MAT4010: Functional Analysis
Homework 2

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Problem 2.7-5. Show that the operator $T : l^\infty \mapsto l^\infty$ defined by $y = (\eta_j) = Tx$, $\eta_j = \xi_j/j$, $x = (\xi_j)$, is linear and bounded.

For any $x, z \in l^\infty$, $x = (\xi_j)$ and $z = (z_j)$, so $x + z = (\xi_j + z_j)$. We know that $[T(x+z)]_j = (\xi_j + z_j)/j$, $(Tx)_j = \xi_j/j$, and $(Tz)_j = z_j/j$. Therefore, $[T(x+z)]_j = (Tx)_j + (Tz)_j$ for all j, so $T(x+z) = Tx + Tz$. For any scalar a, $[T(ax)]_j = (a\xi_j)/j$, while $a(Tx)_j = a(\xi_j)/j$. Thus, $[T(ax)]_j = a(Tx)_j$, which implies that $T(ax) = aTx$. Therefore, T is a linear map.

To prove T is bounded, notice that $\|Tx\|_\infty = \sup_{i \in \mathbb{N}^+} \xi_j/j$ and $\|x\|_\infty = \sup_{i \in \mathbb{N}^+} \xi_j$. Since for each j, we have $|x_j|/j \leq |x_j|$, we conclude that

$$\|Tx\|_\infty = \sup_{i \in \mathbb{N}^+} \xi_j/j \leq \sup_{i \in \mathbb{N}^+} \xi_j = \|x\|_\infty$$

This implies that T is bounded.

Problem 2.7-6. Show that the range $\mathcal{R}(T)$ of a bounded linear operator $T : X \mapsto Y$ need not be closed in Y. Hint: Use T in Problem 2.7-5.

Take $X = Y = l^\infty$, and consider T in Problem 2.7-5. We need to construct an sequence in $\mathcal{R}(T)$ that converges to element in Y but not in $\mathcal{R}(T)$. Consider the vector y defined by $y_i = \frac{1}{\sqrt{i}}$ for $i = 1, 2, \ldots$. It is obvious that $y_i \leq 1$, so $y \in Y$. Also, $y \notin \mathcal{R}(T)$, because if it is in the range, its pre-image should be $(1, \sqrt{2}, \ldots, \sqrt{n}, \ldots)$, but the pre-image vector is not in Y since the entry tends to infinity. This implies that $y \notin \mathcal{R}(T)$. We can consider the sequence of vector $x^{(n)}$ whose first n entries are $1, \sqrt{2}, \ldots, \sqrt{n}$ and others are zero. Then all $x^{(n)} \in X$, so we can apply T to them, their images are $y^{(n)}$ whose first n entries are $1, 1/\sqrt{2}, \ldots, 1/\sqrt{n}$ and others are zero. Then, as $n \to \infty$,

$$\|y^{(n)} - y\|_\infty = \sup_{i=n+1} \frac{1}{\sqrt{i}} = \frac{1}{\sqrt{n+1}} \to 0$$

Therefore, y in Y is a limit point of $y^{(n)} \in \mathcal{R}(T)$, but it is not in range of T, so the range of T is not closed in Y.

Problem 2.7-7. Let T be a bounded linear operator from a normed space X onto a normed space Y. If there is a positive b such that $\|Tx\|_Y \geq b\|x\|_X$ for all $x \in X$, show that then $T^{-1} : Y \mapsto X$ exists and is bounded.

Since we have known that T is onto mapping, we only need to prove that T is injective, that is, the kernel of T is zero vector in X. Consider any x such that $Tx = 0_Y$, $\|Tx\|_Y = 0$, but

$\|Tx\|_Y \geq b\|x\|_X$ for all $x \in X$ and $b > 0$, this impiles that $x = 0_X$. Therefore, T is bijective linear mapping from X to Y, so $T^{-1} : Y \mapsto X$ exists.

To prove it is bounded, we consider any $y \in Y$, there exists $x \in X$ such that $T^{-1}y = x$ and $T(x) = y$. Therefore, we have

$$\|T^{-1}y\|_X = \|x\|_X \leq \frac{1}{b}\|Tx\|_Y = \|y\|_Y$$

Therefore, T^{-1} is bounded.

Problem 2.7-8. Show that the inverse $T^{-1} : \mathcal{R}(T) \mapsto X$ of a bounded linear operator $T : X \mapsto Y$ need not be bounded. Hint. Use T in Problem 2.7-5.

