# MAT4010: Functional Analysis
# Homework 4

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**Problem 3.3-3(a).** Show that the vector space $X$ of all real-valued continuous functions on $[-1,1]$ is the direct sum of the set of all even continuous functions and the set of all odd continuous functions on $[-1,1]$.

For all $f \in \mathcal{C}[-1,1]$, for all $x \in [-1,1]$, we have

$$f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} = g(x) + h(x)$$

where since $-x \in [-1,1]$, $f(-x)$ is also in $\mathcal{C}[-1,1]$, and $h(x), g(x)$ are both continuous function in $\mathcal{C}[-1,1]$. Notice that

$$g(-x) = \frac{f(-x)+f(x)}{2} = g(x), \qquad h(-x) = \frac{f(-x)-f(x)}{2} = -h(x)$$

Therefore, $g$ is even continuous function on $[-1,1]$ and $h$ is odd continuous function on $[-1,1]$. This implies that $X = X_1 + X_2$, where $X_1$ is the set of all odd continuous functions and $X_2$ is the set of all even continuous functions.

Now we need to prove for each $f$, such $g$ and $h$ are unique. Suppose $f(x) = g_1(x) + h_1(x) = g_2(x) + h_2(x)$, where $g_1, g_2 \in X_2$ and $h_1, h_2 \in X_1$. Then we have

$$\phi(x) = g_1(x) - g_2(x) = h_2(x) - h_1(x) = \psi(x)$$

Since $\phi(-x) = g_1(-x) - g_2(-x) = g_1(x) - g_2(x) = \phi(x)$, we know $\phi(x)$ is even continuous function on $[-1,1]$. Similarly, $\psi(-x) = h_2(-x) - h_1(-x) = h_1(x) - h_2(x) = -\psi(x)$. Therefore, $\psi(x)$ is odd continuous function on $[-1,1]$. This implies that $\phi(x)$ and $\psi(x)$ are both odd and even functions on $[-1,1]$. Then we have $\phi(-x) = \phi(x)$ and $\phi(-x) = -\phi(x)$, which yields $\phi(x) = 0$ for all $x \in [-1,1]$. Similarly $\psi(x) = 0$ for all $x \in [-1,1]$. Therefore, $g_1(x) = g_2(x)$ on $[-1,1]$ and $h_1(x) = h_2(x)$ on $[-1,1]$. In conclusion, for all $f \in \mathcal{C}[-1,1]$, there exists a unique even function $g(x)$ in $[-1,1]$ and a unique odd function $h(x)$ in $[-1,1]$ such that $f(x) = g(x) + h(x)$. This gives $X = X_1 \oplus X_2$.

**Problem 3.3-6.** Show that $Y = \{x \,|\, x = (\xi_j) \in l^2, \xi_{2n} = 0, n \in \mathbb{N}\}$ is a closed subspace of $l^2$ and find $Y^\perp$. What is $Y^\perp$ if $Y = \text{span}\{e_1, \dots, e_n\} \in l^2$, where $e_j = (\delta_{jk})$?

For any $x^{(1)}, x^{(2)} \in Y$, and any scalar $a, b$, where $x^{(1)} = (\xi_1^{(1)}, 0, \xi_3^{(1)}, 0, \dots)$ and $x^{(2)} = (\xi_1^{(2)}, 0, \xi_3^{(2)}, 0, \dots)$, we have

$$ax^{(1)} + bx^{(2)} = (a\xi_1^{(1)} + b\xi_1^{(2)}, 0, a\xi_3^{(1)} + b\xi_3^{(2)}, 0, \dots)$$

Since $ax^{(1)} + bx^{(2)} \in l^2$ and $a\xi_{2n}^{(1)} + b\xi_{2n}^{(2)} = 0$ for all $n$, we can conclude that $ax^{(1)} + bx^{(2)} \in Y$. Thus $Y$ is a subspace of $l^2$.

To prove it is closed, we take a convergent sequence $x^{(k)}$ in $Y$, and $x^{(k)} \to x^* \in X$. Suppose $x^* \notin Y$, then there exists $j_0$ such that $x^*_{2j_0} \neq 0$, and

$$\|x^{(k)} - x^*\|_2 \geq \sum_{j=1}^{\infty} |\xi_{2j-1}^{(k)} - \xi_{2j-1}^*|^2 + |x^*_{2j_0}|^2 \geq |x^*_{2j_0}|^2$$

Take $k \to \infty$ on both sides, we have $0 \geq |x^*_{2j_0}|^2 > 0$, which is a contradiction. Therefore, $x^* \in Y$, and $Y$ is closed.

To find $Y^\perp$, we first consider a necessary condition for any element in $Y^\perp$. If $u \in Y^\perp$, where $u = (u_j)$, then $u \perp x$ for all $x \in Y$. Since $e_1, e_2, \ldots$ are in $Y$, so at least $u \perp e_i$ for all odd $i$. Thus, $\langle u, e_i \rangle = u_i = 0$ implies that all odd entries of $u$ must be zero. However, for all $u$ such that $u_{2j-1} = 0$, $\langle u, x \rangle = 0$ because $\xi_{2j} = 0$ for all $x$. This shows the sufficiency of $u_{2j-1} = 0$ for any $u$ to be in $Y^\perp$. Therefore, $Y^\perp = \{u \,|\, u = (u_j) \in l^2, u_{2j-1} = 0, j \in \mathbb{N}^+\}$.

Similarly, if $Y = \text{span}\{e_1, \ldots, e_n\} \in l^2$, then $u \in Y^\perp$ if and only if $u \perp e_1, \ldots, e_n$. Thus, $Y^\perp = \{u \,|\, u = (u_j) \in l^2, u_1 = \cdots = u_n = 0\}$.

**Problem 3.3-7.** Let $A$ and $B \supset A$ be nonempty subsets of an inner product space $X$. Show that $A \subset A^{\perp\perp}$, $B^\perp \subset A^\perp$, and $A^{\perp\perp\perp} = A^\perp$.

We first prove $A \subset A^{\perp\perp}$. Take arbitrary $x_0 \in A$. Consider any $u \in A^\perp$, by definition, $u \perp x$ for all $x \in A$. Thus, we have $u \perp x_0$, and since $x_0 \perp u$ for all $u$, we have $x_0 \in (A^\perp)^\perp$. Notice that the choice of $x_0$ is arbitrary in $A$, we conclude that $A \subset A^{\perp\perp}$.

