MAT3220: Operation Research
Homework 1

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Problem 1. Consider a general form linear program,

$$\begin{aligned}
\max_{x_j} \quad & \sum_{j=1}^{n} c_j x_j \\
s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i \in I \\
& \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i \in E \\
& x_j \ge 0, \; j \in S
\end{aligned}$$

Show that it is possible to equivalently formulate it into the following standard form,

$$\begin{aligned}
\max_{\vec{x}} \quad & \vec{c}^{\mathrm{T}} \vec{x} \\
s.t. \quad & \bar{A}\vec{x} = \vec{b} \\
& \vec{x} \ge 0
\end{aligned}$$

First, we transform the original form into matrix form as follows

$$\begin{aligned}
\max_{\vec{x}} \quad & \vec{c}^{\mathrm{T}} \vec{x} \\
s.t. \quad & A_I \vec{x} \le \vec{b}_I \\
& A_E \vec{x} = \vec{b}_E \\
& \vec{x} \ge 0
\end{aligned}$$

Let $\vec{s} = \vec{b}_I - A_I \vec{x}$, it is equivalent to the following form

$$\begin{aligned}
\max_{\vec{x}} \quad & \vec{c}^{\mathrm{T}} \vec{x} \\
s.t. \quad & A_I \vec{x} + \vec{s} = \vec{b}_I \\
& A_E \vec{x} = \vec{b}_E \\
& \vec{x}, \vec{s} \ge 0
\end{aligned}$$

We can further compact the above two equality constraints into a block matrix form, which is

$$\left[\begin{array}{c:c} A_I & I \\ \hdashline A_E & 0 \end{array}\right] \cdot \begin{bmatrix} \vec{x} \\ \vec{s} \end{bmatrix} = \begin{bmatrix} \vec{b}_I \\ \vec{b}_E \end{bmatrix} \Longleftrightarrow \bar{A} \cdot \vec{\bar{x}} = \vec{b}$$

Similarly, we can replace the objective function as follows

$$\vec{c}^{\mathrm{T}}\vec{x} = \left[\begin{array}{c:c} \vec{c}^{\mathrm{T}} & 0 \end{array}\right] \cdot \begin{bmatrix} \vec{x} \\ \vec{s} \end{bmatrix} = \vec{c}^{\mathrm{T}}\vec{x}$$

Therefore, we could obtain the equivalent form of the original formualtion as follows

$$\begin{aligned} \max_{\vec{x}} \quad & \vec{c}^{\mathrm{T}}\vec{x} \\ s.t. \quad & \bar{A}\vec{x} = \vec{b} \\ & \vec{x} \geq 0 \end{aligned}$$

Problem 2.

- Consider the following primal linear program (P) and its dual problem (D),

$$(P) \quad \begin{aligned} \max_{\vec{x}} \quad & \vec{c}^{\mathrm{T}}\vec{x} \\ s.t. \quad & A\vec{x} + \vec{s} = \vec{b} \\ & \vec{x} \geq 0, \ \vec{s} \geq 0 \end{aligned} \qquad\qquad (D) \quad \begin{aligned} \min_{\vec{y}} \quad & \vec{b}^{\mathrm{T}}\vec{y} \\ s.t. \quad & A^{\mathrm{T}}\vec{y} - \vec{w} = \vec{c} \\ & \vec{y} \geq 0, \ \vec{w} \geq 0 \end{aligned}$$

  Prove the following theorem (aka. the theorem of complementary slackness).

  > Suppose (P) and (D) are both feasible. Then,
  >
  > – (P) and (D) both have optimal solutions.
  >
  > – Denote a pair of their respective optimal solutions to be $(\vec{x}^*, \vec{s}^*)$ and $(\vec{y}^*, \vec{w}^*)$. Show that
  >
  > $$s_i^* y_i^* = 0, \ 1 \leq i \leq m, \text{ and } w_j^* x_j^* = 0, \ 1 \leq j \leq n$$

  The first part is easy to prove. Since both (P) and (D) are feasible, they are either optimal or unbounded. If (P) is unbounded, then by weak duality theorem, all feasible solution of (P) should be less than or equal to any of solutions to (D), which is impossible. This is because positive infinity cannot be bounded by any fixed value. If (D) is unbounded, then the solution to (D) tends to negative infinitity. By weak duality again, this is impossible, because negative infinity cannot be an upper bound of solution to (P).

  Consider the second part, since $(\vec{x}^*, \vec{s}^*)$ and $(\vec{y}^*, \vec{w}^*)$ are feasible solutions, we have

$$\begin{cases} \vec{s}^* = \vec{b} - A\vec{x} \geq \vec{0} \\ \vec{w}^* = A^{\mathrm{T}}\vec{y}^* - \vec{c} \geq \vec{0} \end{cases}$$

Since $\vec{y}^*$ and $\vec{x}^*$ are both nonnegative, we have

$$\begin{cases} \vec{y}^{*\mathrm{T}}(\vec{b} - A\vec{x}) \geq \vec{0} \\ \vec{x}^{*\mathrm{T}}(A^{\mathrm{T}}\vec{y}^* - \vec{c}) \geq \vec{0} \end{cases}$$

which shows that

$$\begin{cases} \vec{y}^{*\mathrm{T}}\vec{b} \geq \vec{y}^{*\mathrm{T}}A\vec{x} \\ \vec{x}^{*\mathrm{T}}A^{\mathrm{T}}\vec{y}^* \geq \vec{x}^{*\mathrm{T}}\vec{c} \end{cases}$$

