MAT4010: Functional Analysis
Homework 13

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Problem 8.4-8. Let $T : l^2 \to l^2$ be defined by $x = (\xi_1, \xi_2, \dots) \mapsto Tx = (\xi_2, \xi_3, \dots)$. Let $m = m_0$ and $n = n_0$ be the smallest numbers such that we have $\mathcal{N}(T^m) = \mathcal{N}(T^{m+1})$ and $T^{n+1}(X) = T^n(X)$. Find $\mathcal{N}(T^m)$. Does there exist a finite m_0? Find n_0.

Notice that $T^m(x) = (\xi_{m+1}, \xi_{m+2}, \dots)$, so by letting it equal to $(0, 0, \dots)$, we can obtain $\mathcal{N}(T^m) = \text{span}(e_1, \cdots, e_m)$, where e_i is the standard basis of l^2. Since $\text{span}(e_1, \cdots, e_m)$ will never equal to $\text{span}(e_1, \cdots, e_m, e_{m+1})$ for finite m, there does not exist finite m_0 such that $\mathcal{N}(T^m) = \mathcal{N}(T^{m+1})$, so $m_0 = +\infty$. Notice that T is surjective because for any $x' = (x_1, x_2, \dots) \in l^2$, we have $(\sum_{i=1}^{\infty} |x_i|^2)^{1/2} < \infty$, and we can find $x = (0, x_1, x_2, \dots)$ such that $Tx = x'$. Also, $x \in l^2$ because $\|x\|_{l^2} = (\sum_{i=1}^{\infty} |x_1|^2)^{1/2} < \infty$. Therefore, $T(X) = X$, and $n_0 = 0$.

Extra Problem 1. Let X be Banach and $K : X \mapsto X$ be compact. Let

$$p = \min\{n \geq 0 \,|\, \mathcal{N}((I-K)^{n+1}) = \mathcal{N}((I-K)^n)\}, \quad q = \min\{m \geq 0 \,|\, \mathcal{R}((I-K)^{m+1}) = \mathcal{R}((I-K)^m)\}$$

Prove that $p = q$.

Suppose $p > q$, then we have

$$\mathcal{N}(S) \subsetneq \cdots \subsetneq \mathcal{N}(S^{q-1}) \subsetneq \mathcal{N}(S^q) \subsetneq \cdots \subsetneq \mathcal{N}(S^{p-1}) \subsetneq \mathcal{N}(S^p) = \mathcal{N}(S^{p+1}) = \cdots$$

$$\mathcal{R}(S) \supsetneq \cdots \supsetneq \mathcal{R}(S^{q-1}) \supsetneq \mathcal{R}(S^q) = \mathcal{R}(S^{q+1}) = \cdots$$

Recall Lemma 6 in lecture, $\dim(\mathcal{N}(S^k)) = \text{codim}(\mathcal{R}(S^k))$ for all $k \geq 1$. Therefore, we know

$$\dim(\mathcal{N}(S^q)) = \text{codim}(\mathcal{R}(S^q)) = \text{codim}(\mathcal{R}(S^{q+1})) = \dim(\mathcal{N}(S^{q+1}))$$

This is a contradiction, since $\mathcal{N}(S^q)$ is a proper subspace of $\mathcal{N}(S^{q+1})$, so they cannot have the same dimension (finite). Notice that $\dim(\mathcal{N}(S^0)) = \text{codim}(\mathcal{R}(S^0))$ because $\mathcal{N}(S^0) = \{0\}$ and $\mathcal{R}(S^0) = X$.

Similarly, suppose $p < q$, we will obtain $\text{codim}(\mathcal{R}(S^p)) = \text{codim}(\mathcal{R}(S^{p+1})) = d < \infty$ and $\mathcal{R}(S^{p+1}) \subsetneq \mathcal{R}(S^p)$. Consider $X = \mathcal{R}(S^{p+1}) \oplus M_1$ and $X = \mathcal{R}(S^p) \oplus M_2$, where $\dim(M_1) = \dim(M_2)$. However, $\mathcal{R}(S^p)$ can be decomposed as $\mathcal{R}(S^p) = \mathcal{R}(S^{p+1}) \oplus M_3$ where $\dim(M_3) > 0$. This implies that $M_3 \oplus M_2 = M_1$, i.e., $\dim(M_3) + \dim(M_2) = \dim(M_1)$, which is a contradiction to $\dim(M_1) = \dim(M_2)$. Therefore, $p = q$.

