MAT4010: Functional Analysis
Homework 1

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Problem 2.1-14. Let Y be a subspace of a vector space X. The coset of an element $x \in X$ with respect to Y is denoted by $x + Y$ and is defined to be the set

$$x + Y = \{v \mid v = x + y, y \in Y\}$$

Show that under algebraic operations defined by

$$(w + Y) + (x + Y) = (w + x) + Y$$

$$\alpha(x + Y) = \alpha x + Y$$

these cosets constitute the elements of a vector space. This space is called the *quotient space* (or sometimes *factor space*) of X by Y (or modulo Y) and is denoted by X/Y. Its dimension is called the codimension of Y and is denoted by codim Y, that is,

$$\text{codim } Y = \dim(X/Y)$$

To prove all cosets consititute a vector space, we only need to check the standard definition. Firstly, consider $x_1 + Y$ and $x_2 + Y$ which are two arbitrary cosets in X/Y and $x_1, x_2 \in X$. We have

$$(x_1 + Y) + (x_2 + Y) = (x_1 + x_2) + Y = (x_2 + x_1) + Y = (x_2 + Y) + (x_1 + Y)$$

where the second equality is because X is a vector space and $x_1, x_2 \in X$. Also, consider arbitrary $x_3 + Y \in X/Y$,

$$\begin{aligned}
x + (y + z) &= (x_1 + Y) + [(x_2 + Y) + (x_3 + Y)] \\
&= (x_1 + Y) + [(x_1 + x_2) + Y] \\
&= [x_1 + (x_2 + x_3)] + Y \\
&= [(x_1 + x_2) + x_3] + Y \\
&= [(x_1 + x_2) + Y] + (x_3 + Y) \\
&= [(x_1 + Y) + (x_2 + Y)] + (x_3 + Y)
\end{aligned}$$

where the fourth equality is because x_1, x_2, x_3 are vectors in vector space X. Then we need to find the zero vector, which in this case is $\mathbf{0} + Y$, where $\mathbf{0} \in X$ is the zero vector of X. Then we have

$$(x_1 + Y) + (\mathbf{0} + Y) = (x_1 + \mathbf{0}) + Y = x_1 + Y$$

where the second equality is because $\mathbf{0}$ is zero vector in X and $x_1 \in X$. Similarly, we have

$$(x_1 + Y) + (-x_1 + Y) = [x_1 + (-x_1)] + Y = \mathbf{0} + Y$$

where $x_1 + (-x_1) = \mathbf{0}$ is due to the fact that X is vector space, and $x_1 \in X$, $\mathbf{0}$ is zero vector.

Now we verify the properties on scalar multiplication. Consider any a, b in the field over which X is defined. We have

$$a[b(x_1 + Y)] = a(bx_1 + Y) = [a(bx_1)] + Y = [(ab)x_1] + Y = (ab)(x_1 + Y)$$

where the third equality is because $x_1 \in X$ and X is a vector space. We also know $1x_1 = x_1$ because 1 is the unit scalar and x_1 is in the vector space X. Thus, we have

$$1(x_1 + Y) = (1x_1) + Y = x_1 + Y$$

Next, we consider the distributive laws

$$\begin{aligned} a[(x_1 + Y) + (x_2 + Y)] &= a[(x_1 + x_2) + Y] = [a(x_1 + x_2)] + Y = (ax_1 + ax_2) + Y \\ &= (ax_1 + Y) + (ax_2 + Y) = a(x_1 + Y) + a(x_2 + Y) \end{aligned}$$

where the third equality is because of the distirbutive law of x_1, x_2 in vector space X. Finally, we have

$$(a+b)(x_1+Y) = [(a+b)x_1]+Y = (ax_1+bx_1)+Y = (ax_1+Y)+(bx_1+Y) = a(x_1+Y)+b(x_1+Y)$$

where the second equality is due to the distributive law of x_1, x_2 in vector space X.

Therefore, X/Y is a vector space because it satisfies all of the defining properties of a vector space.

Problem 2.2-8. There are several norms of practical importance on the vector space of ordered n-tuples of numbers, notably those defined by

$$\|x\|_1 = |\xi_1| + |\xi_2| + \cdots + |\xi_n|$$

$$\|x\|_p = (|\xi_1|^p + |\xi_2|^p + \cdots + |\xi_n|^p)^{1/p}$$

$$\|x\|_\infty = \max\{|\xi_1|, \cdots, |\xi_n|\}$$

In each case, verify that the four properties of norm are satisfied.

Firstly, for the L^1-norm, since $|\xi_i| \geq 0$ and $|\xi_i| = 0 \iff \xi_i = 0$ for all $i = 1, \ldots, n$, we can conclude that $\|x\|_1 \geq 0$ and $\|x\|_1 = 0 \iff |\xi_i| = 0,\ \forall\, i \iff \xi_i = 0,\ \forall\, i \iff x = \mathbf{0}$, where $\mathbf{0}$ is the zero vector. Then consider any scalar a, we have $\|ax\|_1 = |a\xi_1| + \cdots + |a\xi_n|$, but since $|a\xi_i| = a|\xi_i|$, it is easy to conclude that $\|ax\|_1 = a|\xi_1| + \cdots + a|\xi_n| = a(|\xi_1| + \cdots + |\xi_n|) = a\|x\|_1$. For triangle inequality, consider any vector $y = (y_i)_{i=1}^n$,

$$\begin{aligned} \|x + y\|_1 &= |\xi_1 + y_1| + \cdots + |\xi_n + y_n| \\ &\leq (|\xi_1| + |y_1|) + \cdots + (|\xi_n| + |y_n|) \\ &= (|\xi_1| + \cdots + |\xi_n|) + (y_1 + \cdots + y_n) \\ &= \|x\|_1 + \|y\|_1 \end{aligned}$$

where the inequality is due to triangle inequality of absolute value (for number). Thus, $\|x\|_1$ satisfies all properties of a norm.