Let $X = Y = l^\infty$, and since T in Problem 2.7-5. is bounded linear operator, we need to prove T is injective. Consider $Tx = 0_Y$, it is easy to see that $x_i = iy_i = 0$ for all i, thus $x = 0_X$ and T is injective. Therefore, on $\mathcal{R}(T) \mapsto X$, T^{-1} is well-defined. However, consider the standard basis of l^∞, i.e., $e_1, e_2, \ldots$, where e_i means the vector with i-th entry equal to 1 and others all 0. All of them are in $\mathcal{R}(T)$ because their pre-images are just $e_1, 2e_2, \ldots$ and for all i, $ie_i \in l^\infty$. On the other hand, $\|T^{-1}e_i\| = i$ for all i. This implies that T^{-1} maps bounded vectors e_i to unbounded vectors. Therefore, T^{-1} cannot be bounded.

Problem 2.8-3. Find the norm of the linear functional f defined on $\mathcal{C}[-1, 1]$ by

$$f(x) = \int_{-1}^{0} x(t)\, dt - \int_{0}^{1} x(t)\, dt$$

Notice that for $x \in \mathcal{C}[-1, 1]$, we use the maximum norm. First we consider

$$|f(x)| \leq \left|\int_{-1}^{0} x(t)\, dt\right| + \left|\int_{0}^{1} x(t)\, dt\right| \leq \int_{-1}^{0} \|x(t)\|_\infty\, dt + \int_{0}^{1} \|x(t)\|_\infty\, dt \leq 2\|x\|_\infty$$

Therefore, $|f(x)|/\|x\|_\infty \leq 2$. Then we consider a sequence of function $x_n(t)$ on $\mathcal{C}[-1, 1]$ defined by

$$x_n(t) = \begin{cases} 1 & \text{if } t \in [\frac{1}{n}, 1] \\ -1 & \text{if } t \in [-1, -\frac{1}{n}] \\ nx & \text{if } t \in (-\frac{1}{n}, \frac{1}{n}) \end{cases}$$

Then it is obvious that $x_n \in \mathcal{C}[-1, 1]$ and $\|x_n\|_\infty = 1$. Furthermore, for all $n \geq 1$, we have

$$|f(x_n)| = \left|-1 + \frac{1}{2n} - \left(1 - \frac{1}{2n}\right)\right| = 2 - \frac{1}{n}$$

Therefore, $\sup_{\|x\|_\infty = 1} |f(x)| = 2$, so the norm of the linear functional f is 2.

Problem 2.8-4. Show that for $J = [a, b]$,

$$f_1(x) = \max_{t \in J} x(t)$$

$$f_2(x) = \min_{t \in J} x(t)$$

define functionals on $\mathcal{C}[a, b]$. Are they linear? Bounded?

Since $x(t)$ is continuous function defined on compact set $[a, b]$, $x(t)$ must attained its maximum and minimum point at some $t_1, t_2 \in [a, b]$. Therefore, $f_1(x), f_2(x)$ are finite on a field, so they are well-defined functionals.

They are not linear. Consider $x(t) = t$ and $y(t) = -t$ for all $t \in J$. Then $f_1(x) = b$ and $f_1(y) = -a$. Since $x(t)+y(t) \equiv 0$, $f_1(x+y) = 0$, but $f_1(x)+f_1(y) = b-a$. As long as $a \neq b$, this f_1 is not linear. Similarly, since $f_2(x) = a$, $f_2(y) = -b$, and $f_2(x+y) = 0$, as long as $a \neq b$, f_2 is not linear. Therefore, f_1, f_2 is not linear for any $a \neq b$. However, when $a = b$, f_1, f_2 is indeed linear.

They are bounded. For f_1, for any $x \in \mathcal{C}[a, b]$, we have

$$|f_1| = \left|\max_{t\in J} x(t)\right| \leq \max_{t\in J} |x(t)| = \|x\|_\infty$$

For f_2, for any $x \in \mathcal{C}[a, b]$, we have

$$|f_2| = \left|\min_{t\in J} x(t)\right| \leq \max_{t\in J} |x(t)| = \|x\|_\infty$$

Therefore, f_1, f_2 are bounded.

Problem 2.8-7. If f is a bounded linear functional on a complex normed space, is $\bar{f}$ bounded? Linear? (The bar denotes the complex conjugate.)

Since f is bounded and $|f(x)| = |\bar{f}(x)|$ because the norm of complex number a is defined by $\sqrt{a\bar{a}}$ and the complex conjugate of $\bar{a}$ is a. This implies that if $|f(x)| \leq c\|x\|$ then $|\bar{f}(x)| \leq c\|x\|$. Thus, $\bar{f}$ is bounded.

