MAT4010: Functional Analysis
Homework 5

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Problem 3.9-2. Let H be a Hilbert space and $T : H \mapsto H$ a bijective bounded linear operator whose inverse is bounded. Show that $(T^*)^{-1}$ exists and $(T^*)^{-1} = (T^{-1})^*$.

By definition of T^{-1}, we have $TT^{-1} = T^{-1}T = I$. Thus, we have $(TT^{-1})^* = (T^{-1}T)^* = I^*$. By definition, it is obvious that the adjoint operator of identity map is identity map itself. Also, by fact in lecture, $(TT^{-1})^* = (T^{-1})^*T^*$ and $(T^{-1}T)^* = T^*(T^{-1})^*$. Therefore, we have $(T^{-1})^*T^* = T^*(T^{-1})^* = I$. Since T^{-1} is a bounded linear operator, its adjoint operator $(T^{-1})^*$ is also a bounded linear operator. Therefore, we $(T^*)^{-1}$ exists and $(T^*)^{-1} = (T^{-1})^*$.

Problem 3.9-3. If (T_n) is a sequence of bounded linear operators on a Hilbert space and $T_n \to T$, show that $T_n^* \to T^*$.

Consider for all $x, y \in H$,

$$\langle Tx, y\rangle = \left\langle \left(\lim_{n\to\infty} T_n\right) x, y\right\rangle = \lim_{n\to\infty} \langle T_n x, y\rangle = \lim_{n\to\infty} \langle x, T_n^* y\rangle = \left\langle x, \left(\lim_{n\to\infty} T_n^*\right) y\right\rangle$$

Since Hilbert space is complete, bounded linear operators on a Hilbert space is complete, so T is also bounded linear operator and thus T^* exists. Therefore, we can conclude that $T^* = \lim_{n\to\infty} T_n^*$.

Problem 3.9-6. Let H_1 and H_2 be Hilbert spaces and $T : H_1 \mapsto H_2$ a bounded linear operator. If $M_1 = \mathcal{N}(T) = \{x \mid Tx = 0\}$, show that

(a) $T^*(H_2) \subset M_1^\perp$,

For any $y \in H_2$, if $T^*y = 0$, then since $M_1^\perp$ is a vector space, $T^*y \in M_1^\perp$; otherwise $T^*y \neq 0$. Consider $\langle y, TT^*y\rangle = \langle T^*y, T^*y\rangle = \|T^*y\|^2 > 0$. This implie that $T(T^*y) \neq 0$, for all $y \in H_2$ and $T^*y \neq 0$. This shows that $T^*y \notin M_1$, but M_1 is closed subspace of H_1, so $H_1 = M_1 \oplus M_1^\perp$. Therefore, $T^*y \in M_1^\perp$. Therefore, $T^*(H_2) \subset M_1^\perp$.

(b) $[T(H_1)]^\perp \subset \mathcal{N}(T^*)$,

For any $y \in [T(H_1)]^\perp$, y satisfies $\langle y, Tx\rangle = 0$ for all $x \in H_1$. However, since $\langle y, Tx\rangle = \langle T^*y, x\rangle$, we have $\langle T^*y, x\rangle = 0$ for all $x \in H_1$. Notice that $T^*y \in H_1$, so take $x = T^*y$, we have $\|T^*y\|^2 = 0$, which means $T^*y = 0$. This shows that $y \in \mathcal{N}(T^*)$. Since y is arbitrarily chosen in $[T(H_1)]^\perp$, $[T(H_1)]^\perp \subset \mathcal{N}(T^*)$.

(c) $M_1 = [T^*(H_2)]^\perp$.

From (a), we have $T^*(H_2) \subset M_1^\perp$, so $[T^*(H_2)]^\perp \supset M_1^{\perp\perp}$. Since M_1 is closed, $M_1^{\perp\perp} = M_1$, so $[T^*(H_2)]^\perp \supset M_1$.

For any $x \in [T^*(H_2)]^\perp$, x satisfies $\langle x, T^*y \rangle = 0$ for all $y \in H_2$. However, since $\langle x, T^*y \rangle = \langle Tx, y \rangle$, we have $\langle Tx, y \rangle = 0$ for all $y \in H_2$. Notice that $Tx \in H_2$, so take $y = Tx$, we have $\|Tx\|^2 = 0$, which means $Tx = 0$. This shows that $x \in \mathcal{N}(T) = M_1$. Since x is arbitrarily chosen in $[T^*(H_2)]^\perp$, $[T^*(H_2)]^\perp \subset M_1$. In conclusion, $M_1 = [T^*(H_2)]^\perp$.

Problem 3.9-7. Let T_1 and T_2 be bounded linear operators on a complex Hilbert space H into itself. If $\langle T_1x, x \rangle = \langle T_2x, x \rangle$ for all $x \in H$, show that $T_1 = T_2$.