Then, for the L^p-norm where $1 < p < \infty$, since $|\xi_i|^p \geq 0$ and $|\xi_i|^p = 0 \iff \xi_i = 0$ for all $i = 1, \dots, n$, we can conclude that $\|x\|_p \geq 0$ and $\|x\|_p = 0 \iff |\xi_i|^p = 0,\ \forall\, i \iff \xi_i = 0,\ \forall\, i \iff x = \mathbf{0}$, where $\mathbf{0}$ is the zero vector. Then consider any scalar a, we have

$$\begin{aligned}\|ax\|_p &= (|a\xi_1|^p + \cdots + |a\xi_n|^p)^{1/p} \\ &= (|a|^p|\xi_1|^p + \cdots + |a|^p|\xi_n|^p)^{1/p} \\ &= |a|(|\xi_1|^p + \cdots + |\xi_n|^p)^{1/p} = |a|\|x\|_p\end{aligned}$$

For triangle inequality, consider any vector $y = (y_i)_{i=1}^n$,

$$\begin{aligned}\|x+y\|_p^p &= \sum_{i=1}^n |\xi_i + y_i||\xi_i + y_i|^{p-1} \\ &\leq \sum_{i=1}^n |\xi_i||\xi_i + y_i|^{p-1} + \sum_{i=1}^n |y_i||\xi_i + y_i|^{p-1} \\ &\leq \left(\sum_{i=1}^n |\xi_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n (|\xi_i + y_i|^{p-1})^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n (|\xi_i + y_i|^{p-1})^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\ &= \|x\|_p \left(\sum_{i=1}^n |\xi_i + y_i|^p\right)^{\frac{p-1}{p}} + \|y\|_p \left(\sum_{i=1}^n |\xi_i + y_i|^p\right)^{\frac{p-1}{p}} \\ &= (\|x\|_p + \|y\|_p)\|x+y\|_p^{p-1}\end{aligned}$$

where the first inequality is due to triangle inequality of absolute value (for numbers), and the second inequality is due to Hölder's inequality for L^p-space equipped with counting measure. Therefore, we can cancel out $\|x+y\|_p^{p-1}$ on both sides if $x + y \neq \mathbf{0}$, and then we obtain $\|x+y\|_p \leq \|x\|_p + \|y\|_p$. If $x + y = \mathbf{0}$, then $\|x+y\|_p \leq \|x\|_p + \|y\|_p$ will trivially hold. Thus, $\|x\|_1$ satisfies all properties of a norm.

Finally, for the L^∞-norm, since $|\xi_i| \geq 0$, the maximum of all ξ_i must be nonnegative, i.e., $\|x\|_\infty \geq 0$. Also $\|x\|_\infty = 0$ is equivalent to say the largest $|\xi_i|$ is zero, but since all $|\xi_i| \geq 0$, so it is equivalent to say all $|\xi_i| = 0$ and thus $\xi = 0$. Therefore, $\|x\|_\infty = 0 \iff x = \mathbf{0}$. Then consider any scalar a, we have

$$\|ax\|_\infty = \max\{|a\xi_1|, \cdots, |a\xi_n|\} = \max\{|a||\xi_1|, \cdots, |a||\xi_n|\} = |a| \max\{|\xi_1|, \cdots, |\xi_n|\} = |a|\|x\|_\infty$$

For triangle inequality, we need to first prove a claim that for $a_i, b_i \in \mathbb{R}$ for all $i = 1, \dots, n$,

$$\max\{a_1 + b_1, \cdots, a_n + b_n\} \leq \max\{a_1, \cdots, a_n\} + \max\{b_1, \cdots, b_n\}$$

This is because for all $i = 1, \dots, n$, we have $a_i \leq \max\{a_1, \cdots, a_n\}$ and $b_i \leq \max\{b_1, \cdots, b_n\}$, then $a_i + b_i \leq \max\{a_1, \cdots, a_n\} + \max\{b_1, \cdots, b_n\}$. Since for all $i = 1, \dots, n$, this is true, we can take the maximum over all i, it will still hold, and our claim is proved. Then consider any vector $y = (y_i)_{i=1}^n$,

$$\begin{aligned}\|x+y\|_\infty &= \max\{|\xi_1 + y_1|, \cdots, |\xi_n + y_n|\} \leq \max\{|\xi_1| + |y_1|, \cdots, |\xi_n| + |y_n|\} \\ &\leq \max\{|\xi_1|, \cdots, |\xi_n|\} + \max\{|y_1|, \cdots, |y_n|\} = \|x\|_\infty + \|y\|_\infty\end{aligned}$$

Therefore, $\|x\|_\infty$ satisfies all properties of a norm.

Problem 2.2-11. A subset A of a vector space X is said to be *convex* if $x, y \in A$ implies

$$M = \{z \in X \,|\, z = \alpha x + (1-\alpha)y,\ 0 \le \alpha \le 1\} \subset A$$

M is called a *closed segment* with *boundary points* x and y; any other $z \in M$ is called an *interior point* of M. Show that the *closed unit balls*

$$\tilde{B}(0;1) = \{x \in X \,|\, \|x\| \le 1\}$$

in a normed space X is convex.

Take arbitrary point $x, y \in \tilde{B}(0;1)$, for all $\alpha \in [0,1]$, consider

$$\|\alpha x + (1-\alpha)y\| \le \|\alpha x\| + \|(1-\alpha)y\| = |\alpha|\|x\| + |1-\alpha|\|y\|$$

Since α and $1-\alpha$ are both nonnegative, and x, y are both in the closed unit balls, we have

$$\|\alpha x + (1-\alpha)y\| \le |\alpha| + |1-\alpha| = 1$$

Thus $\alpha x + (1-\alpha)y \in \tilde{B}(0;1)$. This shows that $x, y \in \tilde{B}(0;1)$ implies $M \subset \tilde{B}(0;1)$ for M defined in the question, so $\tilde{B}(0;1)$ is a convex subset of X.

Problem 2.3-6. Show that the closure $\bar{Y}$ of a subspace Y of a normed space X is again a vector subspace.

Consider any point $x \in Y$, we can assign a sequence in Y that converges to it, i.e., $x_n \equiv x$ for all positive integer n. For any point $x \in \bar{Y} \setminus Y$, since $\bar{Y}$ is the closure of Y, these x must be a limit point of Y. Therefore, there must exist a sequence $x_n \in Y$ such that $x_n \to x$ as $n \to \infty$. In conclusion, for any $x \in \bar{Y}$, we can find a sequence $x_n \in Y$ such that $x_n \to x$ as $n \to \infty$.

Now we start to prove the subset is a subspace. First we prove the closedness of it under addition. Take arbitary $w, v \in \bar{Y}$, then there exists sequences $w_n, v_n \in Y$ such that $w_n \to w$ and $v_n \to v$. Since Y is a subspace, so $w_n + v_n \in Y$. From $w_n \to w$ and $v_n \to v$, for arbitrary ϵ, there exists N_1, N_2 such that $\|w_n - w\| < \epsilon/2$ for all $n \ge N_1$ and $\|v_n - v\| < \epsilon/2$ for all $n \ge N_2$. Therefore,

$$\|w_n + v_n - (w+v)\| \le \|w_n - w\| + \|v_n - v\| < \epsilon$$

Thus, $w + v$ is a limit point of a sequence $w_n + v_n$ which is in Y, i.e., $w + v \in \bar{Y}$.

