MAT4010: Functional Analysis
Homework 12

李肖鹏 (116010114)

Due date: Dec. 3, 2019

Problem 8.2-6. Define $T : l^2 \mapsto l^2$ by $Tx = y = (\eta_j)$, where $x = (\xi_j)$ and $\eta_j = \sum_{k=1}^{\infty} \alpha_{jk}\xi_k$, $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |\alpha_{jk}|^2 < \infty$. Show that T is compact.

Define $T_N : l^2 \to l^2$ by $T_N x = (\eta_1, \ldots, \eta_N, 0, \ldots)$, where $\eta_j = \sum_{k=1}^{\infty} \alpha_{jk}\xi_k$. It is easy to see T_N is linear. It is also compact because l^2 is Banach, and $\mathrm{Im}(T_N)$ is a finite dimensional vector space, hence Banach. Then the compactness of T_N follows from Example 1 in lecture. Consider

$$\|T_N x - Tx\| = \left(\sum_{j=N+1}^{\infty} |\eta_j|^2\right)^{1/2} = \left(\sum_{j=N+1}^{\infty}\sum_{k=1}^{\infty} |\alpha_{jk}|^2\right)^{1/2} \left(\sum_{k=1}^{\infty} |\xi_k|^2\right)^{1/2} \leq M_j \|x\|$$

where $M_j \to 0$ since $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |\alpha_{jk}|^2 < \infty$. Therefore, $\|T_N - T\| \leq M_j \to 0$, which means $T_N \to T$. By Theorem 8.1-5 in textbook, the limiting operator of T_N, i.e., T is also compact.

Problem 8.2-8. Does there exist a surjective compact linear operator $T : l^\infty \mapsto l^\infty$?

No, actually there does not exist any surjective compact linear operator that maps from Banach space into any infinite dimensional Banach space. Suppose there exists, then by open mapping theorem, T must map a open set U to open set V. However, if T is compact, then V must be precompact set in l^∞. We need to prove that there does not exist precompact open set in any infinite dimensional normed space (l^∞ is an example of infinite dimensional Banach space).

For any open set V in l^∞, it contains an open ball $B(x_0; r)$ for $x_0 \in V$. Take $e_1 \in l^\infty$ with $\|e_1\| = 1$. By Riesz Lemma with $M = \mathrm{span}\{e_1\}$ (which is closed) and $\theta = 1/2$, there exists e_2 such that $\|e_1\| = 1$ and $\|e_2 - e_1\| \geq 1/2$. Continue this process, we obtain $\{e_i\}_{i=1}^{\infty} \subset \overline{B(0;1)}$ such that $\|e_i - e_j\| \geq 1/2$ for all $i \neq j$. Therefore, let $u_i = x_0 + \frac{r}{2}e_i$, then $\{u_i\}_{i=1}^{\infty} \subset B(x_0; r)$ is a bounded sequence in X but has no convergent subsequence. This shows $B(x_0; r)$ is not precompact.

Problem 8.2-9. If $T \in B(X, Y)$ is not compact, can the restriction of T to an infinite dimensional subspace of X be compact?

If T is not compact in $B(X, Y)$, then the restriction of T to an infinite dimensional subspace M of X can be compact, but there does not always exist such M that the restriction of T on M is compact.

To see $T\Big|_M$ can be compact, consider $T : l^2 \to l^2$ defined by $T(x_1, \ldots, x_n, \ldots) = (x_1, 0, x_3, 0, \ldots)$. Then T is obviously linear and bounded. T is not compact, because consider $T : l^2 \to S$ where

$S = \{(x_j)_{j=1}^{\infty} \in l^2 \,|\, x_{2j} = 0, \forall\, j \geq 1\}$ (note that S is a closed subspace of l^2, hence Banach), then T is surjective hence not compact by last problem, i.e., there exists bounded $x^{(n)} \in l^2$ such that $Tx^{(n)} \in S$ does not have any convergent subsequence. Therefore, for $T : l^2 \to l^2$, use the same $x^{(n)}$, $Tx^{(n)}$ still has no convergent subsequence. This shows that T is not compact. However, if we take $M = \{(x_j)_{j=1}^{\infty} \in l^2 \,|\, x_{2j-1} = 0, \forall\, j \geq 1\}$, then $T\Big|_M : M \mapsto l^2$ maps all $x \in l^2$ to $\mathbf{0}_{l^2}$, so it must be compact.

