MAT4010: Functional Analysis
Homework 11

李肖鹏 (116010114)

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Problem 7.5-9. If T is a normal operator, i.e., $T^*T = TT^*$, on a Hilbert space H, show that $r_\sigma(T) = \|T\|$.

Recall HW5, Problem 3.10-15, we have shown that for normal operator T, $\|T^2\| = \|T\|^2$ (This holds for H both real and complex). Now, we need to show T^2 is also a normal operator. This is true because

$$T^2(T^2)^* = T^2(T^*)^2 = T(TT^*)T^* = TT^*TT^* = T^*TT^*T = T^*(T^*T)T = (T^*)^2T^2 = (T^2)^*T^2$$

This in general shows that the square of a normal operator is still normal. By induction, this shows $\|T^{2^k}\| = \|T\|^{2^k}$ for all positive integer k. Recall Gelfand's formula,

$$r_\sigma(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = \lim_{k\to\infty} \|T^{2^k}\|^{1/2^k} = \lim_{k\to\infty} \|T\| = \|T\|$$

Therefore, $r_\sigma(T) = \|T\|$ if T is normal.

Extra Problem 1. Let X be complex Banach space and $T : X \mapsto X$ is linear and bounded. Suppose $R > r(T)$, where $r(T)$ is the spectral radius of T, $\Gamma = \{z \in \mathbb{C} \,|\, |z| = R\}$ is oriented counter-clockwise. Prove that

$$\frac{1}{2\pi i} \oint_\Gamma (zI - T)^{-1}\, dz = I$$

Hint: independence on R and Neumann series.

Since $(zI - T)^{-1}$ is analytic in $|z| > r(T)$, by Cauchy–Goursat theorem, the integral value of a holomorphic is independent of path connecting two same points. Therefore, we can only consider Γ such that $R > \|T\|$, and the integral value remains the same. In $R > \|T\|$, the Neumann series converges uniformly, i.e., we can exchange the order of infinite sum and integral, so

$$\frac{1}{2\pi i} \oint_\Gamma (zI - T)^{-1}\, dz = \frac{1}{2\pi i} \oint_\Gamma \sum_{k=0}^{\infty} z^{-(k+1)} T^k\, dz = \sum_{k=0}^{\infty} T^k \left(\frac{1}{2\pi i} \oint_\Gamma z^{-(k+1)}\, dz \right)$$

By Cauchy's integral formula, $\frac{1}{2\pi i} \oint_\Gamma z^{-(k+1)}\, dz = 0$ when $k \geq 1$, and $\frac{1}{2\pi i} \oint_\Gamma z^{-(k+1)}\, dz = 1$ when $k = 0$. Therefore, we have

$$\frac{1}{2\pi i} \oint_\Gamma (zI - T)^{-1}\, dz = T^0 \frac{1}{2\pi i} \oint_\Gamma z^{-1}\, dz = I$$

Extra Problem 2. Let X and T be given as in last problem. Prove that for all $z_1, z_2 \in \rho(T)$, $R(z_1) - R(z_2) = (z_2 - z_1)R(z_1)R(z_2)$, where $R(z) = (zI - T)^{-1}$.

Consider the identity

$$z_1 I - T = z_2 I - T + (z_1 - z_2)I$$

Multiply $(z_1 I - T)^{-1}$ to the left on both sides, we have

$$I = (z_1 I - T)^{-1}(z_2 I - T) + (z_1 - z_2)(z_1 I - T)^{-1}$$

Multiply $(z_2 - T)^{-1}$ to the right on both sides, we have

$$(z_2 - T)^{-1} = (z_1 I - T)^{-1} + (z_1 - z_2)(z_1 I - T)^{-1}(z_2 - T)^{-1}$$

which is equivalent to

$$R(z_2) = R(z_1) + (z_1 - z_2)R(z_1)R(z_2)$$

Therefore, we verified that $R(z_1) - R(z_2) = (z_2 - z_1)R(z_1)R(z_2)$.

Extra Problem 3. Let $X = L^2(0,1)$, $T : f \in L^2(0,1) \mapsto \int_0^x f(t)\, dt$. Explore this example to show $\lim_{n\to\infty} \|T^n\|^{1/n} \neq \|T\|$.

(i) Show T is linear and bounded as a mapping from X to X.

Since $f \in L^2(0,1)$, $F(x) = \int_0^x f(t)\, dt$ is absolutely continuous, so it is continuous, thus in $L^2(0,1)$. This shows T is from X to X. T is linear because for all scalar a, b and $f, g \in L^2(0,1)$, we have

$$T(af + bg) = \int_0^x (af + bg)(t)\, dt = a\int_0^x f(t)\, dt + b\int_0^x g(t)\, dt = aT(f) + bT(g)$$

T is bounded because by Cauchy-Schwarz, for all $x \in [0,1]$,

$$|T(f)(x)| = \left|\int_0^x f(t)\, dt\right| \le \left(\int_0^x |f(t)|^2\, dt\right)^{1/2}\left(\int_0^x 1^2\, dt\right)^{1/2} \le \|f\|_{L^2(0,1)}$$

Therefore, we have

$$\|T(f)\|_{L^2(0,1)} = \left(\int_0^1 |T(f)(x)|^2\, dx\right)^{1/2} \le \|f\|_{L^2(0,1)}$$

Therefore, T is bounded with norm $\|T\| \le 1$.