Then we prove $B^\perp \subset A^\perp$. Take arbitrary $y_0 \in B^\perp$, for all $v \in B$, we have $y_0 \perp v$. Since $A \subset B$, for all $x \in A$, $x$ is also in $B$, and $y_0 \perp x$. Since $y_0 \perp x$ for all $x \in A$, $y_0 \in A^\perp$. Notice that $y_0$ is arbitrary in $B^\perp$, so $B^\perp \subset A^\perp$.

From $A \subset A^{\perp\perp}$, we can say that for all nonempty subset $S \subset X$, we have $S \subset S^{\perp\perp}$. If $S$ is empty, since empty set is the subset of any set (including empty set), so $S$ still satisfies $S \subset S^{\perp\perp}$, because $\varnothing^\perp = X$ and $X^\perp = \varnothing$. Therefore, we can take $S = A^\perp$, then we have $A^\perp \subset (A^\perp)^{\perp\perp}$.

Similarly, from $B^\perp \subset A^\perp$, we can say for all nonempty subset $S_1 \subset S_2 \subset X$, we have $S_2^\perp \subset S_1^\perp$. If $S_2$ is empty, then $S_1$ is also empty and $S_2^\perp \subset S_1^\perp$ trivially holds. If $S_2$ is nonempty but $S_1$ is empty, then since $S_1^\perp = X$, $S_2^\perp$ must be a subset of the whole spacee $X$. Therefore, we can take $S_1 = A$ and $S_2 = A^{\perp\perp}$, then the conclusion is $(A^{\perp\perp})^\perp \subset A^\perp$. Therefore, combined with $A^\perp \subset (A^\perp)^{\perp\perp}$ proved just now, we can say $A^\perp = A^{\perp\perp\perp}$.

**Problem 3.3-9.** Show that a subspace $Y$ of a Hilbert space $H$ is closed in $H$ if and only if $Y = Y^{\perp\perp}$.

First we prove the "only if" part. If $Y$ is a closed subspace of Hilbert space, then by Corollary in lecture, $H = Y \oplus Y^\perp$. By Problem 3.3-7, we know $Y \subset Y^{\perp\perp}$, so we only need to prove $Y^{\perp\perp} \subset Y$. For all $u \in Y^{\perp\perp}$, $u \perp v$ for all $v \in Y^\perp$. Also, there exists unique $x \in Y$ and $y \in Y^\perp$ such that $u = x + y$. Therefore, for all $v \in Y^\perp$, $\langle x + y, v \rangle = 0$, which means $\langle x, v \rangle + \langle y, v \rangle = 0$. Since $x \in Y$ and $v \in Y^\perp$, we have $\langle x, v \rangle = 0$, thus $\langle y, v \rangle = 0$. Since $v$ is arbitrary in $Y^\perp$, we can take $v = y$.

then $\langle y, y \rangle = 0$ implies $y = 0$. This shows that $u = x \in Y$. Notice that $u$ is arbitrary in $Y^{\perp\perp}$, we conclude that $Y^{\perp\perp} \subset Y$. Combined with the previous result, $Y = Y^{\perp\perp}$.

Then we prove the "if" part. If $Y = Y^{\perp\perp}$ and $Y \subset H$, then we take a convergent sequence $u_n \in Y$ such that $u_n \to u \in H$. Since $u_n \in Y$, so for any $v \in Y^{\perp}$, $\langle u_n, v \rangle = 0$ for all $n$. By the continuity of inner product, we can take $n \to \infty$ on both sides, i.e., $\lim_{n\to\infty} \langle u_n, v \rangle = 0$, which implies $\langle u, v \rangle = 0$. This shows $u \in Y^{\perp\perp} = Y$, so any limit point of $Y$ is in $Y$, and $Y$ is closed.

**Problem 3.3-10.** If $M \neq \varnothing$ is any subset of a Hilbert space $H$, show that $M^{\perp\perp}$ is the smallest closed subspace of $H$ which contains $M$, that is, $M^{\perp\perp}$ is contained in any closed subspace $Y \in H$ such that $Y \supset M$.

First the fact that $M^{\perp\perp}$ is a subspace of $H$ is trivial. Also, from Problem 3.3-9, subspace $M^{\perp\perp}$ is closed in $H$ if and only if $M^{\perp\perp} = (M^{\perp\perp})^{\perp\perp}$. Since we proved for any nonempty subset $M \in H$, $M^{\perp\perp\perp} = M^{\perp}$, so $(M^{\perp\perp})^{\perp\perp} = (M^{\perp})^{\perp}$. This shows $M^{\perp\perp}$ is closed.

To show $M^{\perp\perp}$ is the smallest closed subspace contains $M$, we only need to show $M^{\perp\perp} = \overline{\text{span}(M)}$. This is because $\text{span}(M)$ is the smallest subspace contains $M$, and the smallest closed subspace must contained $\text{span}(M)$, but the smallest closed set containing $\text{span}(M)$ is its closure. Since the closure of a subspace is again a subspace, $\overline{\text{span}(M)}$ is the smallest closed subspace contains $M$.

Consider $M \subset M^{\perp\perp}$, since $M^{\perp\perp}$ is a subspace, $\text{span}(M) \subset M^{\perp\perp}$. Due to the closedness of $M^{\perp\perp}$, $\overline{\text{span}(M)} \subset M^{\perp\perp}$.

Since $M \subset \overline{\text{span}(M)}$, we have $M^{\perp} \supset \overline{\text{span}(M)}^{\perp}$. This further shows $M^{\perp\perp} \subset \overline{\text{span}(M)}^{\perp\perp}$. However, since $\overline{\text{span}(M)}$ is a closed subspace, by Problem 3.3-9, $\overline{\text{span}(M)}^{\perp\perp} = \overline{\text{span}(M)}$. Therefore, $M^{\perp\perp} \subset \overline{\text{span}(M)}$. Therefore, we proved that $M^{\perp\perp} = \overline{\text{span}(M)}$, and this implies $M^{\perp\perp}$ is the smallest closed subspace containing $M$.