To see not all T can have some M such that $T\Big|_M$ is compact, consider $T : l^2 \mapsto l^2$ as $Tx = x$ for $x \in l^2$. No matter what M you choose, as long as it is a infinite dimensional vector space, $T\Big|_M : M \to l^2$ can not be compact because $T\Big|_M : M \to M$ must be an open mapping, hence not compact, but in fact $T\Big|_M$ will map any element into M, so $T\Big|_M : M \to l^2$ is still not compact.

Problem 8.2-10. Let (λ_n) be a sequence of scalars such that $\lambda_n \to 0$ as $n \to \infty$. Define $T : l^2 \mapsto l^2$ by $Tx = y = (\eta_j)$ where $x = (\xi_j)$ and $\eta_j = \lambda_j \xi_j$. Show that T is compact.

Consider $T_N : l^2 \mapsto l^2$ defined by $T_N x = (\lambda_1\xi_1, \ldots, \lambda_N\xi_N, 0, \ldots)$. Then T_N is compact for all fixed N. This is because $\dim(\mathrm{Im}(T_N)) < \infty$, thus Banach, and l^2 is also Banach. Since $\lambda_j \to 0$, we have

$$\|T_N x - Tx\| = \left(\sum_{j=N+1}^{\infty} |\lambda_j \xi_j|^2\right)^{1/2} \leq \left(\sup_{j\geq N+1} |\lambda_j|\right)\left(\sum_{j=N+1}^{\infty} |\xi_j|^2\right)^{1/2} = \left(\sup_{j\geq N+1} |\lambda_j|\right)\|x\|$$

Therefore, we conclude that $\|T_N - T\| \leq \sup_{j\geq N+1} |\lambda_j|$. Therefore, take $N \to \infty$, we have $T_N \to T$. By Theorem 8.1-5 in textbook, the limiting operator of compact operators are still compact operator, T is also compact.

Extra Problem 1. Let X and Y be Banach with $\dim X < \infty$; let $T : X \mapsto Y$ be linear and bounded. Prove that T is compact.

Take any bounded sequence $x_n \in X$, since $\dim(X) < \infty$, by Bolzano-Weierstrass there exists a subsequence of x_n, i.e., $x_{n_k} \to x \in X$. Since T is linear and bounded, hence Lipschitz continuous, so $Tx_{n_k} \to Tx \in Y$. Therefore, T is compact.

Extra Problem 2. Let X be Banach space with $\dim X = \infty$. Prove that if $T : X \mapsto X$ is compact, then $0 \in \sigma(T)$.

Suppose $0 \notin \sigma(T)$, then $-T$ is bijective. Thus, T is bijective and in particular surjective from X to X. If T is compact, then we have a surjective linear operator maps from Banach space into infinite dimensional Banach space, which contradicts Problem 8.2-8. Thus, $0 \in \sigma(T)$.

Extra Problem 3. Let X and Y be Banach, and $K : X \mapsto Y$ be compact. Prove that if $x_n \xrightarrow{w} x_\infty$ in X, then $Kx_n \to Kx_\infty$ strongly in Y.

We first claim that if every subsequence of a sequence $\{x_n\} \subset X$ has a convergent subsequence that converges to x, then this sequence $\{x_n\}$ converges to x. Assume this is true, then for any

$f^* \in Y$, $f^* \circ T$ is a linear functional on X, so $f^* \circ T(x_n) \to f^* \circ T(x)$. This is equivalent to say $f^*(Tx_n) \to f^*(Tx)$, therefore, $Tx_n \xrightarrow{w} Tx$. Since each subsequence x_{n_k} of x_n is bounded, by compactness of T, there exists a further subsequence $x_{n_{k_m}}$ such that $Tx_{n_{k_m}} \to y_{n_k}$. Then $Tx_{n_{k_m}} \xrightarrow{w} y_{n_k}$, but since $Tx_n \xrightarrow{w} Tx$, $Tx_{n_{k_m}} \xrightarrow{w} Tx$. By uniqueness of weak convergence limit, we have $Tx = y_{n_k}$. Thus each subsequence of Tx_n has a further subsequence that converges to Tx, this implies that $Tx_n \to Tx$.

