MAT3220: Operation Research
Homework 2

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Due date: Feb 26, 12 p.m., 2019

Problem 1. Use the strong duality theorem to prove Gordan's theorem: Either $A\vec{x} > \vec{0}$ has a solution, or $A^{\mathrm{T}}\vec{y} = \vec{0}$, $\vec{y} \gneqq \vec{0}$ has a solution.

Denote $\vec{e}$ as a vector whose all entries are equal to one. Consider the following primal-dual problem.

$$(P) \quad \min_{\vec{x},\vec{y}} \quad \vec{e}^{\mathrm{T}}\vec{y} \qquad\qquad (D) \quad \max_{\vec{z}} \quad \vec{e}^{\mathrm{T}}\vec{z}$$

$$s.t. \quad A\vec{x} + \vec{y} \geq \vec{e} \quad \text{(dual variable } \vec{z}\text{)} \qquad\qquad s.t. \quad A^{\mathrm{T}}\vec{z} = \vec{0}$$

$$\vec{y} \geq \vec{0} \qquad\qquad \vec{z} \leq \vec{e}, \quad \vec{z} \geq \vec{0}$$

If $A\vec{x} > \vec{0}$ has a solution, then we can amplify this solution by multiplying a positive real number so that we obtain another solution $\vec{x}^*$ such that $A\vec{x}^* \geq \vec{e}$. Then, if we take $\vec{y}^* = \vec{0}$, we obtain the optimal solution of primal problem, i.e., $\vec{e}^{\mathrm{T}}\vec{y}^* = 0$. By strong duality, the dual problem also has a optimal solution 0, meaning that $\vec{e}^{\mathrm{T}}\vec{z}^* = 0$, hence $\vec{z}^* = \vec{0}$. Such $\vec{z}^*$ is also the only feasible solution, because if we have any other feasible solution $\vec{z} \geq \vec{0}$ must have at least one positive entry, which yields a larger objective value, which is a contradiction. Hence, there does not exist a $\vec{z} \gneqq \vec{0}$ satisfying $A^{\mathrm{T}}\vec{z} = \vec{0}$.

Conversely, if $A\vec{x} > \vec{0}$ does not have a solution, then vector $A\vec{x}$ has at least one entry that is nonpositive. Since the primal problem is obviously feasible and bounded, it has the optimal solution $\vec{y}^* \gneqq \vec{0}$ for all chosen of $\vec{x}$. Hence, the optimal value $\vec{e}^{\mathrm{T}}\vec{y}^* > 0$, and by strong duality, this also implies the optimal value of dual $\vec{e}^{\mathrm{T}}\vec{z}^* > 0$. Thus, we conclude that $\vec{z}^* \gneqq \vec{0}$ is a feasible solution that satisfies $A^{\mathrm{T}}\vec{z} = \vec{0}$. Therefore, Gordan's theorem is proved.

Problem 2.

- Prove the following general form of the separation theorem. Suppose that $S \subseteq \mathbb{R}^n$ is a closed convex set, and that $\vec{u} \notin S$. Then, there exist $\vec{c} \in \mathbb{R}^n$ and $d \in \mathbb{R}$, such that $\vec{c}^{\mathrm{T}}\vec{u} < d$ and $\vec{c}^{\mathrm{T}}\vec{x} > d$ for all $\vec{x} \in S$.

  Denote $\bar{S} = \{\vec{x} - \vec{u} \,|\, \forall\, \vec{x} \in S\}$, since S is closed and convex, it is easy to show that $\bar{S}$ is also closed and convex. Consider the function $f(\vec{x}) = \|\vec{x}\|_2$, since it is continous, we know a continuous function will map a closed set to closed set, so the set

  $$D = \{\|\vec{x} - \vec{u}\|_2 \,|\, \vec{x} \in S\}$$

is a closed set, hence its minimum will be obtained by some $\vec{z} \in S$. Denote $\vec{c} = \vec{z} - \vec{u}$ and $d = \frac{1}{2}\vec{c}^{\mathrm{T}}(\vec{z} + \vec{u})$, we need to prove this hyperplan satisfies our assumption.

We claim that for all $\vec{x} \in S$, we have

$$\vec{c}^{\mathrm{T}}\vec{x} = (\vec{z} - \vec{u})^{\mathrm{T}}\vec{x} \geq (\vec{z} - \vec{u})^{\mathrm{T}}\vec{z} > \frac{1}{2}(\vec{z} - \vec{u})^{\mathrm{T}}(\vec{z} + \vec{u}) = d > (\vec{z} - \vec{u})^{\mathrm{T}}\vec{u} = \vec{c}^{\mathrm{T}}\vec{u}$$