Extra Problem 2. Let $K \in L^2((0,1)\times(0,1))$ with $\|K\|_{L^2((0,1)\times(0,1))} < 1$. Prove that $\forall\, f \in L^2(0,1)$, the integral equation

$$u(x) - \int_0^1 K(x,y)u(y)\,dy = f(x), \quad x \in (0,1)$$

has a solution $u \in L^2(0,1)$.

Define $A : L^2(0,1) \mapsto L^2(0,1)$ by $Au(x) = \int_0^1 K(x,y)u(y)\,dy$. Since we have proved that such A is a compact operator in class, and it suffices to show that $(I - A)$ is surjective on $L^2(0,1)$. We tend to show $(I - A)$ is injective. Consider the integral equation

$$u(x) = \int_0^1 K(x,y)u(y)\,dy, \quad x \in (0,1)$$

By generalized Minkowski inequality,

$$\|u\|_{L^2(0,1)} \le \|K\|_{L^2((0,1)\times(0,1))}\|u\|_{L^2(0,1)}$$

If $\|u\|_{L^2(0,1)} \neq 0$, then by cancel out it on both sides, we obtain $\|K\|_{L^2((0,1)\times(0,1))} \ge 1$, which contradicts $\|K\|_{L^2((0,1)\times(0,1))} < 1$. Thus, $\|u\|_{L^2(0,1)} = 0$, i.e., $u(x) = 0$ a.e. on $(0,1)$. This implies that $(I - A)$ is injective, by Fredholm alternative, $(I - A)$ is surjective, which proved the required statement.

Extra Problem 3. Let X be Banach and $K : X \mapsto X$ be compact. Prove

(i) For all $\hat{x} \in X \setminus \mathcal{N}(I - K)$, there exists $x_0 \in \hat{x}$ such that $\|x_0\| = \|\hat{x}\|_0$.

For simplicity, we denote $S = I - K$, then since K is compact, by Lemma 5, $\dim(\mathcal{N}(S)) < \infty$. For any n, there exists $y_n \in \mathcal{N}(S)$, such that $\|\hat{x}\|_0 \le \|x - y_n\| < \|\hat{x}\|_0 + \frac{1}{n}$. Since $\|y_n\| \le \|x\| + \|y_n - x\| < 2\|x\| + 1$, y_n is a bounded sequence in $\mathcal{N}(S)$, by Bolzano-Weierstrass, y_n has a convergent subsequence $y_{n_k} \to y \in \mathcal{N}(S)$. Therefore, take limit as $k \to \infty$ on left, middle, and right of inequality $\|\hat{x}\|_0 \le \|x - y_{n_k}\| < \|\hat{x}\|_0 + \frac{1}{n_k}$, we have $\|x - y\| = \|\hat{x}\|_0$. Therefore, let $x_0 = x - y$, and such $x_0 \in \hat{x}$ satisfies $\|x_0\| = \|\hat{x}\|_0$.

(ii) If $y \in X$ and $(I - K)x = y$ has a solution x, then there exists solution x_0 which has the smallest norm among all solutions.

If x_1, x_2 are two solutions of $Sx = y$, then $Sx_1 = y$ and $Sx_2 = y$. Since S is linear, we can conclude $S(x_1 - x_2) = \mathbf{0}$. Therefore, all solution of $Sx = y$ are in $\hat{x}$. By (i), there exists $x_0 \in \hat{x}$ such that $\|x_0\| = \|\hat{x}\| = \inf_{z\in\hat{x}}\|z\|$. Thus x_0 has the smallest norm among all solutions.