MAT3220: Operation Research
Homework 7

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Problem 1.

- Formulate the constraint $t \geq \|x\|^2$ as a second order cone constraint.

  Consider the following relation,

  $$\left\| \begin{pmatrix} \frac{t-1}{2} \\ \vec{x} \end{pmatrix} \right\|^2 \leq \frac{t+1}{2} \Longleftrightarrow \frac{(t-1)^2}{4} + \|\vec{x}\|^2 \leq \frac{(t+1)^2}{4}$$
  $$\Longleftrightarrow \|\vec{x}\|^2 \leq t$$

  Thus, the constraint $t \geq \|x\|^2$ is equivalent to

  $$\left\| \begin{pmatrix} \frac{t-1}{2} \\ \vec{x} \end{pmatrix} \right\|^2 \leq \frac{t+1}{2} \Longleftrightarrow \begin{pmatrix} \frac{t+1}{2} \\ \frac{t-1}{2} \\ \vec{x} \end{pmatrix} \in \text{SOC}\,(n+2)$$

  where n is the dimension of $\vec{x}$.

- Formulate the constraint $ts \geq \|x\|^2$, $t \geq 0$, $s \geq 0$ as a second order cone constraint.

  Consider the following relation,

  $$\left\| \begin{pmatrix} \frac{t-s}{2} \\ \vec{x} \end{pmatrix} \right\|^2 \leq \frac{t+s}{2} \Longleftrightarrow \frac{(t-s)^2}{4} + \|\vec{x}\|^2 \leq \frac{(t+s)^2}{4}$$
  $$\Longleftrightarrow \|\vec{x}\|^2 \leq ts$$

  Thus, the constraint $ts \geq \|x\|^2$ is equivalent to

  $$\left\| \begin{pmatrix} \frac{t-s}{2} \\ \vec{x} \end{pmatrix} \right\|^2 \leq \frac{t+s}{2} \Longleftrightarrow \begin{pmatrix} \frac{t+s}{2} \\ \frac{t-s}{2} \\ \vec{x} \end{pmatrix} \in \text{SOC}\,(n+2)$$

  where n is the dimension of $\vec{x}$.

- Formulate the second order cone constraint $t \geq \|x\|$ as an SDP constraint. (*Hint*: Use the Schur complement lemma)

Consider the following relation,

$$t \geq \|x\| \iff \begin{bmatrix} t & \vec{x}^{\mathrm{T}} \\ \vec{x} & tI \end{bmatrix} = \begin{bmatrix} A & B^{\mathrm{T}} \\ B & C \end{bmatrix} \succeq 0$$

The relation is obviously true when $t > 0$, because if $t > 0$, then $C \succ 0$, so we can apply Schur complement lemma, which shows that

$$\begin{bmatrix} t & \vec{x}^{\mathrm{T}} \\ \vec{x} & tI \end{bmatrix} \succeq 0 \iff t - \vec{x}^{\mathrm{T}} t^{-1} \vec{x} \geq 0 \iff t \geq \|x\|$$

However, when $t = 0$, Schur complement lemma cannot be applied, but it suffices to show that $\|\vec{x}\| = 0$ if and only if

$$\begin{bmatrix} 0 & \vec{x}^{\mathrm{T}} \\ \vec{x} & \mathbf{0} \end{bmatrix} \succeq 0$$

Take any vector $\vec{z} \in \mathbb{R}^{n+1}$ with $\vec{z} = (u_0 \in \mathbb{R}, \vec{u} \in \mathbb{R}^n)^{\mathrm{T}}$, consider

$$\vec{z}^{\mathrm{T}} \begin{bmatrix} 0 & \vec{x}^{\mathrm{T}} \\ \vec{x} & \mathbf{0} \end{bmatrix} \vec{z} = 2u_0 \vec{u}^{\mathrm{T}} \vec{x} \geq 0$$

Since $u_0 \vec{u}^{\mathrm{T}} \vec{x} \geq 0$ holds for any $u_0, \vec{u}$, take $u_0 = 1, \vec{u} = -\vec{x}$, we have $\|\vec{x}\| \leq 0$, which implies that $\|\vec{x}\| = 0$. Conversely, if $\|\vec{x}\| = 0$, then $\vec{x} = \vec{0}$, thus what we need to prove is trivial because zero matrix is always PSD.