The nontrivial part of the above relation is

$$(\vec{z} - \vec{u})^{\mathrm{T}}\vec{x} \geq (\vec{z} - \vec{u})^{\mathrm{T}}\vec{z}, \quad \text{and} \quad (\vec{z} - \vec{u})^{\mathrm{T}}\vec{z} > \frac{1}{2}(\vec{z} - \vec{u})^{\mathrm{T}}(\vec{z} + \vec{u}) > (\vec{z} - \vec{u})^{\mathrm{T}}\vec{u}$$

The second one is easy, since

$$(\vec{z} - \vec{u})^{\mathrm{T}}\vec{z} > \frac{1}{2}(\vec{z} - \vec{u})^{\mathrm{T}}(\vec{z} + \vec{u}) \iff \frac{1}{2}(\vec{z} - \vec{u})^{\mathrm{T}}(\vec{z} - \vec{u}) > 0$$

which is obviously correct when $\vec{z} \neq \vec{u}$. Also,

$$\frac{1}{2}(\vec{z} - \vec{u})^{\mathrm{T}}(\vec{z} + \vec{u}) > (\vec{z} - \vec{u})^{\mathrm{T}}\vec{u} \iff \frac{1}{2}(\vec{z} - \vec{u})^{\mathrm{T}}(\vec{z} - \vec{u}) > 0$$

which is obviously correct when $\vec{z} \neq \vec{u}$.

To prove the first relation, we consider any $\lambda \in (0, 1]$, $\vec{x}_\lambda = (1 - \lambda)\vec{z} + \lambda\vec{x} \in S$. Since $\vec{z}$ obtain the minimum distance, we have

$$\begin{aligned}
\|\vec{z} - \vec{u}\|_2^2 \leq \|\vec{x}_\lambda - \vec{u}\|_2^2 &= \|(1 - \lambda)\vec{z} + \lambda\vec{x} - \vec{u}\|_2^2 \\
&= \|(1 - \lambda)(\vec{z} - \vec{u}) + (1 - \lambda)\vec{u} + \lambda\vec{x} - \vec{u}\|_2^2 \\
&= (1 - \lambda)^2\|\vec{z} - \vec{u}\|_2^2 + \lambda^2\|\vec{x} - \vec{u}\|_2^2 + 2(1 - \lambda)\lambda(\vec{x} - \vec{u})^{\mathrm{T}}(\vec{z} - \vec{u})
\end{aligned}$$

Since $\lambda \neq 0$, we have

$$(\lambda - 2)\|\vec{z} - \vec{u}\|_2^2 + 2(1 - \lambda)(\vec{x} - \vec{u})^{\mathrm{T}}(\vec{z} - \vec{u}) + \lambda\|\vec{x} - \vec{u}\|_2^2 \geq 0$$

Since this inequality is continuous with respect to λ, take $\lambda \to 0$, we have

$$\begin{aligned}
0 &\geq \|\vec{z} - \vec{u}\|_2^2 - (\vec{x} - \vec{u})^{\mathrm{T}}(\vec{z} - \vec{u}) \\
&= (\vec{z} - \vec{u})^{\mathrm{T}}(\vec{z} - \vec{u}) - (\vec{x} - \vec{u})^{\mathrm{T}}(\vec{z} - \vec{u}) \\
&= (\vec{z} - \vec{x})^{\mathrm{T}}(\vec{z} - \vec{u}) = (\vec{z} - \vec{u})^{\mathrm{T}}\vec{z} - (\vec{z} - \vec{u})^{\mathrm{T}}\vec{x}
\end{aligned}$$

Hence, we prove that $(\vec{z} - \vec{u})^{\mathrm{T}}\vec{x} \geq (\vec{z} - \vec{u})^{\mathrm{T}}\vec{z}$.

- Prove the following general form of the bipolar theorem by the separation theorem. For a closed convex cone $\mathcal{K} \subseteq \mathbb{R}^n$, we have $(\mathcal{K}^*)^* = \mathcal{K}$.

  By definition, $\mathcal{K}^* = \{\vec{y} \in \mathbb{R}^n \mid \vec{y}^{\mathrm{T}}\vec{x} \geq 0,\ \forall\, \vec{x} \in \mathcal{K}\}$. For any $\vec{z} \in \mathbb{R}^n$ satisfying that for all $\vec{y} \in \mathcal{K}^*$, $\vec{z}^{\mathrm{T}}\vec{y} \geq 0$ will lie in the dual cone of $\mathcal{K}^*$. Obviously, all of $\vec{x} \in \mathcal{K}$ satisfies such condition, hence $(\mathcal{K}^*)^* \supseteq \mathcal{K}$.