Then we prove the closedness of it under scalar multiplication. Take arbitrary scalar a, for any $w \in \bar{Y}$, similarly we can find convergent sequence $w_n \in Y$ such that $w_n \to w$. Since Y is a subspace, so $aw_n \in Y$. Then for arbitrary ϵ, there exists N_1 such that $\|w_n - w\| < \epsilon$ for all $n \ge N_1$. Therefore,

$$\|aw_n - aw\| = |a|\|w_n - w\| < |a|\epsilon$$

Thus aw is a limit point of sequence aw_n, meaning that $aw \in \bar{Y}$.

Finally, we need to prove $\mathbf{0} \in \bar{Y}$, where $\mathbf{0}$ is the zero vector of X. This is trivial because Y is a subspace, so $\mathbf{0} \in Y \subset \bar{Y}$. Therefore, the closure $\bar{Y}$ is also a subspace of X.

Problem 2.3-12. A *seminorm* on a vector space X is a mapping $p : X \mapsto \mathbb{R}$ satisfying all properties of norm except the one $\|x\| = 0 \iff x = \mathbf{0}$. (Some authors call this a *pseudonorm.*) Show that

$$p(\mathbf{0}) = 0$$

$$|p(y) - p(x)| \leq p(y - x)$$

(Hence if $p(x) = 0$ implies $x = \mathbf{0}$, then p is a norm.)

Since p is a seminorm, we have $p(ax) = |a|p(x)$ for all scalar a. Thus we can fix any x and let $a = 0$, then since X is a vector space and $x \in X$, we have $ax = \mathbf{0}$ (zero vector of X) and since $p(x)$ is a real number, so $0 \cdot p(x) = 0$. Therefore, $p(\mathbf{0}) = 0$.

From the definition of seminorm, we also have $p(u + v) \leq p(u) + p(v)$ for all $u, v \in X$. Let $u = x - y$ and $v = y$, then we have $p(x) \leq p(x - y) + p(y)$, which is equivalent to $-p(x - y) \leq p(y) - p(x)$. Also notice that $p(x - y) = p(y - x)$, so we have $-p(y - x) \leq p(y) - p(x)$. Similarly, let $u = x$ and $v = y - x$, then we have $p(y) \leq p(x) + p(y - x)$, i.e., $p(y) - p(x) \leq p(y - x)$. In conclusion, we can obtain $|p(y) - p(x)| \leq p(y - x)$.

Problem 2.3-14. Let Y be a closed subspace of a normed space $(X, \|\cdot\|)$. Show that a norm $\|\cdot\|_0$ on X/Y is defined by

$$\|\hat{x}\|_0 = \inf_{x \in \hat{x}} \|x\|$$

where $\hat{x} \in X/Y$, that is, $\hat{x}$ is any coset of Y. Also prove that if X is complete, then so is X/Y.

Since $\hat{x} \in X/Y$, define $\hat{x} = u + Y$, then for any $x \in \hat{x}$, we can write $x = u + y$ for some $y \in Y$. Since $\|\cdot\|$ is a norm in X, $\|u + y\| \geq 0$ for all $y \in Y$, thus $\|\hat{x}\|_0 \geq 0$, and further, we have $\|\hat{x}\|_0 = \inf_{y \in Y} \|u + y\|$.

If $\hat{x} = \mathbf{0}_{X/Y}$, then $u = \mathbf{0}_X$, which means $\|\hat{x}\|_0 = \inf_{y \in Y} \|\mathbf{0} + y\| = \inf_{y \in Y} \|y\| = 0$ because $\mathbf{0} \in Y$. Conversely, if $\inf_{y \in Y} \|u + y\| = 0$, then there exists $y_n \in Y$ such that $\|y_n + u\| \to 0$, i.e., $y_n \to -u$. Since Y is closed, $-u \in Y$, so $u \in Y$. However, if $u \in Y$, then $u + Y = \mathbf{0}_X + Y = \mathbf{0}_{X/Y}$. Thus, we conclude that $\hat{x} = \mathbf{0}_{X/Y} \iff \|\hat{x}\|_0 = 0$.

Consider any scalar a, then $\|a\hat{x}\|_0 = \inf_{y \in Y} \|au + y\|$. If $a = 0$, then $\|a\hat{x}\|_0 = 0$ and $|a| \inf_{y \in Y} \|u + y\| = 0$, so $\|a\hat{x}\|_0 = |a| \|\hat{x}\|_0$. If $a \neq 0$, then

$$\|a\hat{x}\|_0 = \inf_{y \in Y} \|au + y\| = \inf_{y \in Y} |a| \|u + y/a\| = |a| \inf_{y \in Y} \|u + y/a\|$$

Notice that $\{y/a \,|\, y \in Y\} = Y$ because Y is a vector space, so

$$\|a\hat{x}\|_0 = |a| \inf_{y \in Y} \|u + y/a\| = |a| \inf_{y \in Y} \|u + y\| = |a| \|\hat{x}\|_0$$

Finally, also consider arbitrary $\hat{z} \in X/Y$ where $\hat{z} = v + Y$, $v \in X$. Since $\|\hat{x}\|_0$ is the greatest lower bound of $\|u + y\|$ for $y \in Y$, then for arbitrary small $\epsilon > 0$, there exists $y_1 \in Y$ such that $\|u + y_1\| < \|\hat{x}\|_0 + \epsilon$. Similarly, there exists $y_2 \in Y$ such that $\|v + y_2\| < \|\hat{z}\|_0 + \epsilon$. Therefore,

$$\|u + v + y^*\| \leq \|u + v + y_1 + y_2\| \leq \|u + y_1\| + \|v + y_2\| < \|\hat{x}\|_0 + \|\hat{z}\|_0 + 2\epsilon$$

where $y^* = y_1 + y_2 \in Y$. Therefore, we have

$$\|\hat{x} + \hat{z}\|_0 = \inf_{y \in Y} \|u + v + y\| \leq \|u + v + y^*\| < \|\hat{x}\|_0 + \|\hat{z}\|_0 + 2\epsilon$$

Take $\epsilon \to 0$, it yield $\|\hat{x} + \hat{z}\|_0 \leq \|\hat{x}\|_0 + \|\hat{z}\|_0$. Thus, $\|\cdot\|_0$ is a norm on X/Y.

