MAT3220: Operation Research
Homework 6

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Due date: April 16, 12 p.m., 2019

Consider the following primal and dual problem,

$$(P)\quad \min_{\vec{x}} \quad \vec{c}^{\mathrm{T}}\vec{x} \qquad\qquad (D)\quad \max_{\vec{y}} \quad \vec{b}^{\mathrm{T}}\vec{y}$$
$$s.t. \quad A\vec{x} = \vec{b} \qquad\qquad s.t. \quad A^{\mathrm{T}}\vec{y} + \vec{s} = \vec{c}$$
$$\vec{x} \geq \vec{0} \qquad\qquad \vec{s} \geq \vec{0}$$

Suppose that (P) and (D) both satisfy the Slater condition (the interior point condition).

Problem 1. Let $\vec{w} \in \mathbb{R}^n_{++}$. Consider

$$\begin{aligned} \min_{\vec{x}} \quad & \vec{c}^{\mathrm{T}}\vec{x} - \sum_{i=1}^{n} w_i \ln x_i \\ s.t. \quad & A\vec{x} = \vec{b} \end{aligned}$$

Prove that an optimal solution for the above problem always exists.

To begin with, since (P) and (D) satisfies Slater condition, there exists $\hat{\vec{x}}$, $\hat{\vec{y}}$, and $\hat{\vec{s}}$, such that $A\hat{\vec{x}} = \vec{b}$, $\hat{\vec{x}} > \vec{0}$, $\vec{c} = A^{\mathrm{T}}\hat{\vec{y}} + \hat{\vec{s}}$ and $\hat{\vec{s}} > \vec{0}$. Therefore, consider the objective function for any feasible solution $\vec{x}$ in $\mathcal{F}_P$, we have

$$\begin{aligned} \vec{c}^{\mathrm{T}}\vec{x} - \sum_{i=1}^{n} w_i \ln x_i &= \left(A^{\mathrm{T}}\hat{\vec{y}} + \hat{\vec{s}}\right)^{\mathrm{T}} \vec{x} - \sum_{i=1}^{n} w_i \ln x_i \\ &= \hat{\vec{y}}^{\mathrm{T}} A\vec{x} + \hat{\vec{s}}^{\mathrm{T}}\vec{x} - \sum_{i=1}^{n} w_i \ln x_i \\ &= \hat{\vec{y}}^{\mathrm{T}}\vec{b} + \sum_{i=1}^{n} (\hat{s}_i x_i - w_i \ln x_i) \end{aligned}$$

Since for any i, we have $\hat{s}_i > 0$ and $w_i > 0$, and variable $x_i > 0$, consider the function $\varphi_i(x_i) = \hat{s}_i x_i - w_i \ln x_i$, we have $\varphi_i'(x_i) = \hat{s}_i - w_i x_i^{-1}$, and we can see when $x_i > w_i/\hat{s}_i$, $\varphi_i'(x_i)$ strictly increasing, when $x_i < w_i/\hat{s}_i$, $\varphi_i'(x_i)$ strictly decreasing. Moreover, if $x_i \to \infty$ or $x_i \to 0$ for some i, then the objective function tends to positive infinity. This shows that $\varphi_i(x_i)$ is coercive function and thus has global minimum. Therefore, $\vec{c}^{\mathrm{T}}\vec{x} - \sum_{i=1}^{n} w_i \ln x_i$ is bounded below on $\mathcal{F}_P$.

Consider

$$A = \left\{ \vec{x} \;\middle|\; \vec{c}^{\mathrm{T}}\vec{x} - \sum_{i=1}^{n} w_i \ln x_i \leq \vec{c}^{\mathrm{T}}\hat{\vec{x}} - \sum_{i=1}^{n} w_i \ln \hat{x}_i,\ A\vec{x} = \vec{b} \right\}$$

It is easy to see from previous argument that A is nonempty (at least $\vec{\hat{x}}$ is in A), closed (take a convergent sequence in A and its limit must be also in A) and bounded (from coercive property of objective function). Hence, by Weierstrass theorem, the objective function restricted in A must be able to attain its minimum value at some point in A. However, by definition of A, this point also ensures global minimum of objective function (on $\mathcal{F}_P$). Hence, an optimal solution always exists, and it must lie in subset A of $\mathcal{F}_P$.

Problem 2. Prove Theorem 1 in Topic 6, i.e.,

> Let $\mathcal{F}_P$ be the feasible set for (P), and $\mathcal{F}_D$ be the feasible set for (D). For any $\vec{w} \in \mathbb{R}^n_{++}$ there exist unique $\vec{x} \in \text{int}\,(\mathcal{F}_P)$ and $\vec{s} \in \text{int}\,(\mathcal{F}_D)$ satisfying $\vec{w} = \vec{x} \circ \vec{s}$.

For any $\vec{w} \in \mathbb{R}^n_{++}$, we can consider the optimization model in **Problem 1**. Since the optimal solution $\vec{x}^*$ must exist, so does the optimal solution of its dual problem, which has the same feasible region as (D), so we can obtain the corresponding $\vec{s}^*$. The duality gap of primal barrier and dual barrier is given by

$$x_i^* s_i^* = w_i, \quad \forall\, i = 1, \ldots, n$$

Thus, we will obtain $\vec{w} = \vec{x}^* \circ \vec{s}^*$. However, since both the optimal solution of primal and dual must be strictly positive for all of their entries, i.e., $\vec{x}^* > \vec{0}$ and $\vec{s}^* > \vec{0}$, (otherwise the logarithmic function is not defined), so $\vec{x}^* \in \text{int}\,(\mathcal{F}_P)$ and $\vec{s}^* \in \text{int}\,(\mathcal{F}_D)$.

For the uniqueness, consider the KKT condition of the barrier form in **Problem 1**, if we have $\vec{x} \in \text{int}\,(\mathcal{F}_P)$ and $\vec{s} \in \text{int}\,(\mathcal{F}_D)$, then $\vec{x}$ satisfies primal feasibility. Since this problem only has one equality constriant, the dual feasiblity and complementary condition are satisfied automatically. For the main condition, we have $\vec{c} - \vec{w}/\vec{x} - A^{\mathrm{T}}\vec{y} = \vec{0}$ (here the division is entrywise division). Since $\vec{w} = \vec{x} \circ \vec{s}$, where $\vec{s} = \vec{c} - A^{\mathrm{T}}\vec{y}$, then it is easy to see $\vec{x}$ satisfies the main conditions, and thus $\vec{x}$ must be an optimal solution because that convexity makes KKT conditions sufficient. However, since the objecitve function of primal problem is strictly convex, so the optimal solution is unique, which means $\vec{x}$ and $\vec{s}$ is unique.