**Problem 3.4-6.** Let $\{e_1, \ldots, e_n\}$ be an orthonormal set in an inner product space $X$, where $n$ is fixed. Let $x \in X$ be any fixed element and $y = \beta_1 e_1 + \ldots + \beta_n e_n$. Then $\|x - y\|$ depends on $\beta_1, \ldots, \beta_n$. Show by direct calculation that $\|x - y\|$ is minimum if and only if $\beta_j = \langle x, e_j \rangle$, where $j = 1, \ldots, n$.

Denote $x = (x_1, x_2, \ldots, x_n, \ldots)$, we have $x - y = (x_1 - \beta_1, x_2 - \beta_2, \ldots, x_n - \beta_n, x_{n+1}, \ldots)$, and compute

$$\langle x - y, x - y \rangle = \sum_{k=1}^{n} (x_k - \beta_k)\overline{(x_k - \beta_k)} + \sum_{k=n+1}^{\infty} x_k \overline{x_k} = \sum_{k=1}^{n} |x_k - \beta_k|^2 + \sum_{k=n+1}^{\infty} |x_k|^2$$

Since all $x_k$ are constant, it is easy to see that the value of $\langle x - y, x - y \rangle$ is at least $\sum_{k=n+1}^{\infty} |x_k|^2$. Therefore, the minimum of $\langle x - y, x - y \rangle$ is attained if and only if $\sum_{k=1}^{n} |x_k - \beta_k|^2 = 0$, which is true if and only if $x_k = \beta_k$ for all $k = 1, \ldots, n$. This implies that the minimum of $\langle x - y, x - y \rangle$ is attained if and only if $\beta_k = \langle x, e_k \rangle$ for all $k = 1, \ldots, n$. Also $\|x - y\|$ is minimum if and only if $\langle x - y, x - y \rangle$ is minimum, so we finish the proof.

**Problem 3.4-7.** Let $(e_k)$ be any orthonormal sequence in an inner product space $X$. Show that for any $x, y \in X$,

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle\langle y, e_k\rangle| \le \|x\|\|y\|$$

From Cauchy-Schwarz inequality, we have

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle\langle y, e_k\rangle| \le \left(\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2\right)^{1/2} \left(\sum_{k=1}^{\infty} |\langle y, e_k\rangle|^2\right)^{1/2}$$

By Bessel's inequality, we have

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \le \|x\|^2, \qquad \sum_{k=1}^{\infty} |\langle y, e_k\rangle|^2 \le \|y\|^2$$

Therefore, we have

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle\langle y, e_k\rangle| \le \|x\|\|y\|$$

**Problem 3.4-8.** Show that an element $x$ of an inner product space $X$ cannot have "too many" Fourier coefficients $\langle x, e_k\rangle$ which are "big"; here, $(e_k)$ is a given orthonormal sequence; more precisely, show that the number $n_m$ of $\langle x, e_k\rangle$ such that $|\langle x, e_k\rangle| > 1/m$ must satisfy $n_m < m^2\|x\|^2$.

Suppose the number $n_m$ of $\langle x, e_k\rangle$ such that $|\langle x, e_k\rangle| > 1/m$ satisfy $n_m \ge m^2\|x\|^2$. Then denote the index set of such $k$ as $A$, $|A| = n_m$, and we have

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \ge \sum_{k\in A} |\langle x, e_k\rangle|^2 > \sum_{k\in A} \frac{1}{m^2} \ge \|x\|^2$$

This shows that $\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 > \|x\|$, which contradicts the Bessel's inequality. Therefore, $n_m < m^2\|x\|^2$.

**Problem 3.4-9.** Orthonormalize the first three terms of the sequence $(x_0, x_1, x_2, \ldots)$, where $x_j(t) = t^j$, on the interval $[-1, 1]$, where

$$\langle x, y\rangle = \int_{-1}^{1} x(t)y(t)\, dt$$

Apply Gram-Schmidt process to $x_0 = 1$, we have $u_0' = x_0$, and

$$\|u_0'\| = \sqrt{\int_{-1}^{1} {u_0'}^2(t)\, dt} = \sqrt{2} \Longrightarrow u_0 = \frac{u_0'}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Continue the same process to $x_1 = t$, we have

$$u_1' = x_1 - \langle x_1, u_0\rangle u_0 = t - \frac{\sqrt{2}}{2}\int_{-1}^{1} \frac{\sqrt{2}}{2} t\, dt = t$$

Compute the corresponding norm,

$$\|u_1'\| = \sqrt{\int_{-1}^{1} {u_1'}^2(t)\, dt} = \sqrt{\int_{-1}^{1} t^2\, dt} = \frac{\sqrt{6}}{3} \Longrightarrow u_1 = \frac{u_1'}{\|u_1'\|} = \frac{\sqrt{6}}{2} t$$

Finally, for $x_2 = t^2$, we have

$$u_2' = x_2 - \langle x_2, u_0\rangle u_0 - \langle x_2, u_1\rangle u_1 = t^2 - \frac{1}{2}\int_{-1}^{1} t^2\, dt - \frac{3}{2}t\int_{-1}^{1} t^3\, dt = t^2 - \frac{1}{3}$$

Compute the corresponding norm,

$$\|u_2'\| = \sqrt{\int_{-1}^{1} {u_2'}^2(t)\, dt} = \sqrt{\int_{-1}^{1}\left(t^2 - \frac{1}{3}\right)^2 dt} = \frac{2\sqrt{10}}{15} \Longrightarrow u_2 = \frac{u_2'}{\|u_2'\|} = \frac{\sqrt{10}}{4}(3t^2 - 1)$$

Therefore, $u_0(t) = \frac{\sqrt{2}}{2}$, $u_1(t) = \frac{\sqrt{6}}{2}t$, and $u_2(t) = \frac{\sqrt{10}}{4}(3t^2 - 1)$ is an orthonormal set.

**Problem 3.5-3.** Illustrate with an example that a convergent series $\sum\langle x, e_k\rangle e_k$ need not have the sum $x$.

