Additional Exercises: Convexity

1. Why a real symmetric matrix will always have real (as opposed to complex) eigenvalues?

2. Prove the following Cauchy-Schwarz inequality.

   For any $u, v \in \mathbf{R}^n$, we have
   $$u^{\mathrm{T}} v \leq \|u\|_2 \cdot \|v\|_2.$$

3. Use the Cauchy-Schwarz inequality to prove the so-called triangle inequality for the Euclidean norm:
   $$\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$$
   for all $x, y \in \mathbf{R}^n$.

4. For a square matrix $A \in \mathbf{R}^{n\times n}$, its *trace* is $\mathrm{tr}\,(A) = \sum_{i=1}^n a_{ii}$. Prove: For any $X \in \mathbf{R}^{m\times n}$ and $Y \in \mathbf{R}^{n\times m}$, we have $\mathrm{tr}\,(XY^{\mathrm{T}}) = \mathrm{tr}\,(YX^{\mathrm{T}}) = \sum_{i=1}^m \sum_{j=1}^n X_{ij} Y_{ij}$.

5. Let $X \in \mathbf{R}^{m\times n}$ be a real matrix. The so-called Frobenius norm of X is defined as
   $$\|X\|_F := \left(\sum_{i=1}^m \sum_{j=1}^n X_{ij}^2 \right)^{1/2}$$
   and its spectrum norm is defined as $\|X\|_2 := \Big(\lambda_{\max}(X^{\mathrm{T}} X)\Big)^{1/2}$. Prove: Both $\|\cdot\|_F$ and $\|\cdot\|_2$ are indeed matrix norms.

6. Prove: For any $X \in \mathbf{R}^{m\times n}$ and $y \in \mathbf{R}^m$,
   $$\|Xy\|_2 \leq \|X\|_2 \cdot \|y\|_2.$$

7. Prove: For any X, it holds that $\|X\|_2 \leq \|X\|_F$.

8. Compute the gradient of the quartic function
   $$f(x) = (x^{\mathrm{T}} A x)^2$$
   where $A \in \mathcal{S}^n$.

9. Compute the Hessian matrix of the quartic function
   $$f(x) = (x^{\mathrm{T}} A x)^2$$
   where $A \in \mathcal{S}^n$.

10. Prove: If $h(x)$ is twice continuously differentiable, then $h(x)$ is convex in $\mathbf{R}^n$ is equivalent to $\nabla^2 h(x) \succeq 0$ for all $x \in \mathbf{R}^n$.

11. Prove: $(\prod_{i=1}^n x_i)^{1/n}$ is a concave function in $\mathbf{R}^n_{++}$.

12. Prove:
$$\frac{x_1^n}{x_2 x_3 \cdots x_n}$$
is a convex function in $\mathbf{R}^n_{++}$.

13. Consider $X \in \mathcal{S}^{n\times n}$, and so X has n real eigenvalues as we discussed before. Let them be
$$\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X).$$
Prove: $\lambda_1(X)$ is a convex function.

14. Prove:
$$\ln\left(\sum_{i=1}^n e^{x_i}\right)$$
is a convex function.

15. Suppose that $f(x)$ is convex for $x \in S$, and $g(x) > 0$ is concave for $x \in S$. Prove:
$$\frac{f(x)}{g(x)}$$
is a quasi-convex function.

16. Show that
$$\frac{a^{\mathrm{T}}x + b}{c^{\mathrm{T}}x + d}$$
is quasi-linear in $\{x \mid c^{\mathrm{T}}x + d > 0\}$.

17. Suppose that $f(x)$ is convex for $x \in S$, and $g(x) > 0$ is concave for $x \in S$. Prove:
$$\frac{f(x)^2}{g(x)}$$
is a convex function.

18. Prove: $\prod_{i=1}^n x_i$ is quasi-concave in $\mathbf{R}^n_{++}$.

19. Show that $S := \{x \mid \|x - a\|_2 \leq \|x - b\|_2\}$ is a convex region. Further prove: $\|x - a\|_2 / \|x - b\|_2$ is quasi-convex in S.

20. Prove:
$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$
is a log-concave function.

21. Suppose $Q \in \mathcal{S}^{n\times n}_{++}$. Prove:

$$2x^{\mathrm{T}}y \le x^{\mathrm{T}}Qx + y^{\mathrm{T}}Q^{-1}y$$

for any $x, y \in \mathbf{R}^n$.

22. Suppose $0 < p < 1$. Show that

$$\left(\sum_{i=1}^{n} x_i^p\right)^{1/p}$$

is a *concave* function in $\mathbf{R}^n_{++}$.

23. If $f(x)$ is twice continuously differentiable and quasi-convex, then for any $x \in \mathrm{dom}(f)$:

$$d^{\mathrm{T}}\nabla f(x) = 0 \Longrightarrow d^{\mathrm{T}}\nabla^2 f(x) d \ge 0.$$

24. Prove: If the above condition holds, then there must exist some real value α such that

$$\nabla^2 f(x) + \alpha \nabla f(x)(\nabla f(x))^{\mathrm{T}} \succeq 0.$$

[The Hessian matrix of a quasi-convex function can have at most one negative eigenvalue!]