(ii) Show $\sigma(T) = \{0\}$. (Hint: The integral of function in $L^2(0,1)$ is absolutely continuous and differentiable almost everywhere)

Consider $(\lambda I - T)f = g$ for $g \in L^2(0,1)$, we have

$$\lambda f(x) - \int_0^x f(t)\, dt = g(x)$$

If $\lambda = 0$, then $f(x)$ does not exists for $g(x)$ equal to Cantor function defined on $[0,1]$. This is because Cantor function is in $L^2(0,1)$ but not absolutely continuous. Thus $0 \in \sigma(T)$.

If $\lambda \neq 0$, let $F(x) = \int_0^x f(t)\,dt$, we have $F'(x) - \frac{1}{\lambda}F(x) = \frac{1}{\lambda}g(x)$ almost everywhere and $F(0) = 0$. This is first order linear ODE, so by integrating factor, we have $F(x) = e^{x/\lambda}\int_0^x e^{-t/\lambda}\frac{1}{\lambda}g(t)\,dt$. Since $g(t) \in L^2(0,1)$ and $e^{-t/\lambda}$ is bounded, so the solution exists. Since this is a linear ODE, $F(x)$ is unique up to a zero measure set on $[0,1]$. Therefore, $\sigma(T) = \{0\}$.

Extra Problem 4. Let H be a Hilbert space and $T : H \mapsto H$ is linear and bounded. Let $J : H^* \mapsto H$ be the mapping determined by Riesz Representation Theorem, i.e., $\forall\, h^* \in H^*$, there exists unique $h \in H$ such that $\langle h^*, x\rangle_{H^*,H} = (x, h)_H$ for all $x \in H$, then $Jh^* = h$. Denote the Hilbert adjoint as T' and usual dual operator as $T^* : H^* \mapsto H^*$, then prove that $T^* = J^{-1}T'J$.

For any fixed $f^* \in H^*$, for one thing, we have $\langle T^*f^*, h\rangle_{H^*,H} = \langle f^*, Th\rangle_{H^*,H} = (Th, x)_H$ where $x \in H$ is uniquely determined by $f^* \in H$, and $Jf^* = x$. Therefore, we have $\langle T^*f^*, h\rangle_{H^*,H} = (Th, Jf^*)_H$ for all $h \in H$.

For another thing, $\langle J^{-1}T'Jf^*, h\rangle_{H^*,H} = \langle J^{-1}T'x, h\rangle_{H^*,H}$. Since J is bijective mapping, so there exists a unique $g^* \in H^*$ such that $Jg^* = T'x$, so

$$\langle J^{-1}T'x, h\rangle_{H^*,H} = \langle g^*, h\rangle_{H^*,H} = (h, Jg^*)_H = (h, T'x)_H = (Th, x)_H = (Th, Jf^*)_H$$

Therefore, $\langle J^{-1}T'Jf^*, h\rangle_{H^*,H} = (Th, Jf^*)_H$. Combined with previous result, we have shown that $\langle J^{-1}T'Jf^*, h\rangle_{H^*,H} = \langle T^*f^*, h\rangle_{H^*,H}$ for all $h \in H$, so $T^*f^* = J^{-1}T'Jf^*$. Since f^* is arbitrary in H^*, $T^* = J^{-1}T'J$.

Extra Problem 5. Let X and Y be normed spaces, $T : X \mapsto Y$ is linear and bounded. Prove that $\mathcal{R}(T^*) = X^*$ if and only if there exists $c > 0$ such that $\|Tx\|_Y \geq c\|x\|_X$ for all $x \in X$.

For "only if" part, suppose not, there exists $x_n \in X$ such that $\|Tx_n\|_Y \leq \frac{1}{n}\|x_n\|_X$. Let $y_n = \frac{x_n}{\|x_n\|}$, then $\|Ty_n\| < \frac{1}{n} \to 0$, and $\|y_n\| = 1$. Let $z_n = \frac{y_n}{\|Ty_n\|^{1/2}}$ if $Ty_n \neq \mathbf{0}_Y$, and $z_n = ny_n$ if $Ty_n = \mathbf{0}_Y$. Then we can observe $\|z_n\|_X \to \infty$ while $\|Tz_n\|_Y \to 0$. Since T^* is onto, so for any $x^* \in X$, there exists $y^* \in Y^*$ such that $T^*y^* = x^*$ and $\langle x^*, z_n\rangle_{X^*,X} = \langle T^*y^*, z_n\rangle_{X^*,X} = \langle y^*, Tz_n\rangle_{Y^*,Y} \to 0$. This shows $z_n \xrightarrow{w} \mathbf{0}_X$, so z_n must be bounded, which contradicts to $\|z_n\|_X \to \infty$. Therefore, there exists $c > 0$ such that $\|Tx\|_Y \geq c\|x\|_X$ for all $x \in X$.

