MAT4010: Functional Analysis
Homework 10

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Due date: Nov. 19, 2019

Problem 7.2-5. Let (e_k) be a total orthonormal sequence in a separable Hilbert space H and let $T : H \mapsto H$ be defined at e_k by $Te_k = e_{k+1}$ for all $k \geq 1$, and then linearly and continuously extended to H. Find invariant subspaces. Show that T has no eigenvalues.

Invariance suspaces are $Y_n = \overline{\text{span}\{e_k\}_{k\geq n}}$. For any $x \in Y_n$, $x = \sum_{k=n}^{\infty} a_k e_k$, and

$$T(x) = T\left(\sum_{k=n}^{\infty} a_k e_k\right) = \sum_{k=n}^{\infty} a_k T(e_k) = \sum_{k=n}^{\infty} a_k e_{k+1} \in Y_n$$

To show T has no eigenvalue, only need to show $\lambda I - T$ is injective, i.e., $(\lambda I - T)x = 0$ implies $x = 0$. This is because

$$(\lambda I - T)x = a_1 \lambda e_1 + \sum_{k=2} (a_k \lambda - a_{k-1}) e_k = 0$$

implies that $a_1 \lambda = 0$ and $a_{i+1}\lambda - a_i = 0$ for all $i \geq 1$. If $\lambda = 0$, then $a_i = 0$ automatically for all $i \geq 1$. If $\lambda \neq 0$, then $a_1 = 0$, but $a_2 \lambda = 0$ implies $a_2 = 0$. Therefore, by this process, $a_i = 0$ for all $i \geq 1$, this shows $\lambda I - T$ is injective, so T has no eigenvalue.

Problem 7.3-2. Find a linear operator $T : \mathcal{C}[0, 1] \mapsto \mathcal{C}[0, 1]$ whose spectrum is a given interval $[a, b]$.

Define T by $T(f(x)) = [(b - a)x + a]f(x)$, then it is easy to see T is linear. T is bounded because (denote $\max(|a|, |b|)$ as c)

$$|T(f(x))| \leq |f(x)||(b - a)x + a| \leq c|f(x)| \leq c\|f\||x| \leq c\|f\|$$

For simplicity, denote $\tilde{x} = (b-a)x+a$. Note that $\lambda I - T$ is always injective, since if $(\lambda I - T)f(x) = (\lambda - \tilde{x})f(x) = 0$ implies that $f(x) = 0$ for all $\tilde{x} \neq \lambda$, but $f(x)$ is continuous, so $f(x) = 0$ for all $x \in [0, 1]$. If $\lambda < a$ or $\lambda > b$, $\lambda I - T$ is surjective, because $\lambda - \tilde{x} \neq 0$. If $\lambda \in [a, b]$, then $g(x) = 1$ for $x \in [0, 1]$ is not in range of $\lambda I - T$, because $(\lambda - \tilde{x})f(x) = 0$ at $\tilde{x} = \lambda$. This shows that resolvent of T is $[a, b]^c$ and spectrum is $[a, b]$.

Problem 7.3-3. If Y is the eigenspace corresponding to an eigenvalue λ of an operator T, what is the spectrum of $T\big|_Y$?

Suppose Y corresponding to eigenvalue λ_0, then for all $y \in Y$, we have $(\lambda_0 I - T)y = \mathbf{0}$. This shows that $T\big|_Y = \lambda_0 I$. Therefore, consider $\lambda I - \lambda_0 I$, it is $(\lambda - \lambda_0)I$. This operator is obviously invertible if $\lambda \neq \lambda_0$, and obviously not surjective if $\lambda = \lambda_0$. This implies that the spectrum of $\lambda_0 I$ is $\{\lambda_0\}$, i.e., the spectrum of $T\Big|_Y$ is $\{\lambda_0\}$.

Problem 7.3-4. Let $T : l^2 \mapsto l^2$ be defined by $y = Tx$, $x = (\xi_j)$, $y = (\eta_j)$, $\eta_j = \alpha_j \xi_j$, where (α_j) is dense in $[0, 1]$. Find $\sigma_p(T)$ and $\sigma(T)$.