Now we prove if X is complete, then X/Y is also complete. We first recall canonical projection $\pi(x) : X \mapsto X/Y$ with $\pi(x) = x + Y = \hat{x}$ for all $x \in X$. We can easily show that $\pi(x)$ is bounded, because $\|\pi(x)\|_0 = \inf_{y\in Y}\|x+y\| \leq \|x\|$ with $\mathbf{0}_X \in Y$. Since π is bounded, it must be continuous, i.e., if we have a sequence $x_n \in X$ converges to $x \in X$, then the corresponding sequence $\pi(x_n) \in X/Y$ converges to $\pi(x) \in X/Y$.

Next we construct a Cauchy sequence in X from a Cauchy sequence in X/Y. Take any Cauchy sequence in X/Y, denoted as $\hat{u}_n = u_n + Y$. Since it is Cauchy sequence in X/Y, we can find a subsequence of it such that for all $k \geq 1$,

$$\|\hat{u}_{n_{k+1}} - \hat{u}_{n_k}\|_0 = \inf_{y\in Y}\|u_{n_{k+1}} - u_{n_k} + y\| < \frac{1}{2^k}$$

This implies that $1/(2^k)$ is not a lower bound of $\|u_{n_{k+1}} - u_{n_k} + y\|$ for $y \in Y$. Therefore, for $k = 1$ there exists y_1 such that $\|u_{n_2} - u_{n_1} + y_1\| < \frac{1}{2}$. Define $x_{n_1} = u_{n_1} - y_1 \in X$, then $\hat{x}_{n_1} = u_{n_1} - y_1 + Y \in X/Y$. Notice that in fact $\hat{x}_{n_1} = \hat{u}_{n_1}$, so it belongs to the original sequence $\hat{u}_n$. By the definition of quotient norm and the fact that Y is a vector space, we have

$$\|\hat{u}_{n_2} - \hat{u}_{n_1}\|_0 = \inf_{y\in Y}\|u_{n_2} - u_{n_1} + y\| = \inf_{y\in Y}\|u_{n_2} - y - (u_{n_1} - y_1)\|$$

Therefore, by the same argument, we can find y_2 such that $\|u_{n_2} - y_2 - (u_{n_1} - y_1)\| < \frac{1}{2}$. Define $x_{n_2} = u_{n_2} - y_2 \in X$, then $\hat{x}_{n_2} = u_{n_2} - y_2 + Y \in X/Y$. Again, $\hat{x}_{n_2} = \hat{u}_{n_2}$, so it belongs to the original sequence $\hat{u}_n$. Then let $k = 2$ and obtain y_3 and $\hat{x}_{n_3}$ and so on. Finally, we can obtain a sequence $y_k \in Y$ and a subsequence $\hat{x}_{n_k}$ of $\hat{u}_n$ which satisfies

$$\|u_{n_{k+1}} - y_{k+1} - (u_{n_k} - y_k)\| < \frac{1}{2^k}$$

for $k = 1, 2, \ldots$. This implies that $u_{n_k} - y_k$ is a Cauchy sequence in X, and since X is complete, $u_{n_k} - y_k$ converges to $x \in X$.

Notice that $\pi(u_{n_k} - y_k) = \pi(u_{n_k}) = \hat{u}_{n_k}$, and from our previous argument, $\pi(u_{n_k} - y_k) \to \pi(x) \in X/Y$ since $u_{n_k} - y_k \to x$. Therefore, $\hat{u}_{n_k} = \pi(u_{n_k}) \to \pi(x) \in X/Y$. This shows that for any Cauchy sequence $\hat{u}_n \in X/Y$, it has a convergent subsequence in X/Y. Therefore, the whole sequence will also converge with the same limit as its subsequence, and this shows that X/Y is complete.

Problem 2.3-15. If $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ are normed spaces, show that the product vector space $X = X_1 \times X_2$ becomes a normed space if we define

$$\|x\| = \max(\|x_1\|_1, \|x_2\|_2) \qquad [x = (x_1, x_2)]$$

Since X is a vector space, we only need to prove the norm defined above satisfies the four defining properties. Firstly, since $\|x_1\|_1 \geq 0$ and $\|x_2\|_2 \geq 0$, $\|x\| = \max(\|x_1\|_1, \|x_2\|_2) \geq 0$. Secondly, we have

$$\|x\| = 0 \Longleftrightarrow \max(\|x_1\|_1, \|x_2\|_2) = 0 \Longleftrightarrow \|x_1\|_1 = \|x_2\|_2 = 0 \Longleftrightarrow x_1 = x_2 = 0 \Longleftrightarrow x = \mathbf{0}$$

Thirdly, for any scalar a, we have

$$\|ax\| = \max(\|ax_1\|_1, \|ax_2\|_2) = \max(|a|\|x_1\|_1, |a|\|x_2\|_2) = |a| \max(\|x_1\|_1, \|x_2\|_2) = |a|\|x\|$$

Lastly, according to the claim we proved in Problem 2.2-8, $\max\{a_1+b_1, a_2+b_2\} \le \max\{a_1, a_2\} + \max\{b_1+b_2\}$, thus for any $y = (y_1, y_2) \in X$, we have

$$\begin{aligned}\|x+y\| &= \max\{\|x_1+y_1\|_1, \|x_2+y_2\|_2\} \\ &\le \max\{\|x_1\|_1 + \|y_1\|_1, \|x_2\|_2 + \|y_2\|_2\} \\ &\le \max\{\|x_1\|_1, \|x_2\|_2\} + \max\{\|y_1\|_1, \|y_2\|_2\} = \|x\| + \|y\|\end{aligned}$$

Therefore, X becomes a norm space under the norm defined in the question.

Problem 2.4-1. Give examples of subspaces of l^∞ and l^2 which are not closed.

Consider set A to be the set of all vectors with only finitely many nonzero coordinates, i.e., there exists N such that for $x = (x_1, \cdots, x_n, \cdots) \in A$, $x_n = 0$ for all $n \ge N$. Then A is certainly a subset of l^∞ because $|x_i| \le c_x$ for all $i = 1, 2, \ldots$ where c_x is a constant depending on x. Therefore, we need to first prove A is a subspace and then not closed in l^∞-space.