Suppose $(e_k)$ is orthonormal sequence in $l^2$ space, and $e_1 = (0, 1, 0, 0, 0, \ldots)$, $e_2 = (0, 0, 0, 1, 0, \ldots)$ and so on. Use the usual inner product in $l^2$, then for any $x = (x_1, x_2, \ldots) \in l^2$, we have

$$\sum_{k=1}^{n}\langle x, e_k\rangle e_k = \sum_{k=1}^{n} x_{2k} e_{2k} = (0, x_2, 0, x_4, \ldots, x_{2n}, 0, 0, \ldots)$$

Define $x^* = (0, x_2, \ldots, 0, x_{2n+2}, \ldots)$, consider

$$\left\|\sum_{k=1}^{n}\langle x, e_k\rangle e_k - x^*\right\|^2 = \sum_{k=n+1}^{\infty} |x_{2k}|^2 \to 0$$

as $n \to \infty$ because $\sum_{k=1}^{\infty}|x_k|^2$ is convergent to $\|x\|^2$. Therefore, $\sum_{k=1}^{\infty}\langle x, e_k\rangle e_k$ converges to $x^*$. However, it is clearly $x^* \neq x$ as long as $x_1 \neq 0$. Therefore, convergent series $\sum_{k=1}^{\infty}\langle x, e_k\rangle e_k$ need not have the sum $x$.

**Problem 3.5-4.** If $(x_j)$ is a sequence in an inner product space $X$ such that the series $\|x_1\| + \|x_2\| + \cdots$ converges, show that $(s_n)$ is a Cauchy sequence, where $s_n = x_1 + \ldots + x_n$.

For arbitrary $\epsilon > 0$, consider any $n > m$, we have

$$\|s_n - s_m\| = \left\|\sum_{i=m+1}^{n} x_i\right\| \leq \sum_{i=m+1}^{n}\|x_i\| \leq \sum_{i=m+1}^{\infty}\|x_i\|$$

Since $\sum_{i=1}^{\infty}\|x_i\|$ converges, there exists $N$, such that $\sum_{i=N}^{\infty}\|x_i\| < \epsilon$. Therefore, for all $n > m \geq N$, we have $\|s_n - s_m\| < \epsilon$, so $s_n$ is a Cauchy sequence.

**Problem 3.5-6.** Let $(e_j)$ be an orthonormal sequence in a Hilbert space $H$. Show that if

$$x = \sum_{j=1}^{\infty}\alpha_j e_j, \quad y = \sum_{j=1}^{\infty}\beta_j e_j$$

then $\langle x, y\rangle = \sum_{j=1}^{\infty}\alpha_j\bar{\beta}_j$, given that the series representing $x, y$ are absolutely convergent.

Let $s_n = \sum_{j=1}^{n}\alpha_j e_j$ and $t_n = \sum_{j=1}^{n}\beta_j e_j$, then since the series representing $x, y$ are absolutely convergent, we have $s_n \to x$ and $t_n \to y$ as $n \to \infty$. Consider

$$\langle s_n, t_n\rangle = \left\langle\sum_{j=1}^{n}\alpha_j e_j, \sum_{j=1}^{n}\beta_j e_j\right\rangle = \sum_{j=1}^{n}\alpha_j\bar{\beta}_j$$

Since inner product is continuous, we have

$$\langle x, y\rangle = \lim_{n\to\infty}\langle s_n, t_n\rangle = \lim_{n\to\infty}\sum_{j=1}^{n}\alpha_j\bar{\beta}_j$$

Also, we have

$$\sum_{j=1}^{\infty}|\alpha_j\bar{\beta}_j| \le \left(\sum_{j=1}^{n}|\alpha_j|^2\right)^{1/2}\left(\sum_{j=1}^{n}|\beta_j|^2\right)^{1/2}$$

Since the right hand side is convergent by the absolute convergence of series representing $x, y$, $\sum_{j=1}^{\infty}|\alpha_j\bar{\beta}_j|$ is also convergent.

**Problem 3.5-7.** Let $(e_j)$ be an orthonormal sequence in a Hilbert space $H$. Show that for every $x \in H$, the vector

$$y = \sum_{k=1}^{\infty}\langle x, e_k\rangle e_k$$

exists in $H$ and $x - y$ is orthogonal to every $e_k$.

Let $u_k = \langle x, e_k\rangle e_k$, then $\|u_k\| = |\langle x, e_k\rangle|$. Since by Bessel's inequality,

$$\sum_{k=1}^{\infty}|\langle x, e_k\rangle|^2 \le \|x\|^2$$

for all $n \ge m$, $\sum_{k=m}^{n}|\langle x, e_k\rangle|^2 \to 0$ as $m, n \to \infty$. Consider

$$\left\|\sum_{k=m}^{n}\langle x, e_k\rangle e_k\right\|^2 = \sum_{k=m}^{n}|\langle x, e_k\rangle|^2 \to 0$$

we know that the partial sum $\sum_{k=1}^{n}\langle x, e_k\rangle e_k$ is a Cauchy sequence, and by completeness of $H$, it must be convergent to some point in $H$. Thus $y = \sum_{k=1}^{\infty}\langle x, e_k\rangle e_k$ exists in $H$.

To prove $(x - y) \perp e_j$ for all $j = 1, 2, \ldots$, consider

$$\langle x - y, e_j\rangle = \langle x, e_k\rangle - \sum_{k=1}^{\infty}\langle x, e_k\rangle\langle e_k, e_j\rangle = \langle x, e_k\rangle - \langle x, e_k\rangle = 0$$

because $\langle e_k, e_j\rangle \ne 0$ if and only if $k = j$ and $\langle e_k, e_k\rangle = 1$. Therefore, $(x - y) \perp e_j$ for all $j = 1, 2, \ldots$.

**Problem 3.5-8.** Let $(e_k)$ be an orthonormal sequence in a Hilbert space $H$, and let $M = \mathrm{span}(e_k)$. Show that for any $x \in H$ we have $x \in \bar{M}$ if and only if $x$ can be represented by $\sum_{k=1}^{\infty}\alpha_k e_k$ with coefficients $\alpha_k = \langle x, e_k\rangle$.

First we show the "if" part. If for all $x \in H$, $x = \sum_{k=1}^{\infty}\alpha_k e_k$, then we can let $x_n = \sum_{k=1}^{n}\alpha_k e_k$, and $x_n \to x$. Notice that $x_n \in M$, so $x$ is a limit point of $M$, hence in $\bar{M}$.