25. For $X \in \mathcal{S}^{n\times n}$, its eigenvalues are denoted to be

$$\lambda_1(X) \ge \lambda_2(X) \ge \cdots \ge \lambda_{n-1}(X) \ge \lambda_n(X).$$

Let $1 \le k \le n$. Consider

$$f(X) := \sum_{i=1}^{k} \lambda_i(X).$$

Prove: $f(X)$ is a convex function.

Hint: Show that

$$f(X) = \sup\{\mathrm{tr}(U^{\mathrm{T}}XU) \mid U \in \mathbf{R}^{n\times k},\, U^{\mathrm{T}}U = I_k\}.$$

26. A function $f:\, \mathbf{R}^n_{++} \to \mathbf{R}$

$$h(x) = c x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$$

with $c > 0$ and $\lambda \in \mathbf{R}^n$ is called a *monomial*. Sum of monomials, $f(x) = \sum_{i=1}^{k} h_i(x)$, is called a *posynomial*.

The so-called *geometric programming* problem is as follows:

$$\begin{array}{lll} (G) & \min & f_0(x) \\ & \text{s.t.} & f_i(x) \le 1,\; i = 1, 2, ..., m \\ & & h_j(x) = 1,\; j = 1, 2, ..., p \end{array}$$

where $f_i(x)$ are posynomials ($i = 1, 2, ..., m$), and $h_j(x)$ are monomials ($j = 1, 2, ..., p$).

Show that (G) can be formulated as convex optimization through a variable transformation.

27. Formulate the following L_4-norm approximation problem as QCQP:

$$\min \ \|Ax-b\|_4 = \left(\sum_{i=1}^{m}(a_i^{\mathrm{T}}x-b_i)^4\right)^{1/4}.$$

28. The so-called *Chebyshev center* of a polyhedron is the deepest point inside the polyhedron. Suppose that the polyhedron is given by $P=\{x \mid a_i^{\mathrm{T}}x \le b_i,\, i=1,2,...,m\}$. Formulate the problem of finding the Chebyshev center of P by a convex optimization model.

29. An ellipsoid may be given by the image of a ball under some linear transformation, e.g. $E = \{Bu+b \mid \|u\|_2 \le 1\}$. Without losing generality we can also assume $B \succ 0$. Then the volume of E is proportional to $\det B$.

    Consider again the polyhedron $P=\{x \mid a_i^{\mathrm{T}}x \le b_i,\, i=1,2,...,m\}$. Now the problem is to find the maximum volume ellipsoid inscribed inside P. Formulate the problem by convex optimization.

30. Let $A_i \in \mathcal{S}^{n\times n}$, $i=1,2,...,m$. Therefore, $A_0+x_1A_1+\cdots+x_mA_m$ is a symmetric matrix. We wish to find the values of $x_1,...,x_m$ so as to minimize the gap between the largest and the smallest eigenvalues of $A_0+x_1A_1+\cdots+x_mA_m$. Formulate this problem by SDP.

31. Let

$$\mathcal{K} := \{x \in \mathbf{R}^n \mid x_1 \ge x_2 \ge \cdots \ge x_n \ge 0\}.$$

    Show that $\mathcal{K}$ is a proper cone.

32. Find $A \in \mathbf{R}^{n\times n}$ such that $\mathcal{K} = A\,\mathbf{R}^n_+$.

33. In general, if $\mathcal{K} \subseteq \mathbf{R}^n$ is a proper cone, and $M \in \mathbf{R}^{n\times n}$ is a non-singular matrix, then $M\,\mathcal{K}$ is also a proper cone.

34. Compute $\mathcal{K}^*$.

35. Derive the dual of the following *non-standard* conic optimization problem:

$$\begin{array}{ll} \min & c^{\mathrm{T}}x \\ \text{s.t.} & A_1x+b_1 \in \mathcal{K}_1 \\ & A_2x+b_2 \in \mathcal{K}_2 \\ & \vdots \\ & A_mx+b_m \in \mathcal{K}_m, \end{array}$$

    where $\mathcal{K}_1, \mathcal{K}_2, \cdots, \mathcal{K}_m$ are all closed convex cones.

36. Suppose that $f(x)$ is a convex function, and its conjugate function is known to be $f^*(s)$. Consider the following optimization model

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & Ax \le b. \end{array}$$

Derive the Lagrangian dual of the above problem.

37. The channel capacity optimization problem is:

$$\begin{array}{ll} \min & -c^{\mathrm{T}}x + \sum_{i=1}^m y_i \ln y_i \\ \text{s.t.} & Px = y \\ & x \ge 0,\ e^{\mathrm{T}}x = 1. \end{array}$$

What is the dual of the above problem?

38. The sum of first k largest components of vector $x \in \mathbf{R}^n$ ($k < n$) is known to be a convex function. (Why?) Denote this function to be $f(x)$. Formulate the following portfolio selection problem using $f(x)$: We wish to select from a total of n assets to form a portfolio (no short-selling is allowed). Asset i has an expected rate of return $\mu_i > 0$, and the covariance matrix is Σ. We wish to minimize the variance of the portfolio while requiring that the expected rate of return to the portfolio is at least μ. Moreover, the weight of the first k largest components of investment should not exceed half of the total investment.

39. The condition that $f(x) \le 0.5$ can be formulated by linear programming. How?