MAT4010: Functional Analysis
Homework 3

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Problem 2.10-8. Show that the dual space of the space $c_0 = \{(x_1, x_2, \ldots, x_n, \ldots) \mid x_n \to 0, \text{ as } n \to \infty\}$ is l^1. Also prove that c_0 is Banach under the norm of l^∞, i.e., $\|(x_1, \ldots, x_n, \ldots)\| = \sup_{n \geq 1} |x_n|$.

We first prove that for all $x = (x_1, \ldots, x_n, \ldots) \in c_0$, if $e_i \in c_0$ are standard basis, then $\sum_{i=1}^n x_i e_i \to x$ as $n \to \infty$ under l^∞ norm. This is trivial because

$$\lim_{n \to \infty} |x_n| = 0 \Longrightarrow \lim_{n \to \infty} \sup_{m \geq n} |x_m| = 0 \Longrightarrow \left\| x - \sum_{i=1}^n x_i e_i \right\|_\infty = \sup_{m \geq n} |x_m| \to 0$$

This implies that for all $f \in (c_0)'$, we have

$$f(x) = f\left(\lim_{n \to \infty} \sum_{i=1}^n x_i e_i\right) = \lim_{n \to \infty} f\left(\sum_{i=1}^n x_i e_i\right) = \sum_{i=1}^\infty x_i f(e_i)$$

Then we need to prove $(f(e_1), \ldots, f(e_n), \ldots) \in l^1$. Let $z^{(n)} = (y_1, \ldots, y_n, 0, 0, \ldots)$, where $y_n = \frac{|f(e_n)|}{f(e_n)}$ if $f(e_n) \neq 0$; $y_n = 0$ if $f(e_n) = 0$. Clearly all $z^{(n)} \in c_0$, so we consider

$$|f(z^{(n)})| = \left| \sum_{i=1}^n y_i f(e_i) \right| = \sum_{i=1}^n |f(e_i)| \leq \|f\| \max_{i=1,\ldots,n} |y_n| \leq \|f\|$$

because the maximum of $|y_n|$ can only be 0 or 1. Consider $\sum_{i=1}^n |f(e_i)| \leq \|f\|$, since it is satisifed for all n, so take $n \to \infty$, we have

$$\sum_{i=1}^\infty |f(e_i)| \leq \|f\| \Longrightarrow (f(e_1), \ldots, f(e_n), \ldots) \in l^1$$

Now we can define $T : (c_0)' \mapsto l^1$ as $Tf = (f(e_1), \ldots, f(e_n), \ldots)$. First, we prove T is linear. For all scalar a, b and $f, g \in (c_0)'$, we have

$$\begin{aligned} T(af + bg) &= ((af + bg)(e_1), \ldots, (af + bg)(e_n), \ldots) \\ &= (af(e_1) + bg(e_1), \ldots, af(e_n) + bg(e_n), \ldots) \\ &= a(f(e_1), \ldots, f(e_n), \ldots) + b(g(e_1), \ldots, g(e_n), \ldots) \\ &= aTf + bTg \end{aligned}$$

So T is linear.

Then we prove T is bijective. For injectivity, we only need to prove the kernel of T is the zero map. If $Tf = (0, \ldots, 0, \ldots)$, then $f(e_i) = 0$ for all i. This indeed means f is zero maps on c_0. For surjectivity, we take any $y = (y_1, \dots, y_n, \dots) \in l^1$, then define linear mapping f so that $f(e_i) = y_i$, then such f is in $(c_0)'$, because for all $x = (x_1, \dots, x_n, \dots) \in c_0$, (boundedness)

$$|f(x)| = \left| \sum_{i=1}^{\infty} x_i y_i \right| \leq \sup_i |x_i| \sum_{i=1}^{\infty} |y_i| = \|x\|_\infty \|y\|_{l^1}$$

Finally, we need to prove the isometry of $(c_0)'$ and l^1. Since we have already had for all $f \in (c_0)'$,

$$\sum_{i=1}^{\infty} |f(e_i)| \leq \|f\|$$

However, consider the function f defined above, since $|f(x)| \leq \|x\|_\infty \|y\|_{l^1}$, take the supremum over $\|x\|_\infty = 1$ yields

$$\|f\| \leq \|y\|_{l^1}$$

This means that $\|f\| = \|y\|_{l^1}$. Therefore, the dual space of c_0 is indeed l^1.