It is not linear, because here the field is complex field. If it is linear, then for any $a \in \mathbb{C}$, we need to have $\bar{f}(ax) = a\bar{f}(x)$. Since f is linear, suppose $f(x) = i$ and $f(ix) = if(x) = -1$. Let $a = i$, then $\bar{f}(ix) = \overline{f(ix)} = -1$, but $i\bar{f}(x) = i(-i) = 1$, thus $\bar{f}$ is not linear.

Problem 2.8-13. If Y is a subspace of a vector space X and f is a linear functional on X such that $f(Y)$ is not the whole scalar field of X, show that $f(y) = 0$ for all $y \in Y$.

Suppose $f(y^0) \neq 0$ for some $y^0 \in Y$, then suppose $f(y^0) = p \neq 0$. Since f is a linear functional, assume the scalar field of X is $\mathbb{F}$, then $f : X \mapsto \mathbb{F}$. Since $p \in \mathbb{F}$ and any nonzero element in a field has inverse element, i.e., there exists $p^{-1}p = 1$. Then for any $a \in \mathbb{F}$, there exists a scalar $ap^{-1} \in \mathbb{F}$ such that $f(ap^{-1}y^0) = ap^{-1}f(y^0) = a$. Since Y is a subspace, $ap^{-1}y^0 \in Y$, and this implies that $f(Y) = \mathbb{F}$. Therefore, this contradiction shows that such y^0 does not exist, i.e., $f(y) = 0$ for all $y \in Y$.

Problem 2.8-14. Show that the norm $\|f\|$ of a bounded linear functional $f \neq 0$ on a normed space X can be interpreted geometrically as the reciprocal of the distance $\tilde{d} = \inf\{\|x\|_X \mid f(x) = 1\}$ of the hyperplane $H_1 = \{x \in X \mid f(x) = 1\}$ from the orgin.

We need to show that

$$\sup_{\|x\|_X=1} |f(x)| = \frac{1}{\inf_{f(x)=1} \|x\|_X}$$

To achieve this, we first prove

$$\frac{1}{\inf\limits_{f(x)=1} \|x\|_X} = \sup_{f(x)=1} \frac{1}{\|x\|_X} \tag{1}$$

For any $x \in X$ such that $f(x) = 1$, we have $\|x\|_X \geq \inf_{f(x)=1} \|x\|_X$, this implies that

$$\frac{1}{\inf\limits_{f(x)=1} \|x\|_X} \geq \frac{1}{\|x\|_X}$$

Since the LHS is an upper bound of RHS, it must be larger than or equal to least upper bound of RHS, i.e.,

$$\frac{1}{\inf\limits_{f(x)=1} \|x\|_X} \geq \sup_{f(x)=1} \frac{1}{\|x\|_X}$$

For any $\epsilon > 0$, there exists x_ϵ with $f(x_\epsilon) = 1$, and $\|x_\epsilon\|_X - \epsilon < \inf_{f(x)=1} \|x\|_X$, therefore,

$$\frac{1}{\inf\limits_{f(x)=1} \|x\|_X - \epsilon} < \frac{1}{\|x_\epsilon\|_X} \leq \sup_{f(x)=1} \frac{1}{\|x\|_X}$$

Notice that $\inf_{f(x)=1} \|x\|_X$ is positive fixed number, because if not, then there exists x_n such that $\|x_n\|_X \to 0$. Since f is bounded, so $|f(x_n)| \leq \|f\| \|x_n\|_X \to 0$, but $f(x_n) = 1$ for all n. This is contradiction, so when ϵ is small enough, $\inf_{f(x)=1} \|x\|_X - \epsilon$ will always be positive. Let $\epsilon \to 0$, we have

$$\frac{1}{\inf\limits_{f(x)=1} \|x\|_X} \leq \sup_{f(x)=1} \frac{1}{\|x\|_X}$$

Therefore, the first equality is proved. Now we consider to prove

$$\sup_{\|x\|_X=1} |f(x)| = \sup_{f(x)=1} \frac{1}{\|x\|_X} \tag{2}$$

Notice that by Fact 1 in lecture, we have

$$\sup_{\|x\|_X=1} |f(x)| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|_X} = \sup_{f(x) \neq 0, x \neq 0} \frac{1}{\|\frac{x}{f(x)}\|_X}$$

We only need to prove the two sets are equal, i.e.,

$$\left\{ \frac{x}{f(x)} \,\middle|\, f(x) \neq 0 \right\} = \{x \,|\, f(x) = 1\}$$

For any elements in LHS, it has form $x/f(x)$, and $f(x/f(x)) = f(x)/f(x) = 1$, thus it is in RHS. Similarly, for any elements x in RHS, $x = x/f(x)$, thus in LHS. Therefore, these two sets are equal, then

$$\sup_{\|x\|_X=1} |f(x)| = \sup_{f(x) \neq 0} \frac{1}{\|\frac{x}{f(x)}\|_X} = \sup_{f(x)=1} \frac{1}{\|x\|_X}$$

Therefore, combined (1) and (2), we can conclude the desired result.