Consider $(\lambda I - T)x = \lambda x - Tx = ((\lambda - \alpha_1)\xi_1, (\lambda - \alpha_2)\xi_2, \cdots)$, if $\lambda \in [0, 1]^c$, then $|\lambda - \alpha_i|$ is bounded away from zero, and $\lambda I - T$ is surjective, because for any $y \in l^2$, we can find $z = (\eta_j/(\lambda - \alpha_j))$ such that $Tz = y$. To see $z \in l^2$, consider

$$\sum_{j=1}^{\infty} \frac{|\eta_j|^2}{|\lambda - \alpha_j|^2} \leq \frac{1}{c^2} \sum_{j=1}^{\infty} |\eta_j|^2 < \infty$$

because $y \in l^2$ and given any fixed $\lambda \in [0, 1]^c$, $|\lambda - \alpha_j|^2 \geq c$ for all j. It is obvious that $\lambda I - T$ is injective because each $\lambda - \alpha_j \neq 0$, so to make $(\lambda I - T)x = 0$, each ξ_j must be zero.

For all $\lambda \in [0, 1]$, since α_j is dense in $[0, 1]$, there always exists a subsequence α_{j_k} of α_j such that $\alpha_{j_k} \to \lambda$. For all n, we can select α_{j_k} such that $|\alpha_{j_k} - \lambda| < \frac{1}{2^n}$, denote such α_{j_k} as $\alpha_{j_{k_n}} = b_n$. Then consider y constructed by $\eta_j = 1/2^j$ if $j = j_{k_n}$, and otherwise 0, then it is obvious that $y \in l^2$. However, the preimage of y is $z = (\eta_j)$ satisfies $\sum_{j=1} |\eta_j|^2 \geq \sum_{n=1}^{\infty} 1 = \infty$, i.e., $z \notin l^2$. This shows that as long as $\lambda \in [0, 1]$, T cannot be surjective. In conclusion, $\sigma_p(T) = [0, 1]^c$, and $\sigma(T) = [0, 1]$.

Problem 7.3-9. Let $T : l^\infty \mapsto l^\infty$ be defined by $x \mapsto (\xi_2, \xi_3, \ldots)$, where x is given by $x = (\xi_1, \xi_2, \ldots)$. If $|\lambda| > 1$, show that $\lambda \in \rho(T)$. If $|\lambda| \leq 1$, show that λ is an eigenvalue and find the eigenspace Y.

Notice that $(\lambda I - T)x = (\lambda\xi_1 - \xi_2, \lambda\xi_2 - \xi_3, \cdots)$. If $|\lambda| \leq 1$, then $(\lambda I - T)$ is not injective, because for any element x in $\text{span}\{(1, \lambda, \lambda^2, \cdots)\}$, $(\lambda I - T)x = \mathbf{0}_{l^\infty}$. Thus, if $|\lambda| \leq 1$, λ is an eigenvalue. To find the eigenspace, consider $\lambda\xi_i = \xi_{i+1}$ for all $i \geq 1$, it is easy to see $\xi_n = \lambda^{n-1}\xi_1$ and since $\lambda \leq 1$, so $x \in l^\infty$. Therefore, if $\xi_1 \neq 0$, we conclude that such x is an eigenvector. Thus the eigenspace Y of λ is one dimensional, and $Y = \text{span}\{(1, \lambda, \lambda^2, \cdots)\}$.

However, if $|\lambda| > 1$, if $\xi_1 \neq 0$, then $|\xi_n| \to \infty$, which is impossible, so $\xi_1 = 0$ for all x satisfying $(\lambda I - T)x = \mathbf{0}_{l^\infty}$. This implies that $\xi_j = 0$ for all j, i.e., $x = \mathbf{0}_{l^\infty}$, thus injectivity is verified. To see surjectivity, consider $y \in l^\infty$, to ensure $Tx = y$, we need to find ξ_1 such that $\xi_n = \lambda^{n-1}\xi_1 - \sum_{j=1}^{n-1} y_j \lambda^{n-1-j}$ to be bounded for all n. Take $\xi_1 = \lim_{k\to\infty} \frac{1}{\lambda^{k-1}} \sum_{j=1}^{k-1} y_j \lambda^{k-1-j}$, since $|\lambda| > 1$, this limit exists because $|y_j| \leq M$ for all j, and the summand is geometric series, so the series converges and limit exists. It is not hard to see $|\epsilon_n| \leq \frac{M}{|\lambda|-1} < \infty$ where $M = \sup_j |y_j|$.