To prove c_0 is Banach under l^∞ norm, we only need to prove it is a closed subspace of l^∞-space. This is because we have known l^∞-space is Banach, and any closed subspace of Banach space is also Bananch. According to the question, c_0 is a vector space, and it is obviously a subset of l^∞ because for any element x in c_0, $x_n \to 0$, thus $|x_n|$ must be bounded above, and thus in l^∞. Therefore, the only thing we need to show is that c_0 is closed in l^∞.

Suppose there exists sequence $y^{(n)} = \left(y_1^{(n)}, \dots, y_k^{(n)}, \dots\right) \in c_0$ such that $y^{(n)} \to x$ where $x = (x_1, \dots, x_k, \dots) \notin c_0$. Then there exists some $\epsilon_0 > 0$ such that for all $N \in \mathbb{R}$ and $k \geq N$, we have $|x_k| \geq \epsilon_0$. Since $y^{(n)} \to x$, there exists N_2, such that for $n \geq N_2$, $\left|y_k^{(n)} - x_k\right| \leq \epsilon_0/2$ for all k. This implies that for $k \geq N$ and $n \geq N_2$, we have

$$\left|y_k^{(n)}\right| \geq |x_k| - \left|y_k^{(n)} - x_k\right| \geq \frac{\epsilon_0}{2}$$

This impiles that $y_k^{(n)}$ will not converge to zero as $k \to \infty$, i.e., $y^{(n)} \notin c_0$ for $n \geq N_2$. Contradiction shows that such $y^{(n)}$ and x doesn't exist, which means no limit point of c_0 can exist outside c_0, so it is closed.

Problem 3.1-4. If an inner product space X is real, show that the condition $\|x\| = \|y\|$ implies $\langle x + y, x - y \rangle = 0$. What does this mean geometrically if $X = \mathbb{R}^2$? What does the condition imply if X is complex?

If X is real, then $\langle x, y \rangle = \langle y, x \rangle$. Also, $\|x\| = \|y\|$ implies that $\langle x, x \rangle = \langle y, y \rangle$. Consider

$$\begin{aligned}\langle x + y, x - y \rangle &= \langle x, x - y \rangle + \langle y, x - y \rangle \\ &= \langle x, x \rangle - \langle x, y \rangle + \langle y, x \rangle - \langle y, y \rangle \\ &= \langle x, x \rangle - \langle y, y \rangle = 0\end{aligned}$$

Therefore, $\|x\| = \|y\|$ implies $\langle x + y, x - y \rangle = 0$.

If $X = \mathbb{R}^2$, then this just implies that the two diagonals of any diamond are perpendicular. If X is complex, then we don't have $\langle x, y \rangle = \langle y, x \rangle$. Instead, $\|x\| = \|y\|$ implies that

$$\langle x + y, x - y \rangle = -2i\mathrm{Im}\{\langle x, y \rangle\}, \quad \mathrm{Re}\{\langle x + y, x - y \rangle\} = 0$$

Furthermore, we can say

$$\langle x+y, x-y\rangle + \langle x-y, x+y\rangle = 0$$

Problem 3.1-8. Prove that for a real inner product space we have

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$$

Note that for real inner product space, $\langle x, y\rangle = \langle y, x\rangle$. Therefore, by linearity, we have

$$\begin{aligned}\|x+y\|^2 = \langle x+y, x+y\rangle &= \langle x, x+y\rangle + \langle y, x+y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle\end{aligned}$$

Similarly,

$$\begin{aligned}\|x-y\|^2 = \langle x-y, x-y\rangle &= \langle x, x-y\rangle - \langle y, x-y\rangle \\ &= \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle \\ &= \langle x, x\rangle - 2\langle x, y\rangle + \langle y, y\rangle\end{aligned}$$

Therefore,

$$\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) = \frac{4\langle x, y\rangle}{4} = \langle x, y\rangle$$