We first prove that for bounded linear operators from H to itself, if $\langle Tx, x \rangle = 0$ for all $x \in H$, then $T = 0$. For arbitrary $u, v \in H$, if $\langle Tx, x \rangle = 0$ for all $x \in H$, then we have $\langle T(u+v), (u+v) \rangle = 0$ and $\langle T(u+iv), (u+iv) \rangle = 0$. Therefore,

$$0 = \langle T(u+v), (u+v) \rangle = \langle Tu, u \rangle + \langle Tu, v \rangle + \langle Tv, u \rangle + \langle Tv, v \rangle$$

Since $\langle Tu, u \rangle = 0$ and $\langle Tv, v \rangle = 0$, we have $\langle Tu, v \rangle + \langle Tv, u \rangle = 0$.

Similarly, we have

$$0 = \langle T(u+iv), (u+iv) \rangle = \langle Tu, iv \rangle + \langle T(iv), u \rangle$$

Since $\langle Tu, iv \rangle = -i\langle Tu, v \rangle$ and $\langle T(iv), u \rangle = i\langle Tv, u \rangle$, we have $\langle Tv, u \rangle = \langle Tu, v \rangle$. In conclusion, $\langle Tu, v \rangle = 0$ for all $u, v \in H$. Since $Tu \in H$, take $v = Tu$, we have $\|Tu\|^2 = 0$, which implies that $Tu = 0$ for all $u \in H$. This is exactly the definition of $T = 0$.

If $\langle T_1x, x \rangle = \langle T_2x, x \rangle$ for all $x \in H$, then $\langle (T_1 - T_2)x, x \rangle$ for all $x \in H$. Since T_1, T_2 are bounded linear operators from H to H, $T_1 - T_2$ is also bounded linear operators from H to H. Therefore, $T_1 - T_2 = 0$ by our claim above, which implies that $T_1 = T_2$.

Problem 3.9-8. Let $S = I + T^*T : H \mapsto H$, where T is linear and bounded. Show that $S^{-1} : S(H) \mapsto H$ exists.

We only need to show S is injective. Consider $Sx = (I + T^*T)x = 0$, then $T^*Tx = -x$. Therefore, we have

$$0 \leq \|Tx\|^2 = \langle Tx, Tx \rangle = \langle x, T^*Tx \rangle = \langle x, -x \rangle = -\|x\|^2 \leq 0$$

Therefore, $\|x\|^2 = 0$, which implies that $x = 0$. Thus, S is injective.

Problem 3.9-10. Let (e_n) be a total orthonormal sequence in a separable Hilbert space H and define the right shift operator to be the linear operator $T : H \mapsto H$ such that $Te_n = e_{n+1}$ for $n = 1, 2, \ldots$. Explain the name. Find the range, null space, norm and Hilbert-adjoint operator of T.

If we consider the coordinate of x with respect to basis (e_n), then $x = (x_1, x_2, x_3, \ldots)$ and apply T to x, we obtain $Tx = (0, x_1, x_2, \ldots)$. The coordinate of image of x is "shifted" to right by one entry, so T is called right shift operator.

If $x = x_1e_1 + x_2e_2 + \ldots$, then $Tx = x_1e_2 + x_2e_3 + \ldots$, so $Tx \in \overline{\text{span}(e_2, e_3, \ldots)}$. For any $y \in \overline{\text{span}(e_2, e_3, \ldots)}$, y can be expressed as $y = y_1e_2 + y_2e_3 + \ldots$, whose preimage is $x = y_1e_1 + y_2e_2 + \ldots$. Therefore, $\mathrm{R}(T) = \overline{\text{span}(e_2, e_3, \ldots)}$.

Consider $Tx = 0$, if $x = x_1e_1 + x_2e_2 + \ldots$, then $0 = Tx = x_1e_2 + x_2e_3 + \ldots$. This implies that $x_i = 0$ for all $i = 1, 2, \ldots$ and $x = 0$. Therefore, $\mathcal{N}(T) = \{0\}$.

Since (e_n) is total orthonormal, Parseval's identity gives

$$\|x\|^2 = \sum_{i=1}^{\infty} |x_i|^2 = \|Tx\|^2$$

Therefore, $\|x\| = \|Tx\|$ implies that

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1} \|x\| = 1$$

The adjoint operator T^* is defined by $T^*(e_n) = e_{n-1}$ for all $n \geq 2$, and $T^*(e_1) = 0$. This T^* is bounded because

$$\|T^*x\|^2 = \sum_{i=2}^{\infty} |x_i|^2 \leq \|x\|^2$$

Also, it satisfies the definition of adjoint operator,

$$\langle Tx, y\rangle = \sum_{i=1}^{\infty} x_i \overline{y_{i+1}} = \langle x, T^*y\rangle$$

Therefore, such T^* is well-defined adjoint operator of T.