Extra Problem 1. Let X be a Banach space and M be a closed subspace of X. Let $N : X \mapsto X \setminus M$ be the natural mapping, i.e., $Nx = x + M$. Prove that N is an open mapping.

Since X is Banach, and M is closed subspace of X, so M is Banach. By HW3, $X \setminus M$ is also Banach. Also, N is obviously linear. N is bounded because $\|x + M\|_0 = \inf_y \|x + y\| \leq \|x\|$. Since N is obviously onto, so by open mapping theorem, N maps open sets to open sets.

Extra Problem 2. Let X and Y be Banach, and $T : X \mapsto Y$ be linear, bounded and onto. Prove that if $y_n \to y_0$ in Y as $n \to \infty$, then there exists constant $c > 0$ and $x_n \in X$ such that $Tx_n = y_n$ and $x_n \to x_0$, $\|x_n\| \le c\|y_n\|$, $\forall\, n \ge 1$.

Consider $\bar{T} : X/\mathcal{N}(T) \mapsto Y$ defined by $\bar{T}(x + \mathcal{N}(T)) = Tx$. Then $\bar{T}$ is obviously linear. It is bounded because for all $z \in \mathcal{N}(T)$,

$$\bar{T}(x + \mathcal{N}(T)) = T(x) = T(x + z) \le \|T\|\|x + z\|_X$$

Take infimum over z on both sides, we have

$$\bar{T}(x + \mathcal{N}(T)) \le \|T\| \inf_z \|x + z\|_X = \|T\|\|x + \mathcal{N}(T)\|_{X\setminus\mathcal{N}(T)}$$

Also, $\bar{T}$ is obviously surjective because T is onto. $\bar{T}$ is injective by construction of quotient space. Thus, by bounded inverse mapping theorem, $\bar{T}^{-1}$ exists and is linear and bounded. Since $y_n \to y_0$, there exists $\hat{x}_n \to \hat{x}_0$ where $\hat{x}_n = \bar{T}^{-1}(y_n) \in X \setminus \mathcal{N}(T)$ and $\hat{x}_0 = \bar{T}^{-1}(y_0)$. If $y_0 = \mathbf{0}_Y$, then $\hat{x}_0 = \mathbf{0}_{X\setminus\mathcal{N}(T)}$. This implies,

$$\|x_n\|_X \le 2\|\hat{x}_n\| = 2\|\bar{T}^{-1}(y_n)\| \le 2\|\bar{T}^{-1}\|\|y_n\|_Y$$

In this case $c = 2\|\bar{T}^{-1}\|$. If $y_0 \ne \mathbf{0}_Y$, then $\|y_0\|_Y > 0$. We can choose x_n and x_0 such that $\|x_n - x_0\|_X \le 2\|\hat{x}_n - \hat{x}_0\|$. Suppose c does not exist, and there exists a subsequence x_{n_k} of x_n such that $\|x_{n_k}\|_X > k\|y_{n_k}\|_Y$, i.e., $\frac{\|x_{n_k}\|_X}{k} > \|y_{n_k}\|_Y$. Since x_n is bounded, take $k \to \infty$, we obtain $0 \ge \|y_0\|_Y$. This is a contradiction to $\|y_0\|_Y > 0$. Thus, $\|x_n\|_X \le c\|y_n\|_Y$ for all but finitely many terms. For these finitely many terms, if $y_n \ne \mathbf{0}_Y$, we can choose large c so that $\|x_n\|_X \le c\|y_n\|_Y$ holds; if $y_n = \mathbf{0}_Y$, then we choose $x_n = \mathbf{0}_X$.

Extra Problem 3. Let X and Y be normed spaces and $A : X \mapsto Y$ be linear, closed, and injective. Prove

(i) $A^{-1} : \mathcal{R}(A) \mapsto X$ is also closed, given $\mathcal{R}(A)$ is a normed space.

By definition, $G_A = \{(x, Ax)\,|\, x \in X\}$, and $G_{A^{-1}} = \{(y, A^{-1}y)\,|\, y \in \mathcal{R}(A)\}$. Consider a convergent sequence in $G_{A^{-1}}$, i.e., $(y_n, A^{-1}y_n)$ where $y_n \to y$ and $A^{-1}y_n \to u$. There exists unique $x_n \in X$ such that $Ax_n = y_n \to y$, and $x_n \to u$. Since G_A is closed, we have $(u, y) \in G_A$. This shows $y = Au$, i.e., $u = A^{-1}y$. Therefore, $G_{A^{-1}}$ is also closed.