Problem 3.1-9. Prove that for a complex inner product space we have

$$\text{Re}\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$$

$$\text{Im}\langle x, y\rangle = \frac{1}{4}(\|x+iy\|^2 - \|x-iy\|^2)$$

For complex inner product space, $\langle x, y\rangle = \overline{\langle y, x\rangle}$, $\langle x, y\rangle + \overline{\langle y, x\rangle} = 2\text{Re}\langle y, x\rangle$, and $\langle x, y\rangle - \overline{\langle y, x\rangle} = 2i\text{Im}\langle y, x\rangle$. Therefore, by linearity, we have

$$\begin{aligned}\|x+y\|^2 = \langle x+y, x+y\rangle &= \langle x, x+y\rangle + \langle y, x+y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= \langle x, x\rangle + 2\text{Re}\langle x, y\rangle + \langle y, y\rangle\end{aligned}$$

Similarly,

$$\begin{aligned}\|x-y\|^2 = \langle x-y, x-y\rangle &= \langle x, x-y\rangle - \langle y, x-y\rangle \\ &= \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle \\ &= \langle x, x\rangle - 2\text{Re}\langle x, y\rangle + \langle y, y\rangle\end{aligned}$$

Therefore,

$$\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) = \frac{4\text{Re}\langle x, y\rangle}{4} = \text{Re}\langle x, y\rangle$$

Also consider

$$\begin{aligned}\|x+iy\|^2 = \langle x+iy, x+iy\rangle &= \langle x, x+iy\rangle + i\langle y, x+iy\rangle \\ &= \langle x,x\rangle - i\langle x,y\rangle + i\langle y,x\rangle + i\cdot(-i)\langle y,y\rangle \\ &= \langle x,x\rangle - i\cdot 2i\mathrm{Im}\langle x,y\rangle + \langle y,y\rangle \\ &= \langle x,x\rangle + 2\mathrm{Im}\langle x,y\rangle + \langle y,y\rangle\end{aligned}$$

Similarly,

$$\begin{aligned}\|x-iy\|^2 = \langle x-iy, x-iy\rangle &= \langle x, x-iy\rangle - i\langle y, x-iy\rangle \\ &= \langle x,x\rangle + i\langle x,y\rangle - i\langle y,x\rangle + (-i)\cdot i\langle y,y\rangle \\ &= \langle x,x\rangle + i\cdot 2i\mathrm{Im}\langle x,y\rangle + \langle y,y\rangle \\ &= \langle x,x\rangle - 2\mathrm{Im}\langle x,y\rangle + \langle y,y\rangle\end{aligned}$$

Therefore,

$$\frac{1}{4}(\|x+iy\|^2 - \|x-iy\|^2) = \frac{4\mathrm{Im}\langle x,y\rangle}{4} = \mathrm{Im}\langle x,y\rangle$$

Problem 3.2-3. Let X be the inner product space consisting of the polynomial $x = 0$ and all real polynomials in t, of degree not exceeding 2, considered for real $t \in [a,b]$, with inner product $\langle x,y\rangle = \int_a^b x(t)y(t)\,dt$. Show that X is complete. Let Y consist of all $x \in X$ such that $x(a) = 0$. Is Y a subspace of X? Do all $x \in X$ of degree 2 form a subspace of X?

First, it is easy to see that $\{1, t, t^2\}$ forms a basis of X. Also, $1, t, t^2 \in L^2(a,b)$ and X is a vector space, so X is a subspace of $L^2(a,b)$ with dimension 3. Since $L^2(a,b)$ is a normed space with usual norm, and any finite dimensional subspace of a normed space with the same norm must be closed, X is closed under norm $\|x\| = \sqrt{\langle x,x\rangle}$ in $L^2(a,b)$. Now consider $L^2(a,b)$ equipped with the same inner product $\langle x,y\rangle$ in the question, since it is a Hilbert space, and thus complete, its closed subspace X must be also complete.