To prove the claim, suppose $x_n \not\to x$, then there exists a subsequence x_{n_k} of x_n such that $\|x_{n_k} - x\| \geq \epsilon$. This is because if such subsequence does not exist, then it means all but finitely many x_n satisfies $\|x_n - x\| < \epsilon$, and thus $x_n \to x$. However, if so, then x_{n_k} as a subsequence of x_n contains no further subsequence that converges to x, which leads to contradiction.

Extra Problem 4. Let X be Banach and M be a closed subspace of X. Suppose $K : X \mapsto X$ is compact with $K(M) \subset M$. Prove that $\bar{K} : \hat{x} \in X \setminus M \mapsto \widehat{Kx} \in X \setminus M$ is compact.

For bounded sequence $\hat{x}_n \in X \setminus M$, there exists y_n such that $\|x_n + y_n\| \leq 2 \inf_{y \in M} \|x_n + y\| < \infty$. Let $z_n = x_n + y_n$, then z_n is bounded in X. Therefore by compactness of K, there exists a subsequence z_{n_k} such that $Kz_{n_k} \to x \in X$. Consider

$$\left\|\widehat{Kz_{n_k}} - \hat{x}\right\|_{X \setminus M} = \inf_{y \in M} \|Kz_{n_k} - x + y\| \leq \|Kz_{n_k} - x\| \to 0$$

Therefore, we have $\widehat{Kx_{n_k}} = \widehat{Kx_{n_k}} + \widehat{Ky_{n_k}} = \widehat{Kz_{n_k}} \to \hat{x}$ since $Ky_{n_k} \in M$ for $y_{n_k} \in M$. Therefore, there exists subsequence $\hat{x}_{n_k}$ of $\hat{x}_n$ such that $\bar{K}\hat{x}_{n_k} = \widehat{Kx_{n_k}} \to \hat{x}_n$. This implies that $\bar{K}$ is compact.

Extra Problem 5. Let X, Y, Z be Banach, $X \subset Y \subset Z$. Prove that if $X \hookrightarrow Y$ is compact and $Y \hookrightarrow Z$ is continuous, then $\forall \epsilon > 0$, there exists $C_\epsilon > 0$ such that $\|x\|_Y \leq \epsilon \|x\|_X + C_\epsilon \|x\|_Z$ for all $x \in X$. (Hint: Prove by contradiction)

Suppose not true, then we can find a fixed $\epsilon > 0$ such that for all $n \geq 1$, there exists $x_n \in X$ such that $\|x_n\|_Y > \epsilon \|x_n\|_X + n\|x_n\|_Z$, which implies $\|y_n\|_Y > \epsilon + n\|y_n\|_Z$, for $y_n = \frac{x_n}{\|x_n\|_X}$, where $\|y_n\|_X = 1$. Denote $I : X \mapsto Y$ as the embedded identity map for $X \hookrightarrow Y$, and $J : Y \mapsto Z$ as the embedded identity map for $Y \hookrightarrow Z$. Since $y_n \in X$ is a bounded sequence, by compactness of I, there exists a subsequence y_{n_k} such that $I(y_{n_k})$ converges to y in Y, i.e., $\|y_{n_k} - y\|_Y \to 0$ as $k \to \infty$. Since J is continuous, it is bounded, i.e., $\|J(u)\|_Z \leq c\|u\|_Y$ for all $u \in Y$ and some constant $c \geq 0$. Therefore, we have

$$\|y_{n_k} - y\|_Z = \|J(y_{n_k}) - J(y)\|_Z \leq c\|y_{n_k} - y\|_Y \to 0$$

Therefore, take limit on both sides of $\|y_{n_k}\|_Y > \epsilon + n_k \|y_{n_k}\|_Z$, we obtain $\|y\|_Y \geq \epsilon + \infty \cdot \|y\|_Z$. Since y is bounded, if $\|y\|_Z > 0$, then $\|y\|_Y \geq \infty$, contradiction. This implies that $y = \mathbf{0}_Z$, then we will have $0 \geq \epsilon$, which is a contradiction. Therefore, $\forall \epsilon > 0$, there exists $C_\epsilon > 0$ such that $\|x\|_Y \leq \epsilon \|x\|_X + C_\epsilon \|x\|_Z$ for all $x \in X$.