Problem 2.8-15. Let $f \neq 0$ be a bounded linear functional on a real normed space X. Then for any scalar c we have a hyperplane $H_c = \{x \in X \,|\, f(x) = c\}$, and H_c determines the two half spaces

$$X_{c1} = \{x \,|\, f(x) \leq c\} \qquad \text{and} \qquad X_{c2} = \{x \,|\, f(x) \geq c\}$$

Show that the closed unit ball lies in X_{c1} where $c = \|f\|$, but for no $\epsilon > 0$, the half space X_{c1} with $c = \|f\| - \epsilon$ contains that ball.

To show that the closed unit ball lies in X_{c1}, consider the closed unit ball $B(0;1)$, then any point x satisfying $\|x\|_X \leq 1$ is in this ball. If $\|x\|_X \leq 1$, then $|f(x)| \leq \|f\|\|x\|_X = c$, therefore, any points satisfies $\|x\|_X \leq 1$ are in X_{c1}, and all points in $B(0;1)$ should satisfy $\|x\|_X \leq 1$, thus $B(0;1)$ is contained in X_{c1}.

Since $\|f\| = \sup_{\|x\|_X=1} |f(x)|$, for any $\epsilon > 0$, there exists x with $\|x\|_X = 1$ such that $|f(x)| > \|f\| - \epsilon = c$. However, such x is a point on the closed unit ball $B(0;1)$, while it does not satisfy $f(x) \leq c$. This implies that for any $\epsilon > 0$, the half space X_{c1} with c defined above cannot contain $B(0;1)$.

Problem 2.9-8. If Z is an $(n-1)$-dimensional subspace of an n-dimensional vector space X, show that Z is the null space of a suitable linear functional f on X, which is uniquely determined to within a scalar multiple.

For $(n-1)$-dimensional vector space Z, we can find a basis of it, i.e., $\{e_1, \ldots, e_{n-1}\}$. By basis extension theorem, we can extend this set of independent vectors to the basis of n-dimensional vector space X, i.e., $\{e_1, \ldots, e_{n-1}, u_n\}$. Define a linear functional f such that $f(e_i) = 0$ for all $i = 1, \ldots, n-1$ and $f(u_n) = 1$. Then, by linearity, all $x \in X$ is defined under f.

Now we check whether the null space of f is Z. Let $f(x) = 0$, then since $x = a_1 e_1 + \ldots + a_{n-1}e_{n-1} + b_n u_n$, we have $f(x) = b_n f(u_n) = b_n = 0$. Therefore, $x = a_1 e_1 + \ldots + a_{n-1}e_{n-1}$, which implies that $x \in Z$. In this way, Z is the null space of f.

For the uniqueness, since Z is null space, for all $z \in Z$, $f(z) = 0$. This implies that $f(e_i) = 0$ for all $i = 1, \ldots, n-1$. If $f(u_n) = 0$, then the null space of f is X with n-dimension rather than Z with $(n-1)$-dimension. Therefore, $f(u_n) = p \neq 0$, and this implies that for all $x \in X$, $f(x) = b_n p$. Notice that if we define another $\tilde{f}(x) = b_n \tilde{p}$ for all $x \in X$, then $\tilde{f}(x) = \frac{\tilde{p}}{p} f(x)$. Since $p \neq 0$, $\frac{\tilde{p}}{p}$ is a scalar, and f is defined uniquely up to a scalar multiple.

Problem 2.9-10. Let Z be a proper subspace of an n-dimensional vector space X, and let $x_0 \in X - Z$. Show that there is a linear functional f on X such that $f(x_0) = 1$ and $f(x) = 0$ for all $x \in Z$.