Y is a subspace of X, because for all scalar b, c and $y_1(t), y_2(t) \in Y$, we have $y_1(a) = y_2(a) = 0$. Consider

$$(by_1 + cy_2)(a) = by_1(a) + cy_2(a) = b\cdot 0 + c\cdot 0 = 0$$

we can conclude that $by_1(t) + cy_2(t)$ is in Y, thus Y is a subspace of X. Finally, all $x \in X$ of degree 2 cannot form a subspace of X, because the zero vector of X, i.e., zero polynomial is not of degree two, so all $x \in X$ of degree 2 does not contain zero vector and cannot form a vector space.

Problem 3.2-5. Show that for a sequence (x_n) in an inner product space the conditions $\|x_n\| \to \|x\|$ and $\langle x_n, x\rangle \to \langle x,x\rangle$ imply convergence $x_n \to x$.

Notice that

$$\begin{aligned}
|\langle x_n - x, x_n - x\rangle| &= |\langle x_n, x_n - x\rangle - \langle x, x_n - x\rangle| \\
&= |\langle x_n, x_n\rangle - \langle x_n, x\rangle - \langle x, x_n\rangle + \langle x, x\rangle| \\
&= |(\langle x_n, x_n\rangle - \langle x, x\rangle) - (\langle x_n, x\rangle - \langle x, x\rangle) - (\langle x, x_n\rangle - \langle x, x\rangle)| \\
&\le |\langle x_n, x_n\rangle - \langle x, x\rangle| + |\langle x_n, x\rangle - \langle x, x\rangle| + |\langle x, x_n\rangle - \langle x, x\rangle|
\end{aligned}$$

Since $\|x_n\| \to \|x\|$, we have $\langle x_n, x_n\rangle \to \langle x, x\rangle$. Also, $\langle x_n, x\rangle - \langle x, x\rangle \to 0$. Furthermore,

$$\langle x_n, x\rangle - \langle x, x\rangle = \overline{\langle x, x_n\rangle - \langle x, x\rangle} \Longrightarrow |\langle x_n, x\rangle - \langle x, x\rangle| = |\langle x, x_n\rangle - \langle x, x\rangle|$$

Therefore,

$$|\langle x_n - x, x_n - x\rangle| \le |\langle x_n, x_n\rangle - \langle x, x\rangle| + 2|\langle x_n, x\rangle - \langle x, x\rangle| \to 0 + 2 \cdot 0 = 0$$

We can conclude that $\|x_n - x\| \to 0$, thus $x_n \to x$.

Problem 3.2-7. Show that in an inner product space, $x \perp y$ if and only if we have $\|x + \alpha y\| = \|x - \alpha y\|$ for all scalars α.

Consider the quantity

$$\begin{aligned}
\|x + \alpha y\|^2 - \|x - \alpha y\|^2 &= \langle x + \alpha y, x + \alpha y\rangle - \langle x - \alpha y, x - \alpha y\rangle \\
&= \langle x, x + \alpha y\rangle + \alpha\langle y, x + \alpha y\rangle - [\langle x, x - \alpha y\rangle - \alpha\langle y, x - \alpha y\rangle] \\
&= 2\bar{\alpha}\langle x, y\rangle + 2\alpha\langle y, x\rangle \\
&= 4\text{Re}\{\bar{\alpha}\langle x, y\rangle\}
\end{aligned}$$

If $x \perp y$, then $\langle x, y\rangle = 0$, thus $4\text{Re}\{\bar{\alpha}\langle x, y\rangle\} = 0$. This implies that $\|x + \alpha y\|^2 = \|x - \alpha y\|^2$, so $\|x + \alpha y\| = \|x - \alpha y\|$ for all α.

Conversely, if $\text{Re}\{\bar{\alpha}\langle x, y\rangle\} = 0$ for all α, then take $\alpha = 1$, then $\text{Re}\{\langle x, y\rangle\} = 0$ implies that $\langle x, y\rangle = bi$. However, if we take $\alpha = i$, then $\text{Re}\{\bar{\alpha}\langle x, y\rangle\} = b = 0$. Thus, $\langle x, y\rangle = 0$, and $x \perp y$.

Problem 3.2-8. Show that in an inner product space, $x \perp y$ if and only if $\|x + \alpha y\| \ge \|x\|$ for all scalars α.