Since $\vec{y}^{*\mathrm{T}}A\vec{x}$ is a real number, so its transpose will be equal to itself, i.e.,

$$\vec{y}^{*\mathrm{T}}A\vec{x} = (\vec{y}^{*\mathrm{T}}A\vec{x})^{\mathrm{T}} = \vec{x}^{*\mathrm{T}}A^{\mathrm{T}}\vec{y}^*$$

Hence, we have

$$\vec{y}^{*\mathrm{T}}\vec{b} \geq \vec{y}^{*\mathrm{T}}A\vec{x} \geq \vec{x}^{*\mathrm{T}}A^{\mathrm{T}}\vec{y}^* \geq \vec{x}^{*\mathrm{T}}\vec{c}$$

By strong duality, $\vec{y}^{*\mathrm{T}}\vec{b} = \vec{x}^{*\mathrm{T}}\vec{c}$, thus, we conclude that

$$\vec{y}^{*\mathrm{T}}\vec{b} = \vec{y}^{*\mathrm{T}}A\vec{x} = \vec{x}^{*\mathrm{T}}A^{\mathrm{T}}\vec{y}^* = \vec{x}^{*\mathrm{T}}\vec{c}$$

Hence,

$$\begin{cases} \vec{y}^{*\mathrm{T}}(\vec{b} - A\vec{x}) = \vec{0} \\ \vec{x}^{*\mathrm{T}}(A^{\mathrm{T}}\vec{y}^* - \vec{c}) = \vec{0} \end{cases}$$

which is equivalent to say

$$\begin{cases} \vec{y}^{*\mathrm{T}}\vec{s}^* = \vec{0} \\ \vec{x}^{*\mathrm{T}}\vec{w}^* = \vec{0} \end{cases}$$

Since each entry of $\vec{x}^*$, $\vec{y}^*$, $\vec{s}^*$, and $\vec{w}^*$ is nonnegative, we have

$$s_i^* y_i^* = 0,\ 1 \leq i \leq m, \text{ and } w_j^* x_j^* = 0,\ 1 \leq j \leq n$$

- Apply the same upper bounding process to formulate the dual of the following general linear programming problem,

$$\begin{aligned} \max_{x_j} \quad & \sum_{j=1}^{n} c_j x_j \\ s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i \in I \\ & \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i \in E \\ & x_j \geq 0,\ j \in S \end{aligned}$$

First, we transform the original form into matrix form as follows

$$\begin{aligned} \max_{\vec{x}} \quad & \vec{c}^{\mathrm{T}}\vec{x} \\ s.t. \quad & A_I\vec{x} \leq \vec{b}_I \\ & A_E\vec{x} = \vec{b}_E \\ & \vec{x} \geq 0 \end{aligned}$$

Choose $\vec{y}_I \geq 0$ and $\vec{y}_E$ free, as long as $\vec{y}_I^{\,\mathrm{T}} A_I + \vec{y}_E^{\,\mathrm{T}} A_E \geq \vec{c}^{\,\mathrm{T}}$, from $\vec{x} \geq \vec{0}$, we have

$$\vec{y}_I^{\,\mathrm{T}} A_I \vec{x} + \vec{y}_E^{\,\mathrm{T}} A_E \vec{x} \geq \vec{c}^{\,\mathrm{T}} \vec{x} \Longrightarrow \vec{y}_I^{\,\mathrm{T}} A_I \vec{x} + \vec{y}_E^{\,\mathrm{T}} \vec{b}_E \geq \vec{c}^{\,\mathrm{T}} \vec{x}$$

Also, since $\vec{y}_I \geq \vec{0}$ and $A_I \vec{x} \leq \vec{b}_I$, we have

$$\vec{y}_I^{\,\mathrm{T}} \vec{b}_I + \vec{y}_E^{\,\mathrm{T}} \vec{b}_E \geq \vec{c}^{\,\mathrm{T}} \vec{x}$$

Thus, the dual problem is to minimize the upper bound, i.e.,

$$\begin{aligned} \min_{\vec{y}_I, \vec{y}_E} \quad & \vec{y}_I^{\,\mathrm{T}} \vec{b}_I + \vec{y}_E^{\,\mathrm{T}} \vec{b}_E \\ s.t. \quad & \vec{y}_I^{\,\mathrm{T}} A_I + \vec{y}_E^{\,\mathrm{T}} A_E \geq \vec{c}^{\,\mathrm{T}} \\ & \vec{y}_I \geq 0, \vec{y}_E \ \text{ free} \end{aligned}$$

or equivalently,

$$\begin{aligned} \min_{\vec{y}_I, \vec{y}_E} \quad & \vec{b}_I^{\,\mathrm{T}} \vec{y}_I + \vec{b}_E^{\,\mathrm{T}} \vec{y}_E \\ s.t. \quad & A_I^{\mathrm{T}} \vec{y}_I + A_E^{\mathrm{T}} \vec{y}_E \geq \vec{c} \\ & \vec{y}_I \geq 0, \vec{y}_E \ \text{ free} \end{aligned}$$