Since Z is a proper subspace of an n-dimensional vector space X, denote $\dim(Z) = p < n$. By choosing a basis of Z and extending it to a basis of X, we can use coordinates to express every vector in Z, i.e., $(x^{(1)}, \ldots, x^{(p)}, 0, \ldots, 0)$ and arbitrary vector in X as $(x^{(1)}, \ldots, x^{(n)})$. Define a functional f such that for all $x \in X$,

$$f_{x_0}(x) = \frac{(x_0^{(p+1)}, \ldots, x_0^{(n)}) \cdot (x^{(p+1)}, \ldots, x^{(n)})^{\mathrm{T}}}{(x_0^{(p+1)}, \ldots, x_0^{(n)}) \cdot (x_0^{(p+1)}, \ldots, x_0^{(n)})^{\mathrm{T}}}$$

Therefore, we can see that $f(x_0) = 1$ and $f(x) = 0$ when $x \in Z$. This functional is linear because for scalar a, b and $x, y \in X$, we have

$$\begin{aligned} f_{x_0}(ax+by) &= \frac{(x_0^{(p+1)},\dots,x_0^{(n)})\cdot((ax+by)^{(p+1)},\dots,(ax+by)^{(n)})^{\mathrm{T}}}{(x_0^{(p+1)},\dots,x_0^{(n)})\cdot(x_0^{(p+1)},\dots,x_0^{(n)})^{\mathrm{T}}} \\ &= \frac{(x_0^{(p+1)},\dots,x_0^{(n)})\cdot(ax^{(p+1)}+by^{(p+1)},\dots,ax^{(n)}+bx^{(n)})^{\mathrm{T}}}{(x_0^{(p+1)},\dots,x_0^{(n)})\cdot(x_0^{(p+1)},\dots,x_0^{(n)})^{\mathrm{T}}} \\ &= a\frac{(x_0^{(p+1)},\dots,x_0^{(n)})\cdot(x^{(p+1)},\dots,x^{(n)})^{\mathrm{T}}}{(x_0^{(p+1)},\dots,x_0^{(n)})\cdot(x_0^{(p+1)},\dots,x_0^{(n)})^{\mathrm{T}}} + b\frac{(x_0^{(p+1)},\dots,x_0^{(n)})\cdot(y^{(p+1)},\dots,y^{(n)})^{\mathrm{T}}}{(x_0^{(p+1)},\dots,x_0^{(n)})\cdot(x_0^{(p+1)},\dots,x_0^{(n)})^{\mathrm{T}}} \\ &= af_{x_0}(x)+bf_{x_0}(y) \end{aligned}$$

Therefore, we have construct a linear functional f_{x_0} that satisfies the required property.

Problem 2.9-11. If x and y are different vectors in a finite dimensional vector space X, show that there is a linear functional f on X such that $f(x) \neq f(y)$.

Since X is finite dimensional vector space, we can find a basis of X, i.e., $\{e_1, \dots, e_n\}$. Therefore, any vector can be written as linear combination of basis, i.e., $u = a_1e_1 + \dots, a_ne_n$, where a_i's are scalar. After taking arbitrary distinct vectors x and y in X, we can define $f(z) = (x-y)^{\mathrm{T}}z$ for any $z \in X$. Here the inner product is defined in the same way as before, i.e., for $u = a_1e_1+\dots+a_ne_n$ and $v = b_1e_1 + \dots + b_ne_n$, we have $u^{\mathrm{T}}v = \sum_{i=1}^n a_ib_i$. We need to prove $f(z)$ is linear and $f(x) \neq f(y)$.

First, $f(z)$ is linear, because for $u, v \in X$, suppose $x = x_1e_1+\dots+x_ne_n$ and $y = y_1e_1+\dots+y_ne_n$, we have

$$\begin{aligned} f(u+v) = (x-y)^{\mathrm{T}}(u+v) &= \sum_{i=1}^n (x_i-y_i)(a_i+b_i) = \sum_{i=1}^n (x_i-y_i)a_i + \sum_{i=1}^n (x_i-y_i)b_i \\ &= (x-y)^{\mathrm{T}}u + (x-y)^{\mathrm{T}}v = f(u)+f(v) \end{aligned}$$

For any scalar p_0, consider

$$f(p_0u) = \sum_{i=1}^n (x_i-y_i)(p_0a_i) = p\sum_{i=1}^n (x_i-y_i)a_i = p_0(x-y)^{\mathrm{T}}u = p_0f(u)$$

Since f is linear, $f(x) - f(y) = f(x-y) = \|x-y\|_2^2$. Therefore, $f(x) = f(y)$ if and only if $x = y$, but x, y are distinct, so $f(x) \neq f(y)$.

Problem 2.9-12. If $f_1, \dots, f_p$ are linear functionals on an n-dimensional vector space X, where $p < n$, show that there is a vector $x \neq 0$ in X such that $f_1(x) = 0, \dots, f_p(x) = 0$. What consequences does this result have with respect to linear equations?