For "if" part, for all $x \in X$ such that $Tx = \mathbf{0}_Y$, since $\|Tx\|_Y \geq c\|x\|_X$, we have $\|x\|_X = 0$, so $x = \mathbf{0}_X$, which means T is injective. For $x^* \in X^*$, we can define $f : \mathcal{R}(T) : \mathbb{C}$ by $f(y) = f(Tx) = \langle x^*, x\rangle_{X^*,X}$ where x is uniquely defined for any $y \in \mathcal{R}(T)$ because T is injective. Now we need to prove f is bounded and linear. f is bounded because

$$|f(Tx)| = |\langle x^*, x\rangle_{X^*,X}| \leq \|x^*\|\|x\|_X \leq \frac{\|x^*\|}{c}\|Tx\|_Y$$

f is linear because for all scalar a, b and $y_1, y_2 \in Y$,

$$\begin{aligned} f(ay_1 + by_2) &= f(aTx_1 + bTx_2) = f(T(ax_1 + bx_2)) = \langle x^*, ax_1 + bx_2\rangle_{X^*,X} \\ &= a\langle x^*, x_1\rangle_{X^*,X} + b\langle x^*, x_2\rangle_{X^*,X} = af(y_1) + bf(y_2) \end{aligned}$$

Then by Hahn-Banach, there exists $F \in Y^*$ such that $F\Big|_{\mathcal{R}(T)} = f$. For such F,

$$\langle T^*F, x\rangle_{X^*,X} = \langle F, Tx\rangle_{Y^*,Y} = \langle x^*, x\rangle_{X^*,X}, \quad \forall\, x \in X$$

Therefore, $T^*F = x^*$, which shows T^* is surjective.

Extra Problem 6. Let X and Y be Banach space and $T : X \mapsto Y$ is linear and bounded. Then prove $T : X \mapsto Y$ is bijective if and only if $T^* : Y^* \mapsto X^*$ is bijective. In this case, also prove $(T^{-1})^* = (T^*)^{-1}$.

For "only if" part, by bounded inverse mapping theorem, T^{-1} exists and is bounded, i.e., $\|T^{-1}y\|_X \leq \|T^{-1}\|\|y\|_Y$. This is equivalent to say $c\|x\|_X \leq \|Tx\|_Y$. By Extra Problem 5, T^* is surjective. By Fact 5 in lecture, $\mathcal{N}(T^*) = {}^{\perp\!\!\!\perp}\mathcal{R}(T)$. Since T is surjective, $\mathcal{N}(T^*) = {}^{\perp\!\!\!\perp}Y = \{\mathbf{0}_{Y^*}\}$, so T^* is injective, hence it is bijective.

For "if" part, since by Fact 5 in lecture, $\mathcal{N}(T) \subset \mathcal{R}(T^*)^{\perp\!\!\!\perp} = (X^*)^{\perp\!\!\!\perp} = \{\mathbf{0}_X\}$, so T is injective. Since T^* is surjective, by Extra Problem 5, there exists $c > 0$ such that $\|Tx\| \geq c\|x\|$ for all $x \in X$. Then for any convergent sequence $Tx_n \to y$, since Tx_n is Cauchy, x_n is also Cauchy. Since X is Banach, $x_n \to x \in X$, and by continuity of T, $Tx_n \to Tx = y$. This shows that $y \in \mathcal{R}(T)$, so $\mathcal{R}(T)$ is closed. Suppose T is not surjective, then by Hahn-Banach, there exists $f \in Y^*$ such that $f \neq \mathbf{0}_{Y^*}$, $f\Big|_{\mathcal{R}(T)} = 0$. This shows $\langle f, Tx\rangle = 0$, for all $x \in X$, i.e., $\langle T^*f, x\rangle = 0$. Therefore, $T^*f = \mathbf{0}_{X^*}$, which means $f \in \mathcal{N}(T^*)$. However, T^* is injective, so $\mathcal{N}(T^*) = \{\mathbf{0}_{Y^*}\}$, but $f \neq \mathbf{0}_{Y^*}$, contradiction. Therefore, T is surjective, hence bijective.

Finally, for all $f \in X^*$, $\langle f, T^{-1}y\rangle_{X^*,X} = \langle (T^{-1})^*f, y\rangle_{Y^*,Y}$, let $y = Tx$, then LHS is equal to $\langle f, x\rangle_{X^*,X}$, while RHS is given by

$$\langle (T^{-1})^*f, y\rangle_{Y^*,Y} = \langle (T^{-1})^*f, Tx\rangle_{Y^*,Y} = \langle T^*(T^{-1})^*f, x\rangle_{X^*,X}$$

Therefore, $T^*(T^{-1})^*f = f$ for all $f \in X^*$, so $T^*(T^{-1})^* = I_{Y^*}$. This shows $(T^{-1})^* = (T^*)^{-1}$.