Firstly, the zero vector of l^∞, $(0, 0, \cdots, 0, \cdots)$ is definitely in A because it has finitely many nonzero coordinates. Then consider any two $x, y \in A$, since there exists N_1 such that $x_n = 0$ for all $n \ge N_1$ and N_2 such that $y_n = 0$ for all $n \ge N_2$. Take $N = \max\{N_1, N_2\}$, then for all $x_n, y_n \ge N$, $x_n = y_n = 0$, and $x_n + y_n = 0$. This shows that $x + y$ has only finitely many nonzero coordinates, so $x + y \in A$. Finally, consider any scalar a, $ax = (ax_1, \cdots, ax_n, \cdots)$, since $x_n = 0$ for all $n \ge N_1$, $ax_n = 0$ for all $n \ge N_1$, which shows $ax \in A$. Therefore, A is a subspace of l^∞.

Consider a sequence $x^{(k)}$ defined as $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \cdots, x_n^{(k)}, \cdots)$ where $x_n^{(k)} = \frac{1}{n}$ for all $n \le k$ and $x_n^{(k)} = 0$ for all $n > k$. Then for each k, $x^{(k)} \in A$. Also define x^* as $x^* = (x_1^*, \cdots, x_n^*, \cdots)$ with $x_n^* = \frac{1}{n}$ for all n. Then it is easy to see that

$$\|x^{(k)} - x^*\|_\infty = \max\left\{0, 0, \cdots, 0, \frac{1}{k+1}, \frac{1}{k+2}, \cdots\right\} = \frac{1}{k+1}$$

Therefore, $\lim_{k\to\infty}\|x^{(k)} - x^*\|_\infty = 0$, which shows x^* is a limit point of $x^{(k)} \in A$, but x^* has infinitely many nonzero coordinates so it is not in A, then this implies that A is not closed.

Similarly for l^2-space, we consider the same set A defined as above. It is obvious that A is a subset of l^2-space because any element $x \in A$ satisfies that $|x_1|^2 + \cdots + |x_n|^2 + \cdots$ converges due to only finitely many nonzero x_i. Since we have proved that A is a subspace in l^∞-space, this implies that A is a vector space, and this property is independent on the norm you take, so A is also a subspace of l^2-space because it has been proved to be a subset of l^2-space. Then we only need to prove A is not closed in l^2-space.

Consider the same sequence $x^{(k)}$ and x_n^* defined above. Then it is easy to see that

$$\|x^{(k)} - x^*\|_2 = \left(\sum_{i=1}^{\infty} |x_i^{(k)} - x_i^*|^2\right)^{1/2} = \left(\sum_{i=k+1}^{\infty} \frac{1}{i^2}\right)^{1/2}$$

Since $\sum_{i=1}^{\infty} 1/i^2$ converges, the tail of the series must converge to zero, i.e., $\lim_{k\to\infty}\|x^{(k)} - x^*\|_\infty = 0$. This shows x^* is a limit point of $x^{(k)} \in A$, but x^* has infinitely many nonzero coordinates so it is not in A, then this implies that A is not closed.

Extra Problem 1. Let $\mathcal{C}^1[-1,1]$ be the set of continuously differentiable functions on $[-1,1]$, i.e., $\mathcal{C}^1[-1,1] = \{f \in \mathcal{C}[-1,1] \,|\, f' \text{ exists and is continuous on } [-1,1]\}$. Let

$$\|f\|_1 = \int_{-1}^{1} (|f|^2 + |f'|^2)^{1/2}\, dx$$

(i) Prove that $\| \ \|_1$ is a norm for $\mathcal{C}^1[-1,1]$.

Since for any $f \in \mathcal{C}^1[-1,1]$, $|f|^2 + |f'|^2 \geq 0$ is continuous in $[-1,1]$, so it must be integrable on $[-1,1]$ and thus $\|f\|_1$ is well-defined and nonnegative.

For any continuous function $g(x)$ which is nonnegative on $[-1,1]$, if $\int_{-1}^{1} g(x)\, dx = 0$, then $g(x) \equiv 0$ for $x \in [-1,1]$. Therefore, if $\|f\| = 0$, we have $|f|^2 + |f'|^2 = 0$, which further implies that $f = f' = 0$ for all $x \in [-1,1]$. Conversely if $f = 0$ for all $x \in [-1,1]$, then the integral $\int_{-1}^{1} (|f|^2 + |f'|^2)^{1/2}\, dx = 0$. Thus, $\|f\| = 0 \iff f \equiv 0$ where zero function is the zero vector in $\mathcal{C}^1[-1,1]$.

For any scalar a, consider

$$\|af\|_1 = \int_{-1}^{1} (|af|^2 + |(af)'|^2)^{1/2}\, dx = \int_{-1}^{1} (a^2|f|^2 + a^2|f'|^2)^{1/2}\, dx$$
$$= |a| \int_{-1}^{1} (|f|^2 + |f'|^2)^{1/2}\, dx = |a|\|f\|_1$$

For any function $f, h \in \mathcal{C}^1[-1,1]$, for any fixed x, we have

$$(|f+h|^2 + |f'+h'|^2)^{1/2} \leq (|f|^2 + |f'|^2)^{1/2} + (|h|^2 + |h'|^2)^{1/2}$$

This is because if we fix x, then f and g are constant, and consider Problem 2.2-8, let $p = 2$, then we have $\|u\|_2 = (|u_1|^2 + |u_2|^2)^{1/2}$ is a norm. Since it satisfies $\|u+v\|_2 \leq \|u\|_2 + \|v\|_2$, if we treat $u = (f, f')$ and $v = (h, h')$, then the above inequality will hold automatically. Since for each x the inequality holds, and both sides of them are nonnegative, so

$$\int_{-1}^{1} (|f+h|^2 + |f'+h'|^2)^{1/2}\, dx \leq \int_{-1}^{1} (|f|^2 + |f'|^2)^{1/2}\, dx + \int_{-1}^{1} (|h|^2 + |h'|^2)^{1/2}\, dx$$

Thus, $\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$. Therefore, $\| \ \|_1$ is a norm for $\mathcal{C}^1[-1,1]$.

(ii) Prove that $\mathcal{C}^1[-1,1]$ under the norm $\| \ \|_1$ is not complete. Hint: Let $f_n(x) = \sqrt{x^2 + (1/n)^2}$. Then $f_n(x) \to |x|$ in $\mathcal{C}^1[-1,1]$ under $\| \ \|_1$ as $n \to \infty$.

Consider $f_n(x) = \sqrt{x^2 + (1/n)^2}$, since $f_n'(x) = \frac{x}{\sqrt{x^2+(1/n)^2}}$ is a continuous function on $[-1,1]$, $f_n(x) \in \mathcal{C}^1[-1,1]$. Consider

$$\|f_n - |x|\|_{L^\infty} = \sup_{x\in[-1,1]} |f_n(x) - |x|| = \sup_{x\in[-1,1]} \frac{(1/n)^2}{\sqrt{x^2 + (1/n)^2} + |x|} = \frac{(1/n)^2}{(1/n)} = \frac{1}{n} \to 0$$

as $n \to \infty$. Thus, we also have $\|f_n - |x|\|_{L^1} \to 0$ as $n \to \infty$ because L^∞-norm is stronger than L^1-norm.