  Conversely, if there is some vector $\vec{z}$ in $(\mathcal{K}^*)^*$ but not in $\mathcal{K}$, then by separation theorem, and by choosing origin properly, there is a vector $\vec{c}$ such that $\vec{c}^{\mathrm{T}}\vec{z} < 0$ and $\vec{c}^{\mathrm{T}}\vec{x} > 0$ for all $\vec{x} \in \mathcal{K}$. Since $\vec{c}^{\mathrm{T}}\vec{x} > 0$, $\vec{c}$ is in $\mathcal{K}^*$, but because $\vec{z} \in (\mathcal{K}^*)^*$, $\vec{c}^{\mathrm{T}}\vec{z} \geq 0$, which is a contradiction. Hence, all element in $(\mathcal{K}^*)^*$ must be in $\mathcal{K}$, meaning that $(\mathcal{K}^*)^* \subseteq \mathcal{K}$. Therefore, we finish the proof of bipolar theorem.

Problem 3. Prove the following generalized version of Theorem 5: Suppose that two given polyhedra $P_1 = \{\vec{x} \mid A\vec{x} \le \vec{a}\}$ and $P_2 = \{\vec{x} \mid B\vec{x} \le \vec{b}\}$ (with $\vec{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $\vec{a} \in \mathbb{R}^m$, $B \in \mathbb{R}^{l\times n}$, $\vec{b} \in \mathbb{R}^l$) that are nonempty but they do not intersect. Use the Farkas lemma to prove: there is an affine linear function $f(\vec{x}) = \vec{c}^{\mathrm{T}}\vec{x} + d$ such that

$$f(\vec{x}) > 0 \text{ for } \vec{x} \in P_1, \text{ and } f(\vec{x}) < 0 \text{ for } \vec{x} \in P_2$$

Suppose P_2 consists of vertices $\{\vec{p}_1, \dots, \vec{p}_n\}$ and extreme rays $\{\vec{q}_1, \dots, \vec{q}_m\}$. For any $\vec{x} \in P_2$, since P_1 and P_2 are disjoint, the following system has no solution,

$$\begin{cases} A\vec{x} \le \vec{a} \\ \vec{x} = \left(\sum_{i=1}^{n} \lambda_i \vec{p}_i + \sum_{j=1}^{m} \mu_j \vec{q}_j\right) \\ \sum_{i=1}^{n} \lambda_i = 1 \\ \lambda_i \ge 0 \quad \forall i = 1, \dots, n \\ \mu_j \ge 0 \quad \forall j = 1, \dots, m \end{cases} \iff \begin{cases} A\left(\sum_{i=1}^{n} \lambda_i \vec{p}_i + \sum_{j=1}^{m} \mu_j \vec{q}_j\right) + \vec{\epsilon} = \vec{a} \\ \sum_{i=1}^{n} \lambda_i = 1 \\ \lambda_i \ge 0 \quad \forall i = 1, \dots, n \\ \mu_j \ge 0 \quad \forall j = 1, \dots, m \\ \vec{\epsilon} \ge 0 \end{cases}$$

which can be formulated into matrix form as follows

$$A\vec{u} = \begin{bmatrix} A\vec{p}_1 & \cdots & A\vec{p}_n & A\vec{q}_1 & \cdots & A\vec{q}_m & I \\ 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \\ \mu_1 \\ \vdots \\ \mu_m \\ \vec{\epsilon} \end{bmatrix} = \begin{bmatrix} \vec{a} \\ 1 \end{bmatrix} = \vec{v} \quad \text{and } \vec{u} \ge 0$$

Since it has no solution, by Farkas' lemma, there exists $\vec{z}$ such that $A^{\mathrm{T}}\vec{z} = \vec{0}$ and $\vec{v}^{\mathrm{T}}\vec{z} < 0$. Denote $\vec{z}$ as $(\vec{c}'^{\mathrm{T}}, b)^{\mathrm{T}}$, we have

$$(A\vec{p}_i)^{\mathrm{T}}\vec{c}' + b \ge 0,\ \forall\, i = 1, \dots, n; \quad (A\vec{q}_j)^{\mathrm{T}}\vec{c}' \ge 0,\ \forall\, j = 1, \dots, m; \quad \vec{c}' \ge \vec{0}; \quad \vec{c}'^{\mathrm{T}}\vec{a} + b < 0$$

Let $\vec{c} = A^{\mathrm{T}}\vec{c}'$, then we have for any $\vec{x} \in B$,

$$\vec{c}^{\mathrm{T}}\vec{x} + b = \vec{c}^{\mathrm{T}}\left(\sum_{i=1}^{n} \lambda_i \vec{p}_i + \sum_{j=1}^{m} \mu_j \vec{q}_j\right) + b \ge 0, \quad \text{with } \sum_{i=1}^{n} \lambda_i = 1, \quad \mu_j \ge 0$$

For all $\vec{x} \in A$, we have

$$A\vec{x} \le \vec{a} \Longrightarrow \vec{c}'^{\mathrm{T}}A\vec{x} \le \vec{c}'^{\mathrm{T}}\vec{a} < -b \Longrightarrow \vec{c}^{\mathrm{T}}\vec{x} + b < 0$$

Hence, the proof is finished.