Extra Problem 6. Let X and Y be Banach; let $T : X \mapsto Y$ be bounded; $K : X \mapsto Y$ be compact, with $\mathcal{R}(T) \subset \mathcal{R}(K)$. Prove T is compact.

Define $\bar{K} : X \setminus \mathcal{N}(K) \mapsto \mathcal{R}(K)$ by $\bar{K}(x + \mathcal{N}(K)) = Kx$. $\bar{K}$ is well-defined. $\bar{K}$ is linear. By the same method as Extra Problem 4 to show that $\bar{K}$ is compact. In addition, $\bar{K}$ is bijective. For each fixed $x \in X$, since $\mathcal{R}(T) \subset \mathcal{R}(K)$, we can consider $T : X \mapsto \mathcal{R}(K)$, then the equation $Tx = \bar{K}y$ has unique solution $y \in X \setminus \mathcal{N}(K)$. There exists $A : X \mapsto X \setminus \mathcal{N}(K)$ such that $Ax = y$ and A is linear and closed. By closed graph theorem, A is bounded. Therefore, we can conclude that $T = \bar{K}A$, but $\bar{K}$ is compact and A is bounded, so T is compact.

To prove $\bar{K}$ is well-defined, consider $\hat{x}_1 = \hat{x}_2 \in X \setminus \mathcal{N}(K)$, then $x_1 - x_2 \in \mathcal{N}(K)$, which means $K(x_1 - x_2) = \mathbf{0}_Y$, i.e., $Kx_1 = Kx_2$, so $\bar{K}$ is well-defined.

To prove $\bar{K}$ is linear, consider $\hat{x}_1, \hat{x}_2 \in X \setminus \mathcal{N}(K)$ and scalar a, b, we have

$$\bar{K}(a\hat{x}_1 + b\hat{x}_2) = K(ax_1 + bx_2) = aKx_1 + bKx_2 = a\bar{K}(\hat{x}_1) + b\bar{K}(\hat{x}_2)$$

Thus, $\bar{K}$ is linear.

To prove $\bar{K}$ is compact, For bounded sequence $\hat{x}_n \in X \setminus \mathcal{N}(K)$, there exists $y_n \in \mathcal{N}(K)$ such that $\|x_n + y_n\| \leq 2 \inf_{y \in \mathcal{N}(K)} \|x_n + y\| < \infty$. Let $z_n = x_n + y_n$, then z_n is bounded in X. Therefore by compactness of K, there exists a subsequence z_{n_k} such that $Kz_{n_k} \to u \in \mathcal{R}(K)$. Then it is obvious that $\bar{K}(\hat{z}_{n_k}) \to u$. This implies that $\bar{K}$ is compact. Since $\bar{K}$ is compact, it must be bounded.

To prove $\bar{K}$ is surjective, take any $y = Kx$, we can find $\bar{K}(x + \mathcal{N}(K)) = Kx = y$, thus it is surjective. To prove $\bar{K}$ is injective, let $Kx = \mathbf{0}_Y$, then $x \in \mathcal{N}(K)$, which means $\hat{x} = \mathbf{0}_{X \setminus \mathcal{N}(K)}$. Thus, it is injective.

To prove A is linear, consider any $x_1, x_2 \in X$ and scalar a, b, denote $Ax_1 = y_1$ and $Ax_2 = y_2$, then since $Tx_1 = \bar{K}y_1$ and $Tx_2 = \bar{K}y_2$, we have $T(ax_1 + bx_2) = \bar{K}(ay_1 + by_2)$. Therefore, $A(ax_1 + bx_2) = ay_1 + by_2 = aAx_1 + bAx_2$ by uniqueness of solution. This implies that A is linear.

To prove A is closed, suppose $x_n \to x_\infty$ and $y_n = Ax_n \to y_\infty$. Since $Tx_n = Ky_n$, $Tx_\infty = Ky_\infty$, we have $Ax_\infty = y_\infty$, so A is closed. Notice that K is bounded, so K is closed, and $\mathcal{N}(K) \subset X$ is closed. This together with previous HW, we can see $X \setminus \mathcal{N}(K)$ is Banach.