(ii) If X is Banach, $\mathcal{R}(A)$ is dense in Y and $A^{-1} : \mathcal{R}(A) \mapsto X$ is continuous, then $\mathcal{R}(A) = Y$.

Since $\mathcal{R}(A)$ is dense in Y, for all $y \in Y$, there exists $y_n \in \mathcal{R}(A)$ such that $y_n \to y$. Since $y_n \in \mathcal{R}(A)$, we have $Ax_n = y_n$, where $x_n \in X$. Since A^{-1} is continuous and linear, thus bounded (so Lipschitz continuous), $y_n \to y$ implies $A^{-1}y_n$ is Cauchy. From the fact that X is Banach, $A^{-1}y_n \to x_0 \in X$. Then since A^{-1} is closed, (x_0, y) must be on the graph of A^{-1}, hence $A^{-1}(y) = x_0$. Since y is arbitrarily chosen, $\mathcal{R}(A) = Y$.

Extra Problem 4. Let X and Y be Banach spaces, and $A : X \mapsto Y$ be linear and closed. Prove

(i) $\mathcal{N}(A)$ is closed;

Take a convergent sequence $x_n \in \mathcal{N}(A)$ such that $x_n \in x \in X$, then $Ax_n = 0$. Since A is closed, $(x, 0)$ is also on the graph of A, i.e., $Ax = 0$, so $x \in \mathcal{N}(T)$. This shows $\mathcal{N}(A)$ is closed.

(ii) if A is also injective, then $\mathcal{R}(A)$ is closed in Y is equivalent to that there exists $c > 0$ such that $\|x\|_X \leq c\|Ax\|_Y$, for all $x \in X$.

For "only if" part, since A is closed, A^{-1} is closed. Since $\mathcal{R}(A)$ is closed, it is Banach. Since X is also Banach, A^{-1} is bounded by bounded inverse mapping theorem. Therefore, $\|x\|_X = \|A^{-1}y\| \leq c\|y\| = c\|Ax\|_Y$, for all $x \in X$.

For "if" part, take convergent sequence $Ax_n \to y$. Since we have $\|x\|_X \leq c\|Ax\|_Y$, Ax_n is Cauchy implies that x_n is also Cauchy, but X is Banach, so $x_n \to x \in X$. Since A is closed, $y = Ax$, which means $y \in \mathcal{R}(A)$, so $\mathcal{R}(A)$ is closed.

(iii) there exists constant $c > 0$ such that $\text{dist}(x, \mathcal{N}(A)) \leq c\|Ax\|_Y$, for all $x \in X$ if and only if $\mathcal{R}(A)$ is closed in Y.

Consider $T : X \setminus \mathcal{N}(A) \mapsto Y$ defined by $T(\hat{x}) = Ax$. It is easy to see T is linear and bounded (similar statement have been proved before). Notice that $\text{dist}(x, \mathcal{N}(A)) = \|\hat{x}\|_{X\setminus\mathcal{N}(A)}$, and T is injective. Apply conclusion in (ii) on T, we will obtain the required result.

Extra Problem 5. Let X be a normed space and M be a closed subspace of X. Note that $X = M \oplus N$ is defined as $\forall x \in X$, there exists unique $m \in M$ and $n \in N$ such that $x = m + n$. Prove

(i) If $X = M \oplus N$, then $M \cap N = \{0\}$.

Suppose there exists $a \neq 0$ such that $a \in M$ and $a \in N$, then assume $x = m + n$, we also have $x = (m + a) + (n - a)$, where $m + a \in M$ and $m + a \neq m$. This shows m and n are not unique, which is a contradiction. Hence, $M \cap N = \{0\}$.

(ii) If $X = M + N$ and $M \cap N = \{0\}$, then $X = M \oplus N$.