Compute the quantity

$$\begin{aligned}
\|x + \alpha y\|^2 - \|x\|^2 &= \langle x + \alpha y, x + \alpha y\rangle - \langle x, x\rangle \\
&= \langle x, x + \alpha y\rangle + \alpha\langle y, x + \alpha y\rangle - \langle x, x\rangle \\
&= \bar{\alpha}\langle x, y\rangle + \alpha\langle y, x\rangle + |\alpha|^2\langle y, y\rangle
\end{aligned}$$

If $x \perp y$, $\langle x, y\rangle = \langle y, x\rangle = 0$. This implies that for all α,

$$\|x + \alpha y\| - \|x\| = |\alpha|^2\langle y, y\rangle \ge 0$$

Conversely, since for all α,

$$\bar{\alpha}\langle x, y\rangle + \alpha\langle y, x\rangle + |\alpha|^2\langle y, y\rangle = 2\text{Re}\{\bar{\alpha}\langle x, y\rangle\} + |\alpha|^2\langle y, y\rangle \ge 0$$

Take $\alpha = \pm\frac{1}{k}$ for all $k \in \mathbb{N}^+$, we have

$$\text{Re}\{\langle x, y\rangle\} \geq -\frac{1}{2k}\langle y, y\rangle, \quad \text{Re}\{\langle x, y\rangle\} \leq \frac{1}{2k}\langle y, y\rangle$$

This implies that $|\text{Re}\{\langle x, y\rangle\}| \leq \frac{1}{2k}\langle y, y\rangle$ for all k. Take $k \to \infty$, we conclude that $\text{Re}\{\langle x, y\rangle\} = 0$. Therefore, $\langle x, y\rangle = bi$ for some real number b.

Similarly, take $\alpha = \pm\frac{i}{k}$ for all $k \in \mathbb{N}^+$, we have

$$b \geq -\frac{1}{2k}\langle y, y\rangle, \quad b \leq \frac{1}{2k}\langle y, y\rangle$$

This implies that $|b| \leq \frac{1}{2k}\langle y, y\rangle$ for all k. Take $k \to \infty$, we conclude that $b = 0$. Therefore, $\langle x, y\rangle = 0$, and $x \perp y$.

Extra Problem 1. Let X be a normed space over $\mathbb{C}$, with its norm satisfying the parallelogram rule, i.e., for all $x, y \in X$,

$$2(\|x\|^2 + \|y\|^2) = \|x - y\|^2 + \|x + y\|^2$$

Prove that you can introduce an inner product (x, y) such that $\langle x, x\rangle = \|x\|^2$, for all $x \in X$.

We can define for all $x, y \in X$,

$$\text{Re}\langle x, y\rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2), \quad \text{Im}\langle x, y\rangle = \frac{1}{4}(\|x + iy\|^2 - \|x - iy\|^2)$$

Then by homogeneity of norm, we can easily check $\text{Re}\langle x, x\rangle = \|x\|^2$. For the imaginary part,

$$\begin{aligned}\text{Im}\langle x, x\rangle &= \frac{1}{4}(\|x + ix\|^2 - \|x - ix\|^2) = \frac{1}{4}(\|(1+i)x\|^2 - \|(1-i)x\|^2) \\ &= \frac{1}{4}(|1+i|^2\|x\|^2 - |1-i|^2\|x\|^2) = \frac{1}{4}(4\|x\|^2 - 4\|x\|^2) = 0\end{aligned}$$

This implies that $\langle x, x\rangle = \|x\|^2$, for all $x \in X$.

Then we only need to check the inner product we defined above satisfies all of the defining properties of any inner product. Since $\langle x, x\rangle = \|x\|^2$, we have $\langle x, x\rangle \geq 0$. For all $x \neq 0$, $\|x\| \neq 0$, so $\langle x, x\rangle > 0$.

