diag}(1, 2, 3)$ is a diagonal matrix with its diagonal elements 1, 2 and 3. Denote $\sqrt{\Lambda} = \operatorname{diag}(1, \sqrt{2}, \sqrt{3})$ and $\sqrt{A} = Q\sqrt{\Lambda}Q^{\mathrm{T}}$. Define $D = \{y \in \mathbb{R}^3 \,|\, H(y) < 1\}$, $\phi(x) = (\sqrt{A})^{-1}x$, $H(y) = \|\sqrt{A}y\|_2^2$. It is also easy to see $G = \phi^{-1}(D) = \{x \in \mathbb{R}^3 \,|\, \|x\|_2 < 1\}$. Since $f(y) = e^{\sqrt{H(y)}}$ is continuous and nonnegative, we can apply Corollary 5.8 to obtain

$$\iiint_{H(y)<1} e^{\sqrt{H(y)}}\ dy = \iiint_G e^{\|x\|_2} \det(\sqrt{A}^{-1})\ dx = \frac{1}{\sqrt{6}} \iiint_{\|x\|_2<1} e^{\|x\|_2}\ dx$$

Now we use polar coordinate to do change of variables again. Define

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \tilde{\phi}(\rho, \psi, \theta) = \begin{bmatrix} \rho \sin\psi \cos\theta \\ \rho \sin\psi \sin\theta \\ \rho \cos\psi \end{bmatrix}, \quad \text{where } \psi \in (0, \pi), \theta \in (0, 2\pi), \rho \in (0, 1)$$

$$\tilde{G} = \{(\rho, \psi, \theta) \in \mathbb{R}^3 \mid \psi \in (0, \pi), \theta \in (0, 2\pi), \rho \in (0, 1)\}$$

Then $\tilde{D} = \tilde{\phi}(\tilde{G}) = \{x \in \mathbb{R}^3 \mid \|x\|_2 < 1\} \setminus Z$ where $m(Z) = 0$. Therefore, by Corollary 5.8,

$$\iiint_{\|x\|_2<1} e^{\|x\|_2}\, dx = \iiint_{\tilde{D}} e^{\|x\|_2}\, dx = \iiint_{\tilde{G}} e^{\rho} |\rho^2 \sin\psi|\, d\rho\, d\psi\, d\theta$$

Since $\tilde{G}$ is a rectangle, by FTT-II,

$$\begin{aligned}
\iiint_{\tilde{G}} e^{\rho} |\rho^2 \sin\psi|\, d\rho\, d\psi\, d\theta &= \int_0^{2\pi} \int_0^{\pi} \sin\psi \int_0^1 e^{\rho} \rho^2\, d\rho\, d\psi\, d\theta \\
&= (e-2) \int_0^{2\pi} \int_0^{\pi} \sin\psi\, d\psi\, d\theta \\
&= \int_0^{2\pi} 2(e-2)\, d\theta = 4\pi(e-2)
\end{aligned}$$

Thus, the final answer is

$$\iiint_{H(y)<1} e^{\sqrt{H(y)}}\, dy = \frac{4\pi(e-2)}{\sqrt{6}}$$

$\square$

Chapter 6 Version History

We revised our lecture notes now and then. This section shows the version story of this lecture notes.

2020/12/28 *Updates:release of Version 1.0*

① The first version of this lecture notes was released!

Appendix Solutions Manual for Problem Sets