Problem 3.10-3. Show that if $T : H \mapsto H$ is a bounded self-adjoint linear operator, so is T^n, where n is a positive integer.

We use induction to prove it. Since T is a bounded self-adjoint linear operator, we suppose T^{n-1} is a bounded self-adjoint linear operator, then for any $x, y \in H$, and any scalar a, b, we have

$$T^n(ax+by) = T^{n-1}[T(ax+by)] = T^{n-1}(aT(x)+bT(y)) = aT^{n-1}(T(x))+bT^{n-1}(T(y)) = aT^n(x)+bT^n(y)$$

which shows T^n is also linear. Also,

$$\|T^nx\| = \|T^{n-1}[T(x)]\| \leq \|T^{n-1}\|\|T(x)\| \leq \|T^{n-1}\|\|T\|\|x\|$$

Therefore, T^n is bounded because $\|T^{n-1}\|\|T\|$ is finite. Finally,

$$(T^n)^* = (T^{n-1} \circ T)^* = T^* \circ (T^{n-1})^* = T \circ T^{n-1} = T^n$$

Therefore, T^n is a bounded self-adjoint linear operator.

Problem 3.10-4. Show that for any bounded linear operator T on H, the operators

$$T_1 = \frac{1}{2}(T + T^*), \qquad T_2 = \frac{1}{2i}(T - T^*)$$

are self-adjoint. Show that

$$T = T_1 + iT_2, \qquad T^* = T_1 - iT_2$$

Show uniqueness, that is, $T_1 + iT_2 = S_1 + iS_2$ implies $S_1 = T_1$ and $S_2 = T_2$; here, S_1 and S_2 are self-adjoint by assumption.

It is easy to prove by definition that for all bounded linear operator T_1, T_2 and scalar a, we have $(T_1 + T_2)^* = T_1^* + T_2^*$ and $(aT_1)^* = \bar{a}T_1^*$. Therefore,

$$T_1^* = \left(\frac{1}{2}(T + T^*)\right)^* = \frac{1}{2}(T + T^*)^* = \frac{1}{2}(T^* + (T^*)^*) = \frac{1}{2}(T^* + T) = T_1$$

$$T_2^* = \left(\frac{1}{2i}(T - T^*)\right)^* = -\frac{1}{2i}(T - T^*)^* = -\frac{1}{2i}(T^* - T) = \frac{1}{2i}(T - T^*) = T_2$$

which implies T_1, T_2 are self-adjoint. It is easy to see

$$T_1 + iT_2 = \frac{1}{2}(T + T^*) + \frac{1}{2}(T - T^*) = T$$

$$T_1 - iT_2 = \frac{1}{2}(T + T^*) - \frac{1}{2}(T - T^*) = T^*$$

To show the uniqueness, take the adjoint operator on both sides, we have $(T_1 + iT_2)^* = (S_1 + iS_2)^*$, which implies that $T_1 - iT_2 = S_1 - iS_2$. Add up these two equality, we have $2T_1 = 2S_1$ which implies that $T_1 = S_1$. Similarly, use $T_1 + iT_2 = S_1 + iS_2$ to minus $T_1 - iT_2 = S_1 - iS_2$, we have $2iT_2 = 2iS_2$, which implies that $T_2 = S_2$. Therefore, the uniquenss is proved.

Problem 3.10-9. Show that an isometric linear operator $T : H \mapsto H$ which is not unitary maps the Hilbert space H onto a proper closed subspace of H.

First, the image of a linear operator is a vector space, so $\mathcal{R}(T)$ is a subspace of H. If $\mathcal{R}(T) = H$, then T is surjective. However, if $Tx_1 = Tx_2$, then $T(x_1 - x_2) = 0$. By the isometric property, $\|T(x_1 - x_2)\| = \|x_1 - x_2\|$, thus $\|x_1 - x_2\| = 0$ implies that $x_1 = x_2$ and T is hence injective. Then T is bijective and hence invertible. Notice that for all $x, y \in H$, if H is complex, then since $\|Tx\|^2 = \|x\|^2$ implies that $\langle x, x\rangle = \langle T^*Tx, x\rangle$, we can apply the result of Problem 3.9-7, $T^*T = I$ immediately. However, if H is real, then

$$\|x\|^2 + 2\langle x, y\rangle + \|y\|^2 = \|x + y\|^2 = \|T(x + y)\|^2 = \|Tx\|^2 + 2\langle Tx, Ty\rangle + \|Ty\|^2$$

Since $\|Tx\|^2 = \|x\|^2$ and $\|Ty\|^2 = \|y\|^2$, we have $\langle Tx, Ty\rangle = \langle x, y\rangle$. This implies that $\langle (T^*T - I)x, y\rangle = 0$ for all $x, y \in H$. Take $y = (T^*T - I)x$, we have $\|(T^*T - I)x\|^2 = 0$, then $(T^*T - I)x = 0$ for all x, so $T^*T = I$. Therefore, in any cases, $T^*T = I$, and multiply T^{-1} on both sides, we have $T^* = T^{-1}$, therefore we will have $T^*T = TT^* = I$, which means T is unitary. Therefore, this contradiction shows T cannot be surjective, i.e., $\mathcal{R}(T)$ is a proper subspace of H.