For all $x \in X$, suppose $x = m_1 + n_1 = m_2 + n_2$ where $m_1 \neq m_2$ and $n_1 \neq n_2$. This shows that $m_1 - m_2 = n_1 - n_2$. Since $m_1 - m_2 \in M$ and $n_1 - n_2 \in N$, we obtain $m_1 - m_2 \in M \cap N$, but $m_1 - m_2 \neq 0$, which contradicts the condition $M \cap N = \{0\}$. Therefore, $X = M \oplus N$.

(iii) A mapping $P : X \mapsto M$ is called a projection of X onto M if P is linear and bounded, $P^2 = P$, and $P(X) = M$. Prove that for such P, $Pm = m$ for all $m \in M$.

Since $P(X) = M$, for all $m \in M$, there exists $x \in X$, such that $Px = m$. Then $P^2x = Pm$, and since $P^2 = P$, $Pm = Px$, so this shows that $Pm = m$.

(iv) Suppose P exists for certain M and X, prove that there exists closed linear subspace N of X such that $X = M \oplus N$.

Let $M = \mathcal{R}(P)$ and $N = \mathcal{R}(I - P)$ where I is identity map from X to X. For each $x \in X$, we have $x = Ix = Px + (I - P)x = m + n$, where $m \in M$ and $n \in N$, so $X = M + N$. Suppose $u \in M \cap N$, there exists $x, y \in X$ such that $Px = u = (I - P)y$. Multiple P on both sides,

$$Px = P^2x = P(I - P)y = Py - P^2y = Py - Py = 0$$

Therefore, $u = Px = 0$, which shows $M \cap N = \{0\}$. By part (ii), we obtain $X = M \oplus N$.

(v) Suppose X is Banach and there exists closed subspace N of X such that $X = M \oplus N$. Define $P : X \mapsto M$ by $P(m + n) = m$ for all $m \in M$ and $n \in N$. Prove that P is a projection of X onto M. (Hint: prove P is closed)

It is easy to see P is linear, $P^2 = P$, and $P(X) = M$. We only need to show P is bounded. Since M, N are both closed, X is also Banach, M is Banach. We only need to show P is closed. Suppose $x_k \in X$ and $x_k \to x$, i.e., $m_k + n_k \to m + n$, where $m_k, m \in M$ and $n_k, n \in N$, and $Px_k \to y \in M$. Then since $m_k = P(x_k)$, m_k converges, so n_k converges to $u \in N$. Thus, $y + u = m + n$, i.e., $y - m = n - u$. Since $y - m \in M$ and $n - u \in N$, $y - m = n - u = 0$, which means $y = Px$. Therefore, P is closed, and by closed graph theorem, it is bounded.

(vi) Under assumption in (v), prove that $\max(\|m\|, \|n\|) \leq c\|m + n\|$ for all $m \in M$ and $n \in N$, where c is a constant.

Since P is bounded, we have $m = P(m + n) \leq \|P\|\|m + n\|$. Also, since $I - P$ is also bounded, $n = (I - P)(m + n) \leq \|I - P\|\|m + n\|$. Take $c = 1 + \|P\|$, then we will have $\max(\|m\|, \|n\|) \leq c\|m + n\|$ for all $m \in M$ and $n \in N$.

(vii) Let M be finite dimensional. Prove that there exists closed subspace N of X such that $X = M \oplus N$.

Consider the basis of M as $\{e_i\}_{i=1}^n$ and the dual basis $f_i \in X^*$ such that $f_i(e_j) = \delta_{ij}$. For $x \in M$, $x = \sum_{i=1}^n x_i e_i$. Define $f_1(x) = x_1$ on M, by Hahn-Banach, f_1 can be extended to X, so are f_i's. Therefore, we obtain $f_i \in X^*$ such that $f_i(e_j) = \delta_{ij}$. Define $p(x) = \sum_{i=1}^n f_i(x) e_i$ for all $x \in X$. Check $p(x)$ is linear and bounded, $p^2(x) = p(x)$, and $p(X) = M$. Therefore $p(x)$ is a projection, so there exists N such that $X = M \oplus N$ by part (iv).

(viii) Show that in general, for fixed M, in the decomposition $X = M \oplus N$, N is not unique.

Take $X = \mathbb{R}^2$, and $M = \text{span}(e_1)$. It is easy to see that N is not unique because $N_1 = \text{span}(e_2)$ is a possible choice, $N_2 = \text{span}(e_1 + e_2)$ is another possible choice.