Then we show the "only if" part. If for any $x \in H$ we have $x \in \bar{M}$, then there exists a sequence $x_n \to x$, where $x_n = \sum_{k=1}^{m_n} a_{nk}e_k$. Then by Problem 3.4-6, we have for each fixed $n$,

$$\left\|x - \sum_{k=1}^{m_n}\langle x, e_k\rangle e_k\right\| \le \|x - x_n\|$$

Also notice that for all $l \geq m_n$, we should have

$$\left\| x - \sum_{k=1}^{l} \langle x, e_k \rangle e_k \right\| \leq \left\| x - \sum_{k=1}^{m_n} \langle x, e_k \rangle e_k \right\|$$

Therefore, we can take $c_n = \max\{n, m_n\}$, then as $n \to \infty$, $c_n \to \infty$, and

$$\left\| x - \sum_{k=1}^{c_n} \langle x, e_k \rangle e_k \right\| \leq \|x - x_n\| \to 0 \Longrightarrow \left\| x - \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k \right\| = 0$$

which means $x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$.

**Problem 3.5-9.** Let $(e_n)$ and $(\tilde{e}_n)$ be orthonormal sequences in a Hilbert space $H$, and let $M_1 = \text{span}(e_n)$ and $M_2 = \text{span}(\tilde{e}_n)$. Show that $\bar{M}_1 = \bar{M}_2$ if and only if $e_n = \sum_{m=1}^{\infty} \alpha_{nm} \tilde{e}_m$ and $\tilde{e}_n = \sum_{m=1}^{\infty} \bar{\alpha}_{mn} e_m$ hold simultaneously, where $\alpha_{nm} = \langle e_n, \tilde{e}_m \rangle$.

First we prove the "if" part. If $e_n = \sum_{m=1}^{\infty} \alpha_{nm} \tilde{e}_m$, then by "if" part in Problem 3.5-8, we know $e_n \in \bar{M}_2$. Since this is true for all $n$, we know $M_1 \subset \bar{M}_2$. Since $\bar{M}_2$ is closed and $\bar{M}_1$ is the closure hence the smallest closed set that contained $M_1$, we conclude that $\bar{M}_1 \subset \bar{M}_2$. Similarly, if $\tilde{e}_n = \sum_{m=1}^{\infty} \bar{\alpha}_{mn} e_m$ then $\tilde{e}_n \in \bar{M}_1$, and by the same argument it finally yields $\bar{M}_2 \subset \bar{M}_1$. Therefore, we proved that $\bar{M}_1 = \bar{M}_2$.

Then we prove the "only if" part. If $\bar{M}_1 = \bar{M}_2$, then $e_n \in \bar{M}_2$. Since $\bar{M}_1 = \bar{M}_2$ are closed subspace of $H$, so they are both Hilbert space. Use $\bar{M}_2$ as the Hilbert space in Problem 3.5-8, by "only if" part, we know that $e_n = \sum_{k=1}^{\infty} \langle e_n, \tilde{e}_k \rangle \tilde{e}_k$. Similarly, use $\bar{M}_1$ as the Hilbert space in Problem 3.5-8, since $\tilde{e}_n \in \bar{M}_1$, we have $\tilde{e}_n = \sum_{k=1}^{\infty} \langle \tilde{e}_n, e_k \rangle e_k$.

**Problem 3.6-4.** Derive from Parseval's identity, i.e., $\sum_k |\langle x, e_k \rangle|^2 = \|x\|^2$, the following formula

$$\langle x, y \rangle = \sum_k \langle x, e_k \rangle \overline{\langle y, e_k \rangle}$$

Since $\text{Re}\langle x, y \rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$, we have

$$\begin{aligned} \|x+y\|^2 &= \sum_k |\langle x+y, e_k \rangle|^2 = \sum_k \langle x+y, e_k \rangle \overline{\langle x+y, e_k \rangle} \\ &= |\langle x, e_k \rangle|^2 + |\langle y, e_k \rangle|^2 + 2\text{Re}\langle x, e_k \rangle \overline{\langle y, e_k \rangle} \end{aligned}$$

$$\begin{aligned} \|x-y\|^2 &= \sum_k |\langle x+y, e_k \rangle|^2 = \sum_k \langle x+y, e_k \rangle \overline{\langle x+y, e_k \rangle} \\ &= |\langle x, e_k \rangle|^2 + |\langle y, e_k \rangle|^2 - 2\text{Re}\langle x, e_k \rangle \overline{\langle y, e_k \rangle} \end{aligned}$$

Therefore, $\text{Re}\langle x, y \rangle = \text{Re}\langle x, e_k \rangle \overline{\langle y, e_k \rangle}$. Similarly, we have $\text{Im}\langle x, y \rangle = \frac{1}{4}(\|x+iy\|^2 - \|x-iy\|^2)$, so

$$\begin{aligned} \|x+iy\|^2 &= \sum_k |\langle x+iy, e_k \rangle|^2 = \sum_k \langle x+iy, e_k \rangle \overline{\langle x+iy, e_k \rangle} \\ &= |\langle x, e_k \rangle|^2 + |\langle y, e_k \rangle|^2 + 2\text{Im}\langle x, e_k \rangle \overline{\langle y, e_k \rangle} \end{aligned}$$

$$\begin{aligned}\|x - iy\|^2 &= \sum_k |\langle x - iy, e_k\rangle|^2 = \sum_k \langle x - iy, e_k\rangle\overline{\langle x - iy, e_k\rangle} \\ &= |\langle x, e_k\rangle|^2 + |\langle y, e_k\rangle|^2 - 2\text{Im}\langle x, e_k\rangle\overline{\langle y, e_k\rangle}\end{aligned}$$

Therefore, $\text{Im}\langle x, y\rangle = \text{Im}\langle x, e_k\rangle\overline{\langle y, e_k\rangle}$. This shows that $\langle x, y\rangle = \sum_k \langle x, e_k\rangle\overline{\langle y, e_k\rangle}$.

**Problem 3.8-6.** Show that Riesz's Theorem defines an isometric bijection $T : H \mapsto H'$, $z \mapsto f_z = \langle \cdot, z\rangle$ which is not linear but *conjugate linear*, that is, $\alpha z + \beta v \mapsto \bar{\alpha} f_z + \bar{\beta} f_v$.