If $t < 0$ then $t \geq \|\vec{x}\|_2$ is not true, and at the same time,

$$\begin{bmatrix} t & \vec{x}^{\mathrm{T}} \\ \vec{x} & tI \end{bmatrix} \prec 0$$

In conclusion, the equivalence is proved.

Problem 2.

- Formulate the following minimization problem as SDP,

$$\min_x \quad x^6 + 2x^5 + 3x^4 + 4x^3 + 5x^2 + 6x$$

and solve the problem numerically using CVX.

The SDP formulation is as follows,

$$\begin{aligned} \max_t \quad & t \\ s.t. \quad & -t = Z_{11} \\ & 6 = Z_{12} + Z_{21} \\ & 5 = Z_{13} + Z_{22} + Z_{31} \\ & 4 = Z_{14} + Z_{23} + Z_{32} + Z_{41} \\ & 3 = Z_{24} + Z_{33} + Z_{42} \\ & 2 = Z_{34} + Z_{43} \\ & Z_{44} = 1 \\ & Z \in \mathcal{S}_+^{4\times 4} \end{aligned}$$

The CVX code is as follows,

```
cvx_begin sdp
    variables Z(4,4) t;
    maximize t
    subject to
        -t == Z(1,1);
        6 == Z(1,2)+Z(2,1);
        5 == Z(1,3)+Z(2,2)+Z(3,1);
        4 == Z(1,4)+Z(2,3)+Z(3,2)+Z(4,1);
        3 == Z(2,4)+Z(3,3)+Z(4,2);
        2 == Z(3,4)+Z(4,3);
        Z(4,4) == 1;
        Z == semidefinite(4);
cvx_end
```

CVX Code

The optimal value of t is $t^* = -3$, which shows that the minimum value of the original polynomial is -3.

- Formulate the following problem by SOCP,

$$\begin{aligned} \min_{\vec{x}} \quad & \sum_{i=1}^{m} 1/(\vec{a}_i^{\mathrm{T}}\vec{x} - b_i) \\ s.t. \quad & \|x\| \leq 1 \end{aligned}$$

where we assume that $\{\vec{x} \mid \|\vec{x}\| \leq 1\} \subset \{\vec{x} \mid \vec{a}_i^{\mathrm{T}}\vec{x} > b_i,\ i = 1, 2, \ldots, m\}$.

The above problem can be reformulated (by change of variable and relaxation) into

$$\begin{aligned} \min_{\vec{x}} \quad & \sum_{i=1}^{m} t_i \\ s.t. \quad & \|\vec{x}\| \leq 1 \\ & \vec{a}_i^{\mathrm{T}}\vec{x} - b_i = u_i \quad \forall\, i = 1, 2, \ldots, m \\ & u_i \cdot t_i \geq 1 \quad \forall\, i = 1, 2, \ldots, m \end{aligned}$$

For the first constraint, use the first part of Problem 1 with $t = 1$, we have

$$\|\vec{x}\| \leq 1 \Longleftrightarrow \begin{pmatrix} 1 \\ \vec{x} \end{pmatrix} \in \mathrm{SOC}\,(n+1)$$

For the second constraint, it is linear, so keep it unchanged. For the third constraint, use the second part of Problem 1 with $\vec{x} = 1$ (the $\vec{x}$ in that part, not the $\vec{x}$ here), we have

$$u_i \cdot t_i \geq 1 \Longleftrightarrow \begin{pmatrix} \frac{u_i + t_i}{2} \\ \frac{u_i - t_i}{2} \\ 1 \end{pmatrix} \in \mathrm{SOC}\,(3) \quad \forall\, i = 1, 2, \ldots, m$$

Thus, the SOCP formulation of original problem is

$$\begin{aligned} \min_{\vec{x},u_i,t_i} \quad & \sum_{i=1}^{m} t_i \\ s.t. \quad & \vec{a}_i^{\mathrm{T}} \vec{x} - b_i = u_i \quad \forall\, i = 1,2,\ldots,m \\ & \begin{pmatrix} 1 \\ \vec{x} \end{pmatrix} \in \mathrm{SOC}\,(n+1), \quad \begin{pmatrix} \frac{u_i+t_i}{2} \\ \frac{u_i-t_i}{2} \\ 1 \end{pmatrix} \in \mathrm{SOC}\,(3) \quad \forall\, i = 1,2,\ldots,m \end{aligned}$$