Assume $(a_1, \dots, a_n)$ as a basis of X. For any $x \in X$, $x = x_1a_1+\dots+x_na_n$. For any $k = 1, \dots, p$,

$$f_k(x) = f_k(x_1a_1+\dots+x_na_n) = x_1f_k(a_1)+\dots+x_nf_k(a_n)$$

Therefore, we can obtain a linear system, $Ax = \mathbf{0}$, where

$$A = \begin{bmatrix} f_1(a_1) & f_1(a_2) & \cdots & f_1(a_n) \\ f_2(a_1) & f_2(a_2) & \cdots & f_2(a_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_p(a_1) & f_p(a_2) & \cdots & f_p(a_n) \end{bmatrix}$$

and $x = (x_1, \dots, x_n)$. By using Gaussian Elimination, the RREF of A has at most p pivots. Therefore, the dimension of the null space is $n - p \geq 1$. This implies that there must be nontrivial solution to linear system $Ax = \mathbf{0}$.

The consequence is that every homogeneous system of linear equations in which the number of variables is larger than the number of the equations has a nontrivial solution.

Problem 2.10-4. Let X and Y be normed spaces and $T_n : X \mapsto Y$ $(n = 1, 2, \dots)$ bounded linear operators. Show that convergence $T_n \to T$ implies that for every $\epsilon > 0$ there is an N such that for all $n > N$ and all x in any given closed ball we have $\|T_n x - Tx\| < \epsilon$.

Fix the radius of the given closed ball as $r > 0$. Since $T_n \to T$, for $\epsilon > 0$, there exists N such that for all $n > N$, we have $\|T_n - T\| < \epsilon/r$. For all x in the ball, $\|x\|_X \leq r$. Thus, we have for

$$\|T_n x - Tx\|_Y = \|(T_n - T)x\|_Y \leq \|T_n - T\|\|x\|_X$$

For any $\epsilon > 0$, for $n > N$, we have

$$\|T_n x - Tx\|_Y \leq \|T_n - T\|\|x\|_X < \epsilon/r \cdot r = \epsilon$$

Therefore, we obtain the desired result.

Problem 2.10-13. Let $M \neq \varnothing$ be any subset of a normed space X. The annihilator M^a of M is defined to be the set of all bounded linear functionals on X which are zero everywhere on M. Thus M^a is a subset of the dual space X' of X. Show that M^a is a vector subspace of X' and is closed. What are X^a and $\{0\}^a$?

For any $f, g \in M^a$ and scalar α, β, we have $f(x) = g(x) = 0$ for all $x \in M$. Then for every $x \in M$,

$$(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \alpha 0 + \beta 0 = 0$$

Therefore, $\alpha f + \beta g \in M^a$, so M^a is a vector subspace of X'.

To prove M^a is closed, for each $x \in X$, we can define a set $P_x = \{f \in X' \,|\, f(x) = 0\}$. Then we first prove each P_x is closed. Consider any convergent sequence in P_x, i.e., $f_n \to f \in X'$. Since $f_n \to f$, then for any fixed $u \in X$, $|f_n(u) - f(u)| \leq \|f_n - f\|\|u\|_X \to 0$ as $n \to \infty$. This implies $f_n(x) \to f(x)$, but since $f_n(x)$ is constant zero, so $f(x) = 0$ meaning that $f \in P_x$ and P_x is closed. Notice that $M^a = \bigcap_{x \in M} P_x$, and any intersection of closed sets are closed, so M^a is closed.

It is easy to see X^a is a singleton of zero function defined on X, i.e., a set only contains zero vector of X'. For $\{0\}^a$, it is just X' itself, because for all function in X', it must satisfy $f(0) = 0$.

Extra Problem 1. Let X be a compact metric space. Prove that X is separable, i.e., there exists an at most countable subset of X that is dense in X. Hint: $\forall n \geq 1$, since X is compact, there exists finitely many balls of radius $\frac{1}{n}$, covering X. Denote the centers of these balls by $x_1^n, \dots, x_{k_n}^n$. Define $S_n = \{x_1^n, \dots, x_{k_n}^n\}$ and $S = \bigcup_{n=1}^{\infty} S_n$.

For any integer $n \geq 1$, denote $B(x; \frac{1}{n})$ where $x \in X$ as the open ball centered at x with radius $\frac{1}{n}$. Clearly, the collection of $B(x; \frac{1}{n})$ for all $x \in X$ forms an open cover of X. Since X is compact, there exists an finite subcover $B(x_1^n; \frac{1}{n}), \ldots, B(x_{k_n}^n; \frac{1}{n})$ that covers X, where k_n denote the number of the open ball with radius $\frac{1}{n}$. Define $S_n = \{x_1^n, \ldots, x_{k_n}^n\}$ for all $n \geq 1$, and $S = \bigcup_{n=1}^{\infty} S_n$. Since S_n is finite, so the countable union of finite set is at most countable. Therefore, we need to prove S is dense in X.