Riesz's Theorem says that if $H$ is Hilbert, then for all $f \in H'$, there exists a unique $y \in H$, such that $f_y(x) = \langle x, y\rangle$. Moreover, $\|f\|_{H'} = \|y\|_H$. This directly implies that $T$ is bijective and isometric. Therefore, we only need to prove $T$ is conjugate linear.

Consider any $z, v \in H$ and scalar $\alpha, \beta$, we have

$$\langle \cdot, \alpha z + \beta v\rangle = \overline{\langle \alpha z + \beta v, \cdot\rangle} = \overline{\alpha\langle z, \cdot\rangle + \beta\langle v, \cdot\rangle} = \bar{\alpha}\overline{\langle z, \cdot\rangle} + \bar{\beta}\overline{\langle v, \cdot\rangle} = \bar{\alpha}\langle \cdot, z\rangle + \bar{\beta}\langle \cdot, v\rangle$$

Therefore, $T(\alpha z + \beta v) = \bar{\alpha} Tz + \bar{\beta} Tv$, and $T$ is conjugate linear. It is easy to see at least when $\beta = 0$ and $\alpha = i$, $T(iz) = -iTz \neq iTz$, so $T$ cannt be linear as long as $T$ is not the zero map.

**Problem 3.8-7.** Show that the dual space $H'$ of a Hilbert space $H$ is a Hilbert space with inner product $\langle \cdot, \cdot\rangle$ defined by

$$\langle f_z, f_v\rangle = \overline{\langle z, v\rangle} = \langle v, z\rangle$$

where $f_z(x) = \langle x, z\rangle$.

Notice that the dual space of any normed space is complete, but $H$ is Hilbert thus normed space, so $H'$ is complete. Therefore, we only need to prove $H'$ is equipped with an inner product defined in the question. By Riesz' Theorem, each element $f$ in $H'$ can be represented by $f_z(x) = \langle x, z\rangle$, where $f_z$ is obviously a linear bounded (Riesz's Therorem) functional. Therefore, we only need to show $\langle f_z, f_v\rangle = \langle v, z\rangle$ is an inner product in $H'$.

First, $\langle f_z, f_z\rangle = \langle z, z\rangle \geq 0$ and since $\langle z, z\rangle = 0 \iff z = 0$, we have $\langle f_z, f_z\rangle = 0 \iff f_z = \langle x, 0\rangle \equiv 0$.

Then, $\langle f_z, f_v\rangle = \langle v, z\rangle = \overline{\langle z, v\rangle} = \overline{\langle f_v, f_z\rangle}$.

Consider any scalar $a, b$ and any $f_y \in H'$, we have

$$\langle af_z + bf_y, f_v\rangle = \langle f_{\bar{a}z + \bar{b}y}, f_v\rangle = \langle v, \bar{a}z + \bar{b}y\rangle = a\langle v, z\rangle + b\langle v, y\rangle = a\langle f_z, f_v\rangle + b\langle f_y, f_v\rangle$$

Therefore, $\langle f_z, f_v\rangle = \langle v, z\rangle$ is an inner product in $H'$. This implies that $H'$ is a Hilbert space.

**Problem 3.8-8.** Show that any Hilbert space $H$ is isomorphic with its second dual space $H'' = (H')'$.

Since $H'$ is Hilbert (as we proved in Problem 3.8-7), by Riesz's Theorem, any element $F$ in $H''$ can be expressed as $F_f = \langle g, f\rangle$ with unique $f \in H'$ for all $g \in H'$. By Problem 3.8-7, $H''$ is also Hilbert with inner product $\langle F_f, F_g\rangle = \langle g, f\rangle$. Therefore, we can define a map $\phi : H \mapsto H''$ by $z \mapsto F_{f_z}(h) = \langle h, f_z\rangle$ for all $h \in H'$. Then we need to prove $\phi$ is bijective linear mapping that preserves inner product.

Firstly, we need to show $\phi$ is well-defined function, i.e., for $z = v$, we must have $\phi(z) = \phi(v)$. We only need to show that for all $h \in H'$, $F_{f_z}(h) = F_{f_v}(h)$. Therefore, we need to show $\langle h, f_z\rangle = \langle h, f_v\rangle$ for all $h \in H'$. Since by Riesz's Theorem, we have unique $y \in H$ for each $h$ such that $h = h_y(x) = \langle x, y\rangle$ for all $x$, we only need to show $\langle h_y, f_z\rangle = \langle h_y, f_v\rangle$. Since $\langle h_y, f_z\rangle = \langle z, y\rangle$ and $\langle h_y, f_v\rangle = \langle v, y\rangle$, and $z = v$, so $\langle z, y\rangle = \langle v, y\rangle$. This implies that $F_{f_z}(h) = F_{f_v}(h)$ and $\phi$ is well-defined.

Then we show $\phi$ is linear. Consider any scalar $a, b$ and and $z, v \in H$, we need to show $\phi(az + bv) = a\phi(z) + b\phi(v)$, which means $\langle h, f_{az+bv}\rangle = a\langle h, f_z\rangle + b\langle h, f_v\rangle$. Notice that

$$\langle h, f_{az+bv}\rangle = \langle h_y, f_{az+bv}\rangle = \langle az+bv, y\rangle = a\langle z, y\rangle + b\langle v, y\rangle = a\langle f_y, f_z\rangle + b\langle f_y, f_v\rangle = a\langle h, f_z\rangle + b\langle h, f_v\rangle$$

Therefore, $\phi$ is linear.

Then we show $\phi$ is bijective. Surjectivity is trivial because of Riesz's Theorem. For injectivity, if $\phi(z) = 0$, we have $\langle h, f_z\rangle = 0$ for all $h \in H'$. This further implies that $\langle z, y\rangle = 0$ for all $y \in H$. Then take $y = z$, we immediately have $\|z\|^2 = 0$ and hence $z = 0$. This shows the kernel of $\phi$ is trivial and $\phi$ is injective.

Finally we show $\phi$ preserves the inner product. We can see that $\langle \phi(z), \phi(v)\rangle = \langle F_{f_z}, F_{f_v}\rangle = \langle f_v, f_z\rangle = \langle z, v\rangle$. Thus, we conclude that $H$ and $H''$ are isomorphic.