Now we suppose $\mathcal{C}^1[-1,1]$ under the norm $\| \ \|_1$ is complete, then any Cauchy sequence under the norm $\| \ \|_1$ is convergent. For all $\epsilon > 0$, consider all integers $n \geq m \geq 1/(2\epsilon)$, $x \in [-1,1]$,

$$|f_n(x) - f_m(x)| = \frac{|(1/n)^2 - (1/m)^2|}{\sqrt{x^2 + (1/n)^2} + \sqrt{x^2 + (1/m)^2}} \leq \frac{|(1/n)^2 - (1/m)^2|}{\sqrt{(1/n)^2} + \sqrt{(1/m)^2}} = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \epsilon$$

Similarly, consider

$$|f_n'(x) - f_m'(x)| = \frac{1}{\sqrt{1+1/(n^2x^2)}\sqrt{1+1/(m^2x^2)}} \cdot \frac{|1/m^2 - 1/n^2|}{\sqrt{x^2+(1/n)^2}+\sqrt{x^2+(1/m)^2}}$$
$$< 1 \cdot \left|\frac{1}{n} - \frac{1}{m}\right| \leq \epsilon$$

Therefore, under the norm $\| \ \|_1$,

$$\|f_n(x) - f_m(x)\|_1 < \int_{-1}^{1} (|f_n - f_m|^2 + |f_n' - f_m'|^2)^{1/2} \, dx \leq \int_{-1}^{1} (\epsilon^2 + \epsilon^2)^{1/2} \, dx = 2\sqrt{2}\epsilon$$

which shows f_n is Cauchy sequence under the norm $\| \ \|_1$. Then assume it converges to $g(x) \in \mathcal{C}^1[-1,1]$, we have $\|f_n(x) - g(x)\|_1 \to 0$ as $n \to \infty$. Notice that

$$\|f_n(x)-g(x)\|_{L^1} = \int_{-1}^{1} (|f_n-f_m|^2)^{1/2} \, dx \leq \int_{-1}^{1} (|f_n-f_m|^2+|f_n'-f_m'|^2)^{1/2} \, dx = \|f_n(x)-g(x)\|_1$$

Therefore, $\|f_n(x) - g(x)\|_{L^1} \to 0$ as $n \to \infty$. Combined with $\|f_n - |x|\|_{L^1} \to 0$, we have

$$\|g(x) - |x|\|_{L^1} \leq \|f_n(x) - g(x)\|_{L^1} + \|f_n(x) - |x|\|_{L^1}$$

Take limit on both sides as $n \to \infty$, $\|g(x) - |x|\|_{L^1} = 0$. Since $g(x)$ and $|x|$ are continuous function, and $g(x) = |x|$ almost everywhere, so it implies that $g(x) = |x|$, which is a contradiction, since $g(x)$ is continuously differentiable on $[-1,1]$, but $|x|$ is even not differentiable at $x = 0$. Thus, $\mathcal{C}^1[-1,1]$ under the norm $\| \ \|_1$ is not complete.

(iii) Prove that $\mathcal{C}^1[-1,1]$ under $\|f\| = \max_{x\in[-1,1]} |f(x)| + \max_{x\in[-1,1]} |f'(x)|$ is a Banach space.

Consider any Cauchy sequence $f_n \in \mathcal{C}^1[-1,1]$ under $\| \ \|$ defined above, for any $\epsilon > 0$, there exists M, for $n \geq m \geq M$, we have

$$\|f_n - f_m\|_{L^\infty} + \|f_n' - f_m'\|_{L^\infty} = \max_{x\in[-1,1]} |f_n - f_m| + \max_{x\in[-1,1]} |f_n' - f_m'| = \|f_n - f_m\| < \epsilon$$

Therefore, f_n and f_n' are both Cauchy sequence under L^∞-norm. Since we know $\mathcal{C}[-1,1]$ under L^∞-norm is complete, so $f_n \to f$ and $f_n' \to g$ uniformly with $f, g \in \mathcal{C}[-1,1]$. Since f_n' and g are continuous, they are integrable, and for $x \in [-1,1]$,

$$\int_{-1}^{x} g(t) \, dt = \lim_{n\to\infty} \int_{-1}^{x} f_n'(t) \, dt = \lim_{n\to\infty} f_n(x) - f(-1) = f(x) - f(-1)$$

Take the derivative with respect to x on both sides, we can obtain $g(x) = f'(x)$.

Therefore, we have $\|f_n - f\|_{L^\infty} \to 0$ and $\|f_n' - f'\|_{L^\infty} \to 0$ as $n \to \infty$. Notice that $f' = g \in \mathcal{C}[-1,1]$, so $f \in \mathcal{C}^[-1,1]$, and thus we can consider

$$\|f_n - f\| = \|f_n - f\|_{L^\infty} + \|f_n' - f'\|_{L^\infty}$$

Since the right hand sides tends to zero as $n \to \infty$, we have $\|f_n - f\| \to 0$ as well, thus $f_n \to f$ under norm $\|f\| = \max_{x\in[-1,1]} |f(x)| + \max_{x\in[-1,1]} |f'(x)|$. Therefore, normed space $\mathcal{C}^1[-1,1]$ is Banach under such norm.

Extra Problem 2. Given a function $f \in L^p(a,b)$ $(1 \le p \le \infty)$, suppose we want to approximate f by a polynomial $a_0 + a_1 x + \cdots + a_n x^n$ of degree $\le n$, in L^p-norm. Then what should be the values of $a_0, a_1, \cdots, a_n$? In other words, let $f_0(x) = 1$, $f_1(x) = x$, $\cdots$, $f_n(x) = x^n$. We want to find a_0, a_1, $\cdots$, a_n that minimize $\|f - (a_0 f_0 + a_1 f_1 + \cdots + a_n f_n)\|_{L^p(a,b)}$. In general, let X be a normed space over $\mathbb{R}$, and let $e_1, \cdots, e_n \in X$ be linearly independent. Given $x \in X$, find $\bar{a}_1, \cdots, \bar{a}_n \in \mathbb{R}$ such that

$$\|x - (\bar{a}_1 e_1 + \cdots + \bar{a}_n e_n)\| = \min_{a_1,\cdots,a_n \in \mathbb{R}} \|x - (a_1 e_1 + \cdots + a_n e_n)\|$$

It is not possible to find a formula for the minimizer $(\bar{a}_1, \cdots, \bar{a}_n)$ for a general normed space X (it would be easy, if X is a Hilbert space). But it is possible to prove the existence of $(\bar{a}_1, \cdots, \bar{a}_n)$. Define $F(a_1, \cdots, a_n) = \|x - (a_1 e_1 + \cdots + a_n e_n)\|$, $\forall\, (a_1, \cdots, a_n) \in \mathbb{R}^n$.