To prove $\langle x, y\rangle = \overline{\langle y, x\rangle}$, we only need to prove $\text{Re}\langle x, y\rangle = \text{Re}\langle y, x\rangle$ and $\text{Im}\langle x, y\rangle = -\text{Im}\langle y, x\rangle$. Consider the real part, since $x + y = y + x$, $\|x + y\| = \|y + x\|$; also since $x - y = -(y - x)$, we have

$$\|x - y\| = \|-(y - x)\| = |-1|\|y - x\| = \|y - x\|$$

This implies that

$$\text{Re}\langle x, y\rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2) = \frac{1}{4}(\|y + x\|^2 - \|y - x\|^2) = \text{Re}\langle y, x\rangle$$

For the imaginary part,

$$\begin{aligned}\text{Im}\langle x, y\rangle &= \frac{1}{4}(\|x + iy\|^2 - \|x - iy\|^2) = \frac{1}{4}(\|i(y - ix)\|^2 - \|(-i)(y + ix)\|^2) \\ &= \frac{1}{4}(|i|\|y - ix\|^2 - |-i|\|y + ix\|^2) = \frac{1}{4}(\|y - ix\|^2 - \|y + ix\|^2) \\ &= -\frac{1}{4}(\|y + ix\|^2 - \|y - ix\|^2) = -\text{Im}\langle y, x\rangle\end{aligned}$$

Thus, we conclude that $\langle x, y\rangle = \overline{\langle y, x\rangle}$.

To prove the linearity, firstly, by definition we can observe that as long as $x = 0$ or $y = 0$,

$$\text{Re}\langle 0, y\rangle = \text{Re}\langle x, 0\rangle = \text{Im}\langle 0, y\rangle = \text{Im}\langle x, 0\rangle = 0$$

Consider arbitrary $x, y, z \in X$, and apply parallelogram rule, we have

$$\begin{aligned} 8(\text{Re}\langle x, z\rangle + \text{Re}\langle y, z\rangle) &= 2\|x+z\|^2 - 2\|x-z\|^2 + 2\|y+z\|^2 - 2\|y-z\|^2 \\ &= \|x+y+2z\|^2 - \|x+y-2z\|^2 = 4\text{Re}\langle x+y, 2z\rangle \end{aligned}$$

Let $y = 0$, then we have $2\text{Re}\langle x, z\rangle = \text{Re}\langle x, 2z\rangle$ for all $x, z \in X$. This implies that

$$2(\text{Re}\langle x, z\rangle + \text{Re}\langle y, z\rangle) = \text{Re}\langle x, 2z\rangle + \text{Re}\langle y, 2z\rangle = \text{Re}\langle x+y, 2z\rangle$$

This simply means for all $x, y, z \in X$, we have $\text{Re}\langle x, z\rangle + \text{Re}\langle y, z\rangle = \text{Re}\langle x+y, z\rangle$. Similarly, we can prove $\text{Im}\langle x, z\rangle + \text{Im}\langle y, z\rangle = \text{Im}\langle x+y, z\rangle$. This implies that $\langle x, z\rangle + \langle y, z\rangle = \langle x+y, z\rangle$.

Finally we need to prove $\langle ax, y\rangle = a\langle x, y\rangle$ for any $a \in \mathbb{C}$. Since we have additive property now, we have

$$2\langle x, y\rangle = \langle x, y\rangle + \langle x, y\rangle = \langle 2x, y\rangle$$

Hence by induction, we can derive that for all $n \in \mathbb{N}$, we have $n\langle x, y\rangle = \langle nx, y\rangle$. Furthermore, for all $m \in \mathbb{N}^+$, regard x/m as x above, we have

$$\frac{n}{m}\langle x, y\rangle = \frac{n}{m}m\left\langle \frac{x}{m}, y\right\rangle = n\left\langle \frac{x}{m}, y\right\rangle = \left\langle \frac{n}{m}x, y\right\rangle$$

This implies that for $q \in \mathbb{Q}^+$, we have $q\langle x, y\rangle = \langle qx, y\rangle$. Since every positive real number $r \in \mathbb{R}^+$ is a limit point of positive rational number set, for each r, we have $q_n \in \mathbb{Q}^+$ such that $q_n \to r$. By continuity of norm, we have the continuity of inner product defined above, i.e., $\langle q_n x, y\rangle \to \langle rx, y\rangle$ as $q_n \to r$. Therefore,

$$r\langle x, y\rangle = \lim_{n\to\infty} q_n\langle x, y\rangle = \lim_{n\to\infty}\langle q_n x, y\rangle = \left\langle \lim_{n\to\infty} q_n x, y\right\rangle = \langle rx, y\rangle$$