The last thing is to prove $\mathcal{R}(T)$ is closed. Take a convergent sequence y_n in $\mathcal{R}(T)$, then since T is injective in any case, there exists unique $x_n \in H$ such that $Tx_n = y_n$. Since y_n is Cauchy, and for all $\epsilon > 0$, there exists M such that for all $n \geq m \geq M$, we have

$$\|x_n - x_m\| = \|T(x_n - x_m)\| = \|Tx_n - Tx_m\| = \|y_n - y_m\| < \epsilon$$

Therefore, x_n is also Cauchy and hence convergent in H. Also, $x_n \to x$ implies that $Tx_n \to Tx$, so $Tx = y$. This shows that y has a pre-image $x \in H$, thus $y \in \mathcal{R}(T)$. We can conclude that $\mathcal{R}(T)$ is closed.

Problem 3.10-10. Let X be an inner product space and $T : X \mapsto X$ an isometric linear operator.If $\dim X < \infty$, show that T is unitary.

From Problem 3.10-9, we have derived that any isometric linear operator is injective. Since X is finite dimensional, T is injective mapping from X to $\mathcal{R}(T)$, so $\mathcal{R}(T)$ must have the same dimension as X, but $\mathcal{R}(T)$ is also a subspace of X, so $\mathcal{R}(T) = X$. This implies that T is surjective, and if T is bijective then T^{-1} exists. By exactly the same argument as Problem 3.10-9, we can also derive that any isometric linear operator satisfies $T^*T = I$, but if we multiply T^{-1} to the right on both sides, we have $T^* = T^{-1}$, this implies that $T^*T = TT^* = I$. Therefore, T is unitary.

Problem 3.10-15. Show that a bounded linear operator $T : H \mapsto H$ on a complex Hilbert space H is normal if and only if $\|T^*x\| = \|Tx\|$ for all $x \in H$. Using this, show that for a normal linear operator, $\|T^2\| = \|T\|^2$.

Since $\|T^*x\| = \|Tx\|$ is equivalent to $\langle T^*x, T^*x\rangle = \langle Tx, Tx\rangle$, we have $\langle x, TT^*x\rangle = \langle x, T^*Tx\rangle$. By Problem 3.9-7 again, $\langle x, TT^*x\rangle = \langle x, T^*Tx\rangle$ if and only if $TT^* = T^*T$ given that H is complex. Therefore, $\|T^*x\| = \|Tx\|$ is equivalent to $TT^* = T^*T$ given that H is complex. This shows T is normal if and only if $\|T^*x\| = \|Tx\|$.

Since we have $\|T^2x\| = \|T \circ Tx\| \leq \|T\|\|Tx\| \leq \|T\|^2\|x\|$, take the supremum over $\|x\| = 1$ on both sides,

$$\|T^2\| = \sup_{\|x\|=1} \|T^2x\| \leq \|T\|^2 \implies \|T^2\| \leq \|T\|^2$$

For the reverse inequality, consider

$$\begin{aligned}
\|Tx\|^4 &= \langle Tx, Tx\rangle^2 = \langle x, T^*Tx\rangle^2 \\
&\leq \|x\|^2\|T^*Tx\|^2 = \|x\|^2\langle T^*Tx, T^*Tx\rangle = \|x\|^2\langle Tx, TT^*Tx\rangle \\
&= \|x\|^2\langle Tx, T^*TTx\rangle = \|x\|^2\langle Tx, T^*T^2x\rangle = \|x\|^2\langle T^2x, T^2x\rangle \\
&= \|x\|^2\|T^2x\|^2 \leq \|x\|^4\|T^2\|^2
\end{aligned}$$

Therefore, take the 4-th root on both sides, we have $\|Tx\| \leq \sqrt{\|T^2\|}\|x\|$. Take supremum over $\|x\| = 1$ on both sides, we have

$$\|T\| = \sup_{\|x\|=1} \|Tx\| \leq \sqrt{\|T^2\|}$$

which implies that $\|T\|^2 \leq \|T^2\|$. Therefore, combined with our previous conclusion, we have $\|T\|^2 = \|T^2\|$.