(i) Prove that F is continuous on $\mathbb{R}^n$;

For linearly independent e_i in normed space X, there exists $c_1, c_2 > 0$ such that for all $(a_1, \ldots, a_n) \in \mathbb{R}^n$, $c_1(|a_1| + \cdots + |a_n|) \le \|a_1 x_1 + \cdots + a_n x_n\| \le c_2(|a_1| + \cdots + |a_n|)$. Thus, any two norms of $\mathbb{R}$ and $\mathbb{R}^n$ are equivalent, then W.O.L.G., we only consider Taxicab norm $\|a\|_1 = |a_1| + \cdots + |a_n|$ for $\mathbb{R}$ and $\mathbb{R}^n$. Fixed $(a_1^*, \cdots, a_n^*) \in \mathbb{R}^n$, for all $(a_1, \cdots, a_n) \in \mathbb{R}^n$ such that $\|(a_1, \cdots, a_n) - (a_1^*, \cdots, a_n^*)\|_1 < \delta$, we have

$$\begin{aligned}
\|F(a_1^*, \cdots, a_n^*) - F(a_1, \cdots, a_n)\|_1 &= \big| \|x - (a_1^* e_1 + \cdots + a_n^* e_n)\| - \|x - (a_1 e_1 + \cdots + a_n e_n)\| \big| \\
&\le \|(a_1 - a_1^*) e_1 + \cdots + (a_n - a_n^*) e_n)\| \\
&\le C_2(|a_1 - a_1^*| + \cdots + |a_n - a_n^*|) \\
&= C_2 \|(a_1 - a_1^*, \cdots, a_n - a_n^*)\|_1 < C_2 \delta
\end{aligned}$$

Thus, for arbitrary $\epsilon > 0$, there exists $\delta = \epsilon / C_2$, where C_2 is a constant only depend on $e_1, \cdots, e_n$, such that $\|F(a_1^*, \cdots, a_n^*) - F(a_1, \cdots, a_n)\|_1 < \epsilon$. This implies that F is continuous on $\mathbb{R}^n$.

(ii) Prove that $F(a_1, \cdots, a_n) \to \infty$ as $|(a_1, \cdots, a_n)| \to \infty$, where $|(a_1, \cdots, a_n)| = |a_1| + \cdots + |a_n|$ is a norm of $\mathbb{R}^n$.

Using the LHS of the fact, there exists $C_1 > 0$ such that

$$F(a_1, \cdots, a_n) = \|x - (a_1 e_1 + \cdots + a_n e_n)\| \ge \|a_1 e_1 + \cdots + a_n e_n\| - \|x\| \ge C_1(|a_1| + \cdots + |a_n|) - \|x\|$$

Since $\|x\|$ is fixed and $|(a_1, \cdots, a_n)| \to \infty$, it is east to see that $F(a_1, \cdots, a_n) \to \infty$.

(iii) Prove that F has a global minimum point $(\bar{a}_1, \cdots, \bar{a}_n)$.

Since F is continuous on $\mathbb{R}^n$, then $F(0, \cdots, 0) = \|x\|$ is a finite and fixed number. Thus, the set

$$A = \{(a_1, \cdots, a_n) \in \mathbb{R}^n \,|\, F(a_1, \cdots, a_n) \le \|x\|\}$$

is nonempty. Also, A is bounded because if not, then there exists $(a_1, \cdots, a_n) \in A$ and $|(a_1, \cdots, a_n)| \to \infty$ implies $F(a_1, \cdots, a_n) \to \infty$, which contradicts to $F(a_1, \cdots, a_n) \le \|x\|$. Furthermore, A is closed because A is pre-image of a closed set under continuous function F.

Therefore, in $\mathbb{R}^n$, closed and bounded set A is compact. Continuous function F on compact set A must have its global minimum. Since F on $\mathbb{R}^n \setminus A$ has value larger than $\|x\|$, which is the upper bound of any value of F on A, the global minimum of F on A is the global minimum of it over $\mathbb{R}^n$.

Extra Problem 3. Let X be a vector space equipped with a metric $d(x, y)$. Suppose $d(x, y)$ satisfies

- Translation invariance: $d(x + z, y + z) = d(x, y)$, $\forall\, x, y, z \in X$.
- Degree 1 homogenity: $d(\alpha x, \alpha y) = |\alpha| d(x, y)$, $\forall\, \alpha \in F$, where $F = \mathbb{R}$ or $F = \mathbb{C}$ is the field over which X is a vector space.

Prove that X is a normed space.

Since X is a vector space, take zero vector $\mathbf{0} \in X$, and we can define the norm $\|x\| = d(x, \mathbf{0})$ for any $x \in X$. Then we need to check the norm satisfies four defining properties. Firstly, $\|x\|$ because metric function $d(x, \mathbf{0}) \geq 0$. Secondly, $\|x\| = 0 \iff d(x, \mathbf{0}) = 0 \iff x = \mathbf{0}$ by the definition of metric function d. Thirdly, consider any scalar a, we have $\|ax\| = d(ax, \mathbf{0}) = |a| d(x, \mathbf{0}) = |a| \|ax\|$, where the second equality is due to degree 1 homogenity of d. Finally, for any $x, y \in X$, we have

$$\|x + y\| = d(x + y, \mathbf{0}) = d(x, -y) \leq d(x, \mathbf{0}) + d(\mathbf{0}, -y) = \|x\| + d(y, \mathbf{0}) = \|x\| + \|y\|$$

where the second and third equality is due to translation invariance, and the inequality here is due to triangle inequality of metric function. Thus, X is a normed space.

