MAT3220: Operation Research
Homework 3

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Problem 1. Denote the input-length of an integer n to be $\ell(n)$. Consider two integers a and b. Derive an upper bound for the input-length of ab.

We can take $\ell(n) = [\log_2(|n| + 1)] + 1$, and here $[x]$ is a function which take the largest integer that will not exceed x. Recall the property of it, we know $[x]$ is nondecreasing function and $[x + y] \le [x] + [y] + 1$. Thus, substitute ab into it, we will have

$$\begin{aligned}\ell(ab) = [\log_2(|ab| + 1)] + 1 &\le [\log_2(|a| + 1) + \log_2(|b| + 1)] + 1 \\ &\le [\log_2(|a| + 1)] + [\log_2(|b| + 1)] + 1 + 1 \\ &= \ell(a) + \ell(b)\end{aligned}$$

Thus a suitable upper bound for the input-length of ab is $\ell(a) + \ell(b)$.

Problem 2. Consider a matrix $A = [a_{ij}]_{n\times n}$, where a_{ij}'s are all integers. The determinant for A is defined as

$$\det(A) = \sum_{\sigma\in\Pi} \operatorname{sign}(\sigma) a_{1\sigma_1} a_{2\sigma_2} \cdots a_{n\sigma_n}$$

where Π is the set of all permutations, and $\operatorname{sign}(\sigma)$ is the sign of the permutation σ. Derived an upper bound for the input-length of $\det(A)$.

Denote the integer with the largest absolute value in A to be Δ, and $\ell(n)$ means the input length of integer n. Since

$$|\det(A)| \le \sum_{\sigma\in\Pi} |\operatorname{sign}(\sigma)||a_{1\sigma_1}||a_{2\sigma_2}| \cdots |a_{n\sigma_n}| \le \sum_{\sigma\in\Pi} |\Delta|^n = n!|\Delta|^n$$

Also, for nonnegative integer $a \le b$, we have $\ell(a) \le \ell(b)$, hence

$$\ell(\det(A)) \le \ell(n!|\Delta|^n) \le \ell(n!) + n\ell(|\Delta|)$$

Of course, we can also write $n\ell(n) + n\ell(|\Delta|)$ as an upper bound.

Problem 3. Consider a system of n linear equations for n unknowns, represented in matrix multiplication form as follows:

$$A\vec{x} = \vec{b}$$

where the $n \times n$ matrix A has a nonzero determinant. The so-called Cramer's rule states that in this case the system has a unique solution given by:

$$x_i = \frac{\det(A_i)}{\det(A)}, \qquad i = 1, \dots, n$$

where A_i is the matrix formed by replacing the i-th column of A by the column vector $\vec{b}$. Suppose that A and $\vec{b}$ are all integer-valued. Let

$$\Delta := \max\{|a_{ij}|, |b_i| : i, j = 1, \dots, n\}$$

Clearly, the input-length of A and $\vec{b}$ is lower bounded by the input-length of Δ. Derive an upper bound for the input-length of x_i using the input-length of Δ, where $i = 1, \dots, n$.

We define the input length of a fractional number to be the sum of the input length of its numerator and denominator.

Thus, we have

$$\ell(x_i) = \ell(\det(A_i)) + \ell(\det(A))$$

From Problem 4, we have

$$\ell(x_i) \leq 2\ell(n!) + 2n\ell(\Delta)$$

If you like, you could also consider for any positive integer i,

$$\ell(i) = \log_2(i+1) + 1 \leq \log_2(2^i) + i = i\log_2(1+1) + i = i\ell(1)$$

Hence, we have

$$\ell(x_i) \leq (n^2+n)\ell(\Delta) + 2n\ell(\Delta) = (n^2+3n)\ell(\Delta)$$

Problem 4. Let the input-length of the following integer-valued linear program

$$\begin{aligned} \min_{\vec{x}} \quad & \vec{c}^{\,\mathrm{T}}\vec{x} \\ s.t. \quad & A\vec{x} = \vec{b} \\ & \vec{x} \geq \vec{0} \end{aligned}$$

be L. Let $\vec{x}^{\text{bfs}}$ be an arbitrary basic feasible solution for the above problem. Derive an upper bound on the input-length of $\vec{x}^{\text{bfs}}$ in terms of L.

According to Fundemental Therorem of Linear Programming, if there exists a feasible solution, then there must exist a basic feasible solution $\vec{x}^{\text{bfs}}$. Also, if we suppose $A \in M_{m\times n}(\mathbb{Z})$ $(n > m)$, $\vec{b} \in \mathbb{Z}^m$ and $\vec{c} \in \mathbb{Z}^n$, then there exists a basis $\mathcal{B}$ such that $\vec{x}_{\mathcal{B}} = A_{\mathcal{B}}^{-1}\vec{b}$. Notice that $\vec{x}_{\mathcal{N}}$ is a $(n-m)$-dimensional zero vector.

Therefore,

$$\ell(\vec{x}^{\text{bfs}}) = \ell(\vec{x}_{\mathcal{B}}) + \ell(\vec{x}_{\mathcal{N}}) \leq \ell(\vec{x}_{\mathcal{B}}) + n - m = \sum_{i=1}^{m} \ell(x_i) + n - m$$

By Problem 3, we have

$$\ell(\overrightarrow{x}^{\text{bfs}}) \leq 2m(m\ell(m) + m\ell(\Delta)) + n - m$$

Since $L \geq \ell(\Delta)$ and $L \geq mn + n + m \geq m^2$, we have

$$\ell(\overrightarrow{x}^{\text{bfs}}) \leq 2L(\ell(m) + \ell(\Delta)) + L$$

Since $\ell(m) = \log_2(m+1) + 1 \leq m + 1 \leq L$, we have

$$\ell(\overrightarrow{x}^{\text{bfs}}) \leq 2L(L + L) + L = 5L^2$$

Therefore, a reasonable upper bound of input length of $\overrightarrow{x}^{\text{bfs}}$ is $5L^2$.