Now we have for $r \in \mathbb{R}^+$, $r\langle x, y\rangle = \langle rx, y\rangle$. For $r = 0$, this is trivially correct. Recall the definition again, it is trivial that $\text{Re}\langle -x, y\rangle = -\text{Re}\langle x, y\rangle$ by taking out a factor -1. Similarly, for imaginary part, take out a factor -1, and we have $\text{Im}\langle -x, y\rangle = -\text{Im}\langle x, y\rangle$. Thus for all x, y, we have $\langle -x, y\rangle = -\langle x, y\rangle$. For all $r < 0$, $-r > 0$, thus $(-r)\langle x, y\rangle = \langle (-r)x, y\rangle = -\langle rx, y\rangle$ implies that $r\langle x, y\rangle = \langle rx, y\rangle$. Therefore, for all real number $r \in \mathbb{R}$, $r\langle x, y\rangle = \langle rx, y\rangle$.

For $a \in \mathbb{C}$, we only need to prove $i\langle x, y\rangle = \langle ix, y\rangle$, then the same conclusion will hold for arbitrary complex number a. Consider

$$4\text{Re}\langle ix, y\rangle = \|ix+y\|^2 - \|ix-y\|^2 = \|x-iy\|^2 - \|x+iy\|^2 = -4\text{Im}\langle x, y\rangle$$

Similary, we will obtain $\text{Im}\langle ix, y\rangle = \text{Re}\langle x, y\rangle$. Therefore,

$$i\langle x, y\rangle = -\text{Im}\langle x, y\rangle + i\text{Re}\langle x, y\rangle = \text{Re}\langle ix, y\rangle + i\text{Im}\langle ix, y\rangle = \langle ix, y\rangle$$

Lastly, suppose $a = r_1 + ir_2$, we have

$$\begin{aligned} a\langle x, y\rangle &= r_1\langle x, y\rangle + r_2 i\langle x, y\rangle = \langle r_1 x, y\rangle + r_2\langle ix, y\rangle \\ &= \langle r_1 x, y\rangle + \langle r_2 ix, y\rangle = \langle (r_1 + ir_2)x, y\rangle = \langle ax, y\rangle \end{aligned}$$

Therefore, for all $a \in \mathbb{C}$, we have $a\langle x, y\rangle = \langle ax, y\rangle$.

Extra Problem 2. Let X be a pre-Hilbert space over $\mathbb{C}$. Prove that for all $x \in X$,

$$\|x\| = \sup_{y\in X, y\neq 0} \frac{|\langle x, y\rangle|}{\|y\|} = \sup_{y\in X, y\neq 0} \frac{\mathrm{Re}\langle x, y\rangle}{\|y\|} = \sup_{y\in X, y\neq 0} \frac{\mathrm{Im}\langle x, y\rangle}{\|y\|}$$

First we consider if $x = 0$, then all of the above are zero, so the equalities hold trivially. Thus, we only consider $x \neq 0$. By Cauchy Schwarz inequality, we have $|\langle x, y\rangle| \leq \|x\|\|y\|$. Since this is true for all $y \neq 0$, thus, $\|x\| \geq \frac{|\langle x,y\rangle|}{\|y\|}$. Since $\|x\|$ is an upper bound, so it is larger than or equal to the least upper bound, i.e.,

$$\|x\| \geq \sup_{y\in X, y\neq 0} \frac{|\langle x, y\rangle|}{\|y\|}$$

However, if we take $y = x$, since $x \neq 0$, so is y, thus,

$$\sup_{y\in X, y\neq 0} \frac{|\langle x, y\rangle|}{\|y\|} \geq \frac{|\langle x, x\rangle|}{\|x\|} = \|x\|$$