Consider any point u in $X \setminus S$, we are going to prove it is a limit point of S. Take $n = 1$, then since S_1 covers X, there exists $y_1 \in \{1, \ldots, k_1\}$ such that $d(x_{y_1}^1, u) \leq 1$, where d is the metric function defined on X. Similarly, for $n = 2$, we can find $y_2 \in \{1, \ldots, k_2\}$ such that $d(x_{y_2}^2, u) \leq \frac{1}{2}$. Continue doing this, we can find a sequence y_n such that $d(x_{y_n}^n, u) \leq \frac{1}{n}$. Therefore, there exists sequence $x_{y_n} \in S$ such that $x_{y_n} \to u$, so u is a limit point of S. In conclusion, S is at most countable and dense in X, so X is separable.

Extra Problem 2. Let $\mathbb{R}^m$ $(m \geq 1)$ be equipped with the standard norm

$$\|(x_1, \ldots, x_m)\|_{\mathbb{R}^m} = \sqrt{x_1^2 + \ldots + x_m^2}$$

Let $A = (a_{ij})_{n \times m}$ be a $n \times m$ real matrix. Define mapping $T : \mathbb{R}^m \mapsto \mathbb{R}^n$ by $Tx = A(x_1 \cdots x_m)^{\mathrm{T}}$. Prove that T is linear and

$$\|T\| \leq \sqrt{\sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij}^2}$$

First, we prove T is linear. For $x, y \in \mathbb{R}^m$, by the distributive law of matrix multiplication, we have

$$T(x+y) = A(x_1+y_1, \ldots, x_m+y_m)^{\mathrm{T}} = A(x+y) = Ax+Ay = A(x_1 \cdots x_m)^{\mathrm{T}}+A(y_1 \cdots y_m)^{\mathrm{T}} = Tx+Ty$$

For any scalar $a \in \mathbb{R}$, we have

$$T(ax) = A(ax_1 \cdots ax_m)^{\mathrm{T}} = A(ax) = aAx = aA(x_1 \cdots x_m)^{\mathrm{T}} = aTx$$

Therefore, T is obviously linear operator.

Now, we consider the norm of operator T. Since for $\|x\|_{\mathbb{R}^m} = 1$, we have

$$\begin{aligned}
\|Tx\|_{\mathbb{R}^n} = \|Ax\|_{\mathbb{R}^n} &= \left\| \left(\sum_{k=1}^{m} a_{1k}x_k, \ldots, \sum_{k=1}^{m} a_{nk}x_k \right)^{\mathrm{T}} \right\|_{\mathbb{R}^n} \\
&= \sqrt{\sum_{j=1}^{n} \left(\sum_{k=1}^{m} a_{jk}x_k \right)^2} \leq \sqrt{\sum_{j=1}^{n} \left(\sum_{k=1}^{m} a_{jk}^2 \sum_{k=1}^{m} x_k^2 \right)} \\
&\leq \sqrt{\sum_{j=1}^{n} \left(\sum_{k=1}^{m} a_{jk}^2 \|x\|_{\mathbb{R}^m}^2 \right)} = \sqrt{\sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij}^2}
\end{aligned}$$

Therefore, if we take the supremum on both sides, we have

$$\|T\| = \sup_{\|x\|_{\mathbb{R}^m}=1} \|Tx\|_{\mathbb{R}^n} \leq \sqrt{\sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij}^2}$$

Extra Problem 3. Let $X = \mathcal{C}[-1,1]$ and $f \in X^*$ be the bounded functional defined by

$$f(x) = \int_{-1}^{0} x(t)\,dt - \int_{0}^{1} x(t)\,dt$$

Let $Y = \mathcal{N}(f)$ (null space of f). Thus Y is a closed subspace of X (why?). Let $u = u(t) = -2t \implies f(u) = 2$. Observe that $\inf_{y\in Y}\|u-y\| = \inf_{z\in X, f(z)=2}\|z\|$. Prove that the latter inf is not attained and so the former inf is also not attained.