**Extra Problem 1.** Let $X$ and $Y$ be two normed spaces. We say $X$ is continuously embedded into $Y$ if $X \subset Y$ and if the identity map $i : X \mapsto Y$, $i(x) = x$ is injective and bounded, i.e., there exists constant $C > 0$, such that $\|x\|_Y \leq C\|x\|_X$, for all $x \in X$. Denote it as $X \hookrightarrow Y$. Let $H$ and $V$ be real Hilbert spaces (with their own inner products $(\cdot, \cdot)_H$ and $(\cdot, \cdot)_V$). Suppose $V$ is continuously embedded into $H$ and $V$ is dense in $H$. Prove that $H' \hookrightarrow V'$ and that $H'$ is dense in $V'$.

First we prove $H' \subset V'$. For each $f \in H'$, $f$ is a linear functional defined on $H$. Since $V \subset H$, so $f$ is also a linear funcitonal defined on $V$, thus $f \in V'$. Therefore, $H' \subset V'$.

Then we prove that the map $i : H' \mapsto V'$ given by $i(f) = f\Big|_V$ is injective and bounded. Consider $f\Big|_V(v) \equiv 0$ for all $v \in V$, by Riesz's Theorem, we can identify $f\Big|_V(v)$ as $\langle v, y\rangle$ for unique $y \in V$. Since $V$ is dense in $H$, for all $u \in H$, we have $v_n \in V$ such that $v_n \to u$ (If $u \in V$, then $v_n$ is constant sequence $u$). For all $u \in H$, $\langle u, y\rangle = \lim_{n\to\infty}\langle v_n, y\rangle = 0$, thus $f(u) \equiv 0$ and the pre-image of $f\Big|_V(v) \equiv 0$ is $f(u) = \langle u, y\rangle \equiv 0$. Therefore, $i(f)$ is injective.

Consider

$$\left\|f\Big|_V\right\|_{V'} = \sup_{\|v\|_V=1}\left|f\Big|_V(v)\right| = \sup_{\|v\|_V=1}|f(v)| \leq \|f\|_{H'}\sup_{\|v\|_V=1}\|v\|_H$$

Since $V \hookrightarrow H$, we have $\|v\|_H \leq C\|v\|_V$, thus,

$$\left\|f\Big|_V\right\|_{V'} \leq \|f\|_{H'}\sup_{\|v\|_V=1}C\|v\|_V = C\|f\|_{H'}$$

Therefore, $H' \hookrightarrow V'$.

To prove $H'$ is dense in $V'$, we need to prove ${H'}^{\perp} = \{0_{V'}\}$. This is because if so, ${H'}^{\perp\perp} = \overline{H'} = V'$ immediately implies that $H'$ is dense in $V'$. Consider any $f \in H'$, then there exists unique $y \in H$, such that $f_y(x) = \langle x, y\rangle$ for all $x \in H$. If a $g \in V'$ satisfies $\langle f, g\rangle = 0$, then we can find $v \in V$ such that $g_v(z) = \langle z, v\rangle$ for all $z \in V$. From Problem 3.8-7, we have $\langle f, g\rangle = \langle v, y\rangle = 0$ for all

$y \in H$ and fixed $v \in V$. Since $V$ is dense in $H$, $V^\perp = \{0_H\}$, and now $v \in V$ and $v \in V^\perp$, so $v = 0$. This implies that $g = 0_{V'}$, therefore $H'^\perp = \{0_{V'}\}$.

**Extra Problem 2.** Given that any $f \in L^2(-l, l)$ can be expanded as

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left( a_k \cos \frac{k\pi x}{l} + b_k \sin \frac{k\pi x}{l} \right)$$

(i) Use Euler's formula $e^{i\theta} = \cos\theta + i \sin\theta$, prove that $f(x) = \sum_{k=-\infty}^{\infty} c_k e^{\frac{ik\pi x}{l}}$, where $\sum_{k=-\infty}^{\infty} c_k e^{\frac{ik\pi x}{l}}$ is understood as the limit of $\sum_{k=-n}^{n} c_k e^{\frac{ik\pi x}{l}}$ in $L^2(-l, l)$; give the formula for Fourier coefficient $c_k$.

Since $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$, we have

$$\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} \left( a_k \cos \frac{k\pi x}{l} + b_k \sin \frac{k\pi x}{l} \right) \\
&= \frac{a_0}{2} + \sum_{k=1}^{\infty} \left( a_k \frac{1}{2}(e^{\frac{ik\pi x}{l}} + e^{-\frac{ik\pi x}{l}}) + b_k \frac{1}{2i}(e^{\frac{ik\pi x}{l}} - e^{-\frac{ik\pi x}{l}}) \right) \\
&= \frac{a_0}{2} + \sum_{k=1}^{\infty} \frac{1}{2}(a_k - ib_k) e^{\frac{ik\pi x}{l}} + \sum_{k=-\infty}^{-1} \frac{1}{2}(a_{-k} + ib_{-k}) e^{\frac{ik\pi x}{l}} \\
&= \sum_{k=-\infty}^{\infty} c_k e^{\frac{ik\pi x}{l}}
\end{aligned}$$

where $c_k$ is defined by

$$\begin{cases}
c_0 = \frac{a_0}{2} = \frac{1}{2l}\int_{-l}^{l} f(t)\, dt \\
c_k = \frac{1}{2}(a_k - ib_k) = \frac{1}{2l}\int_{-l}^{l} f(t) e^{\frac{-ik\pi t}{l}}\, dt & k \geq 1 \\
c_k = \frac{1}{2}(a_{-k} + ib_{-k}) = \frac{1}{2l}\int_{-l}^{l} f(t) e^{\frac{-ik\pi t}{l}}\, dt & k \leq -1
\end{cases}$$

or more compactly, for all $k \in \mathbb{Z}$, $c_k$ is defined by

$$c_k = \frac{1}{2l}\int_{-l}^{l} f(t) e^{\frac{-ik\pi t}{l}}\, dt$$

(ii) Denote $c_k$ as $\hat{f}(k)$. Prove that if $l = \pi$ and if $f \in \mathcal{C}^1[-\pi, \pi]$ and is $2\pi$-periodic, then $\widehat{f'}(k) = ik\hat{f}(k)$.