Therefore, we conclude that

$$\|x\| = \sup_{y\in X, y\neq 0} \frac{|\langle x, y\rangle|}{\|y\|}$$

Similarly, since $\mathrm{Re}\{\langle x, y\rangle\} \leq |\langle x, y\rangle| \leq \|x\|\|y\|$, we have $\|x\| \geq \frac{\mathrm{Re}\{\langle x,y\rangle\}}{\|y\|}$ if $y \neq 0$. Therefore, by the same argument, we obtain

$$\|x\| \geq \sup_{y\in X, y\neq 0} \frac{\mathrm{Re}\langle x, y\rangle}{\|y\|}$$

Take $y = x \neq 0$ again, we have

$$\sup_{y\in X, y\neq 0} \frac{\mathrm{Re}\langle x, y\rangle}{\|y\|} \geq \frac{\mathrm{Re}\langle x, x\rangle}{\|x\|} = \|x\|$$

Therefore, we have

$$\|x\| = \sup_{y\in X, y\neq 0} \frac{\mathrm{Re}\langle x, y\rangle}{\|y\|}$$

Again, since $\mathrm{Im}\{\langle x, y\rangle\} \leq |\langle x, y\rangle| \leq \|x\|\|y\|$, we have $\|x\| \geq \frac{\mathrm{Im}\{\langle x,y\rangle\}}{\|y\|}$ if $y \neq 0$. Therefore, by the same argument, we obtain

$$\|x\| \geq \sup_{y\in X, y\neq 0} \frac{\mathrm{Im}\langle x, y\rangle}{\|y\|}$$

This time take $y = -ix \neq 0$, we have

$$\sup_{y\in X, y\neq 0} \frac{\mathrm{Im}\langle x, y\rangle}{\|y\|} \geq \frac{\mathrm{Im}\langle x, -ix\rangle}{\|-ix\|} = \frac{\mathrm{Im}\{i\|x\|^2\}}{\|x\|} = \|x\|$$

Therefore, we have

$$\|x\| = \sup_{y\in X, y\neq 0} \frac{\mathrm{Im}\langle x, y\rangle}{\|y\|}$$

In conclusion, we proved that for all $x \in X$,

$$\|x\| = \sup_{y\in X, y\neq 0} \frac{|\langle x, y\rangle|}{\|y\|} = \sup_{y\in X, y\neq 0} \frac{\mathrm{Re}\langle x, y\rangle}{\|y\|} = \sup_{y\in X, y\neq 0} \frac{\mathrm{Im}\langle x, y\rangle}{\|y\|}$$

Extra Problem 3. Let $L^p(a,b)$ be equipped with the usual norm, where $p \neq 2$ and $1 \leq p \leq \infty$. Prove that $L^p(a,b)$ is not pre-Hilbert. Hint: construct examples of $f, g \in L^p(a,b)$ such that the parallelogram rule is violated.

Consider the function defined on (a,b),

$$f(x) = I_{(a,\frac{a+b}{2})}(x), \quad g(x) = I_{(\frac{a+b}{2},b)}(x)$$

where $I_A(x)$ is the indicator function on A, i.e., if $x \in A$, $I_A(x) = 1$; elsewhere $I_A(x) = 0$. We only need to show that the parallelogram rule is violated for $p \neq 2$.

If $p = \infty$, then we have $\|f\|_\infty = \|g\|_\infty = 1$. Also, $\|f - g\|_\infty = \|f + g\|_\infty = 1$. Therefore,

$$2(\|f\|^2 + \|g\|^2) = 4 \neq 2 = \|f - g\|^2 + \|f + g\|^2$$

If $p < \infty$, then we have $\|f\|_p = \|g\|_p = (\frac{b-a}{2})^{1/p}$. Also, $\|f - g\|_p = \|f + g\|_p = (b - a)^{1/p}$. Therefore, as long as $a \neq b$, and $2 \neq 2^{2/p}$, then

$$2(\|f\|_p^2 + \|g\|_p^2) = 4\left(\frac{b-a}{2}\right)^{2/p} \neq 2(b-a)^{2/p} = \|f - g\|_p^2 + \|f + g\|_p^2$$

However, $2 \neq 2^{2/p}$ if and only if $p \neq 2$, thus we finish the proof. This also implies that if $p \neq 2$, $L^p(a,b)$ is not a inner product space under the usual norm, and it is not pre-Hilbert.