Extra Problem 4. Consider $L^p(a, b)$ when $0 < p < 1$, $p' = \frac{p}{p-1} < 0$. Let's agree that if $g = 0$ a.e. on (a, b), then $\|g\|_{L^{p'}} = 0$. With this agreement,

(i) Prove the reversed Hölder's inequality

$$\|fg\|_{L^1(a,b)} \geq \|f\|_{L^p(a,b)} \|g\|_{L^{p'}(a,b)}, \quad \forall f \in L^p(a, b), g \in L^{p'}(a, b) \tag{$*$}$$

Hint: If $g = 0$ a.e. on (a, b), then it is trivial to prove $(*)$. Assume $|g| > 0$ a.e. on (a, b). Let $q = 1/p$, $q' = q/(q - 1)$, and $u = |fg|^p$ and $v = |g|^{-p}$. Then $uv = |f|^p$. Apply Hölder's inequality to $\int_a^b uv\, dx$ using q and q'.

According to the hint, if $g = 0$ a.e. on (a, b), then it is trivial to prove the inequality because both left and right sides are zero. Assume $|g| > 0$ on a subset of (a, b) with positive measure, and let $q = 1/p$, $q' = q/(q - 1)$, and $u = |fg|^p$ and $v = |g|^{-p}$. Since $q > 1$, from Hölder's inequality, we have

$$\|uv\|_{L^1(a,b)} \le \|u\|_{L^q(a,b)}\|v\|_{L^{q'}(a,b)} \iff \int_a^b uv\,dx \le \left(\int_a^b u^q\,dx\right)^{\frac{1}{q}} \left(\int_a^b v^{q'}\,dx\right)^{\frac{1}{q'}}$$
$$\iff \int_a^b |f|^p\,dx \le \left(\int_a^b |fg|\,dx\right)^p \left(\int_a^b |g|^{p'}\,dx\right)^{1-p}$$
$$\iff \left(\int_a^b |f|^p\,dx\right)^{\frac{1}{p}} \le \left(\int_a^b |fg|\,dx\right) \left(\int_a^b |g|^{p'}\,dx\right)^{-\frac{1}{p'}}$$
$$\iff \left(\int_a^b |f|^p\,dx\right)^{\frac{1}{p}} \left(\int_a^b |g|^{p'}\,dx\right)^{\frac{1}{p'}} \le \int_a^b |fg|\,dx$$

where the last step is valid because $|g| > 0$ on a subset of (a, b) with positive measure, so the integral of $|g|^{p'}$ is positive. Therefore, we obtain the reversed Hölder inequality $\|f\|_{L^p(a,b)}\|g\|_{L^{p'}(a,b)} \le \|fg\|_{L^1(a,b)}$ for all $f \in L^p(a, b)$ and $g \in L^{p'}(a, b)$.

(ii) Prove the reversed Minkowski inequality

$$\|f\|_{L^p} + \|g\|_{L^p} \le \|f + g\|_{L^p}, \quad \forall f, g \in L^p(a, b), f \ge 0, g \ge 0$$

Notice that for $f \ge 0, g \ge 0$, we have $|f + g| = |f| + |g|$, and then we have

$$\begin{aligned}\int_a^b |f+g|^p &= \int_a^b |f||f+g|^{p-1} + \int_a^b |g||f+g|^{p-1} \\ &\ge \left(\int_a^b |f|^p\,dx\right)^{\frac{1}{p}} \left(\int_a^b |f+g|^p\,dx\right)^{\frac{p-1}{p}} + \left(\int_a^b |g|^p\,dx\right)^{\frac{1}{p}} \left(\int_a^b |f+g|^p\,dx\right)^{\frac{p-1}{p}} \\ &= \|f\|_{L^p}\|f+g\|_{L^p}^{p-1} + \|g\|_{L^p}\|f+g\|_{L^p}^{p-1}\end{aligned}$$

where the inequality follows from reversed Hölder inequality. If $f + g > 0$ on a subset with positive measure, then $\|f + g\|_{L^p}^{p-1} > 0$ and by cancelling that factor, we obtain the reversed Minkowski inequality. If $f + g = 0$ a.e., since $f \ge 0$ and $g \ge 0$, $f = 0$ a.e. and $g = 0$ a.e.. In this case $\|f\|_{L^p} = \|g\|_{L^p} = 0$ and $\|f + g\|_{L^p} = 0$, so the both sides are equal. In conclusion, the reversed Minkowski inequality holds for all $f, g \in L^p(a, b), f \ge 0, g \ge 0$.

(iii) Let I_1, I_2 be finite subintervals of (a, b) such that $I_1 \cap I_2 = \varnothing$ and $m(I_1) = m(I_2)$. Let $f = \chi_{I_1}$ be the characteristic function of I_1, and $g = \chi_{I_2}$. Prove the reversed strict Minkowski inequality.

$$\|f\|_{L^p} + \|g\|_{L^p} < \|f + g\|_{L^p}$$

Since $0 < p < 1$, $2^{1/p} > 2$. Let $m(I_1) = m(I_2) = l > 0$ assuming I_1, I_2 are not empty. Therefore, $\|f\|_{L^p} = \left(\int_{I_1} 1^p\,dx\right)^{1/p} = l^{1/p}$, and similarly, $\|g\|_{L^p} = l^{1/p}$. Also, since I_1 and I_2 are disjoint, $\|f + g\|_{L^p} = (2l)^{1/p}$. Therefore, $2l^{1/p} < 2^{1/p}l^{1/p} = (2l)^{1/p}$, meaning that the reversed strict Minkowski inequality holds.

(iv) Prove $\forall\, f, g \in L^p(a,b)$,

$$\int_a^b |f-g|^p \; dx \le \int_a^b |f|^p \; dx + \int_a^b |g|^p \; dx$$

then use this to prove $L^p(a,b)$ is metric space with metric $d(f,g) = \int_a^b |f-g|^p \; dx$. Hint: $\forall\, a, b \ge 0,\; a^p + b^p \ge (a+b)^p$.

Since $|f-g| \le |f| + |g|$, we have $|f-g|^p \le (|f|+|g|)^p \le |f|^p + |g|^p$ for $0 < p < 1$. Therefore,

$$\int_a^b |f-g|^p \; dx \le \int_a^b |f|^p \; dx + \int_a^b |g|^p \; dx$$

holds automatically. Firstly, $d(f,g) \ge 0$, and $d(f,g) = 0$ if and only if $f = g$ almost everywhere. Trivially, $d(f,g) = d(g,f)$ because $|f-g| = |g-f|$ for all $x \in [a,b]$. For triangle inequality, consider $h \in L^p(a,b)$,

$$d(f,g) = \int_a^b |f-g|^p \; dx \le \int_a^b |f-h|^p \; dx + \int_a^b |g-h|^p \; dx = d(f,h) + d(g,h)$$

because $f - g = (f-h) - (g-h)$. Therefore, $L^p(a,b)$ with $0 < p < 1$ under metric d defined in the question is a metric space.