Since $f \in \mathcal{C}^1[-\pi, \pi]$, $f'$ is in $L^2(-\pi, \pi)$. Also, since $f$ is $2\pi$-periodic differentiable function, $f'(x)$ is also $2\pi$-periodic. Therefore, $f'(x)$ has a convergent Fourier series expansion, the only thing we need to do is to determine the coefficient. By the formula derived in part (i), for $k \geq 0$, apply integration by part, and we have,

$$\widehat{f'}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(t) e^{-ikt}\, dt = \frac{1}{2\pi} e^{-ikt} f(t)\Big|_{-\pi}^{\pi} + \frac{1}{2\pi}\int_{-\pi}^{\pi} ikf(t) e^{-ikt}\, dt = ik\hat{f}(k)$$

because the first term vanishes.

(iii) For all $f, g \in L^2(-\pi,\pi)$, prove $\langle f, g\rangle_{L^2(-\pi,\pi)} = 2\pi \sum_{k=-\infty}^{\infty} \hat{f}(k)\overline{\hat{g}(k)}$.

We have known that $e^{inx}$ forms a basis of $L^2(-\pi,\pi)$, so next we need to prove $e^{inx}$ is orthogonal basis. This is trivial because

$$\int_{-\pi}^{\pi} e^{imx}e^{inx}\, dx = \int_{-\pi}^{\pi} \cos mx \cos nx\, dx + i\int_{-\pi}^{\pi} \sin mx \cos nx\, dx + i\int_{-\pi}^{\pi} \cos mx \sin nx\, dx - \int_{-\pi}^{\pi} \sin mx \sin nx\, dx$$

Since $\cos mx, \sin mx$ are all orthognal to each other, $\int_{-\pi}^{\pi} e^{imx}e^{inx}\, dx = 0$. However, notice that $e^{inx}$ is not orthonormal basis, because

$$\|e^{inx}\|^2_{L^2(-\pi,\pi)} = \int_{-\pi}^{\pi} |e^{inx}|^2\, dx = 2\pi$$

Therefore, we can take $u_k = \frac{1}{\sqrt{2\pi}}e^{ikx}$, then $u_k$ forms an orthonormal basis. By Problem 3.6-4, we have

$$\langle f, g\rangle_{L^2(-\pi,\pi)} = \sum_k \langle f, u_k\rangle \overline{\langle g, u_k\rangle}$$

Notice that

$$\langle f, u_k\rangle = \int_{-\pi}^{\pi} f(x)\frac{1}{\sqrt{2\pi}}e^{ikx} = \sqrt{2\pi}\hat{f}(k)$$

Similarly, we have $\langle g, u_k\rangle = \hat{g}(k)$, and this implies that

$$\langle f, g\rangle_{L^2(-\pi,\pi)} = 2\pi \sum_{k=-\infty}^{\infty} \hat{f}(k)\overline{\hat{g}(k)}$$

**Extra Problem 3.** Let $P$ be a simple closed curve in the $x-y$ plane. Suppose $P$ is $\mathcal{C}^1$-smooth, i.e., $P$ can be parameterized by $x = x(s)$ and $y = y(s)$, where $s \in [0, 2\pi]$ is the arclength variable, $2\pi$ is the arclength of $P$, and $x(s)$, $y(s)$ in $\mathcal{C}^1([0, 2\pi])$. Prove the isoperimetric inequality, $A \leq \pi$, where $A$ is the area of the region enclosed by $P$. Hint: By Green's formula, if $P$ is oriented counter-clockwise, then

$$A = \frac{1}{2}\int_P xdy - ydx = \frac{1}{2}\int_0^{2\pi} (x(s)y'(s) - y(s)x'(s))\, ds = \frac{1}{2}[\langle x, y'\rangle_{L^2(0,2\pi)} - \langle y, x'\rangle_{L^2(0,2\pi)}]$$

Since $x(s), y(s) \in \mathcal{C}^1$ and $2\pi$-periodic, we can express them in $x(s) = \sum_{k=-\infty}^{\infty} a_k e^{iks}$ and $y(s) = \sum_{k=-\infty}^{\infty} b_k e^{iks}$. By the last problem, we have $x'(s) = \sum_{k=-\infty}^{\infty} a_k(ik)e^{iks}$ and $y'(s) = \sum_{k=-\infty}^{\infty} b_k(ik)e^{iks}$. Therefore,

$$\langle x, y'\rangle_{L^2(-\pi,\pi)} = 2\pi \sum_{k=-\infty}^{\infty} a_k\overline{(ik)b_k}, \qquad \langle y, x'\rangle_{L^2(-\pi,\pi)} = 2\pi \sum_{k=-\infty}^{\infty} b_k\overline{(ik)a_k}$$

The hint implies that

$$A = \pi\left|(-i)\sum_{k=-\infty}^{\infty} k(a_k\overline{b_k} - \overline{a_k}b_k)\right| \leq \pi \sum_{k=-\infty}^{\infty} 2k|a_k||b_k| \leq \pi \sum_{k=-\infty}^{\infty} k(|a_k|^2 + |b_k|^2)$$

Also, since the curve $P$ is parametrized by arc length, we have

$$\int_0^{2\pi} \sqrt{(x'(s))^2 + (y'(s))^2}\, ds = \int_0^{2\pi} (x'(s))^2 + (y'(s))^2\, ds = 2\pi$$

Since $\|x'\|^2 = \int_0^{2\pi} (x'(s))^2\, ds = 2\pi \sum_{k=-\infty}^{\infty} k^2 |a_k|^2$, and $\|y'\|^2 = \int_0^{2\pi} (y'(s))^2\, ds = 2\pi \sum_{k=-\infty}^{\infty} k^2 |b_k|^2$, we have $\sum_{k=-\infty}^{\infty} k^2 |a_k|^2 + \sum_{k=-\infty}^{\infty} k^2 |b_k|^2 = 1$. Therefore, we have

$$A \leq \pi \sum_{k=-\infty}^{\infty} k(|a_k|^2 + |b_k|^2) \leq \pi \sum_{k=-\infty}^{\infty} k^2(|a_k|^2 + |b_k|^2) = \pi$$

This implies that the isoperimetric inequality $A \leq \pi$ holds.