First, Y is subspace. Take any scalar a, b and $x, y \in Y$, then $f(x) = 0$ and $f(y) = 0$. Consider linear combination

$$\begin{aligned} f(ax+by) &= \int_{-1}^{0} ax(t) + by(t)\,dt - \int_{0}^{1} ax(t)+by(t)\,dt \\ &= a\int_{-1}^{0} x(t)\,dt + b\int_{-1}^{0} y(t)\,dt - a\int_{0}^{1} x(t)\,dt - b\int_{0}^{1} y(t)\,dt \\ &= a\left(\int_{-1}^{0} x(t)\,dt - \int_{0}^{1} x(t)\,dt\right) + b\left(\int_{-1}^{0} y(t)\,dt - \int_{0}^{1} y(t)\,dt\right) = af(x)+bf(y) = 0 \end{aligned}$$

Therefore, $ax + by \in Y$, which means Y is a subspace of X. Then, to prove Y is closed, take a convergent sequence $x_n(t) \in Y$ where $x_n(t) \to x(t) \in X$, we need to show $f(x) = 0$. This is true because

$$f(x) = f\left(\lim_{n\to\infty} x_n\right) = \int_{-1}^{0} \lim_{n\to\infty} x_n(t)\,dt - \int_{0}^{1} \lim_{n\to\infty} x_n(t)\,dt$$

Since norm on X is maximum norm, $x_n(t) \to x(t)$ uniformly, and we can exchange the order of integral and limit, i.e.,

$$\int_{-1}^{0} \lim_{n\to\infty} x_n(t)\,dt - \int_{0}^{1} \lim_{n\to\infty} x_n(t)\,dt = \lim_{n\to\infty}\left(\int_{-1}^{0} x_n(t)\,dt - \int_{0}^{1} x_n(t)\,dt\right) = \lim_{n\to\infty} f(x_n)$$

Therefore, $f(x) = \lim_{n\to\infty} f(x_n) = 0$, and $x \in Y$, Y is closed.

Next, we need to prove for $y(t), z(t) \in \mathcal{C}[-1,1]$,

$$E = \{u(t) - y(t)\,|\,u(t) = -2t, f(y) = 0\} = \{z(t)\,|\,f(z) = 2\} = F$$

This is easy, since for any $p(t) \in E$, $p(t) = u(t) - y(t)$, so $f(p) = f(u) - f(y) = 2 - 0 = 2$, and $p(t) \in F$. For any $q(t) \in F$, since $f(q) = 2$, let $y(t) = u(t) - q(t)$, then $f(y) = 0$ and $y \in \mathcal{C}[-1,1]$, so $q(t) \in E$. Thus, $E = F$.

Finally, we consider $\inf_{z\in X, f(z)=2}\|z\|$. Since from Problem 2.8.3, we have $|f(x)| \le 2\|x\|_\infty$, for all z satisfying $f(z) = 2$, $\|z\|_\infty \ge 1$. Consider the sequence of function $z_k(t)$ defined by

$$z_k(t) = \begin{cases} \frac{2k}{2k-1} & \text{if } t \in [-1, -\frac{1}{k}] \\ -\frac{2k}{2k-1} & \text{if } t \in [\frac{1}{k}, 1] \\ -\frac{2k^2}{2k-1}x & \text{if } t \in (-\frac{1}{k}, \frac{1}{k}) \end{cases}$$

It is easy to see that $f(z) = 2$ by calculating the integral in the definition of f. Notice that $\|z\|_\infty = \frac{2k}{2k-1}$, so when $k \to \infty$, $\|z\|_\infty \to 1$. This implies that $\inf_{z\in X, f(z)=2}\|z\| = 1$.

However, there does not exist $z(t) \in \mathcal{C}[-1,1]$ such that $f(z) = 2$ and $\|z\| = 1$. Since

$$\left|\int_{-1}^{0} z(t)\,dt\right| \le \int_{-1}^{0} \|z(t)\|_\infty\,dt = 1 \implies \int_{-1}^{0} z(t)\,dt \in [-1,1]$$

Similarly, we have $\int_0^1 x(t)\ dt \in [-1,1]$. However, $f(z) = 2$ implies that $\int_{-1}^0 z(t)\ dt = 1$ and $\int_0^1 z(t)\ dt = -1$. From $\|z\| = 1$ we have $|z(t)| \leq 1$ for all $t \in [-1,1]$, and combined with $\int_{-1}^0 z(t)\ dt = 1$, we can conclude that $z(t) = 1$ almost everywhere on $(-1,0)$. By continuity of $z(t)$, $z(t) = 1$ on $(-1,0)$. Similarly from $\int_0^1 z(t)\ dt = -1$ we can imply that $z(t) = -1$ on $(0,1)$. This is a contradiction since $z(t)$ has jump discontinuity at 0, but we assume $z(t) \in \mathcal{C}[-1,1]$. Therefore, $\|z\| = 1$ can not be attained, $\|z\|$ can be arbitrarily closed to 1 but must be strictly larger than 1.