MAT4010: Functional Analysis
Homework 6

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Due date: Oct. 22, 2019

Problem 4.1-5. Prove that a finite partially ordered set A has at least one maximal element.

We prove by induction on the cardinality of A. Suppose $|A| = 1$, then the only element in A is maximal element. Assume for all $|A| \leq n$, A has at least one maximal elememt. Consider $|A| = n+1$, pick arbitrary fixed $a \in A$, if a is maximal, then there is nothing to prove. If not, then there exists $b \in A$ such that $b \geq a$ but $b \neq a$. Denote it as $b > a$, and consider the set $B = \{x \,|\, x > a\}$, then B is nonempty, and $a \notin B$, so $|B| \leq n$. By induction hypothesis, B has at least one maximal element b^*. We need to show b^* is also maximal in A. If there exists $c \in A$ and $c > b^*$, then since $b^* \in B$, $b^* > a$, so $c > a$, then $c \in B$. This contradicts to the fact that b^* is maximal in B, hence such c doesn't exist, which means b^* is maximal in A. Therefore, we can conclude that for all partially ordered set A with finite cardinality, A has at least one maximal element.

Problem 4.1-6. Show that a partially ordered set M can have at most one element a such that $a \leq x$ for all $x \in M$ and at most one element b such that $x \leq b$ for all $x \in M$. [If such an a (or b) exists, it is called the least element (greatest element, respectively) of M.]

Suppose there are two least elements a, b in M, then since a satisfies that for all $x \in M$, $a \leq x$, we have $a \leq b$ because $b \in M$. Similarly, since b satisfies that for all $x \in M$, $b \leq x$, we have $b \leq a$. In a partially ordered set, if $a \leq b$ and $b \leq a$, then $a = b$. Therefore, there can exist at most one least element in M.

Similarly, if there are two greatest elements a, b in M, then since a satisfies that for all $x \in M$, $x \leq a$, we have $b \leq a$, because $b \in M$. Similarly, since b satisfies that for all $x \in M$, $x \leq b$, we have $a \leq b$. In a partially ordered set, if $b \leq a$ and $a \leq b$, then $a = b$. Therefore, there can exist at most one greatest element in M.

Problem 4.1-8. A *greatest lower bound* of a subset $A \neq \varnothing$ of a partially ordered set M is a lower bound x of A such that $l \leq x$ for any lower bound l of A; we write $x = \text{g.l.b. } A = \inf A$. Similarly, a *least upper bound* y of A, written $y = \text{l.u.b. } A = \sup A$, is an upper bound y of A such that $y \leq u$ for any upper bound u of A.

(a) If A has a g.l.b., show that it is unique.

Suppose A has two g.l.b., denoted as $x_1, x_2 \in M$. Since x_1 is g.l.b., it must be a lower bound, but x_2 is g.l.b., so $x_1 \le x_2$. Similarly, x_2 is g.l.b., so it is a lower bound, but x_1 is g.l.b., so $x_2 \le x_1$. In a partially ordered set, if $x_1 \le x_2$ and $x_2 \le x_1$, then $x_1 = x_2$. Therefore, there can exist at most one g.l.b. of A in M.

(b) What are g.l.b. $\{A, B\}$ and l.u.b. $\{A, B\}$ if $A, B \in \mathcal{P}(X)$ where $\mathcal{P}(X)$ is the power set of X and $A \le B$ ($A \subset B \subset X$)?

The g.l.b. of $\{A, B\}$ is A and the l.u.b. of $\{A, B\}$ is B. In Problem 4.1-9, we will prove a more general case, that is, the g.l.b. of $\{A, B\}$ is $A \cap B$ and l.u.b. of $\{A, B\}$ is $A \cup B$. Here is a special case, because we have $A \subset B$, so $A \cap B = A$ and $A \cup B = B$.

Problem 4.1-9. A lattice is a partially ordered set M such that any two elements x, y of M have a g.l.b. (written $x \wedge y$) and a l.u.b. (written $x \vee y$). Show that the power set $\mathcal{P}(X)$ of a given set X is a lattice, where $A \wedge B = A \cap B$ and $A \vee B = A \cup B$, and $A, B \in \mathcal{P}(X)$.

We have proved that $M = \mathcal{P}(X)$ is partially ordered with respect to $\subset$. We also know that for any two elements A, B in $\mathcal{P}(X)$, the union and intersection of them, i.e., $A \cup B$ and $A \cap B$ are both well-defined. Now we only need to show that $A \cup B$ is the supremum of $\{A, B\}$ and $A \cap B$ is the infimum of $\{A, B\}$.

First, it is easy to show that $A \cup B$ is an upper bound, because $A \le A \cup B$ and $B \le A \cup B$. For any upper bound C of $\{A, B\}$, we must have $A \subset C$ and $B \subset C$, then for all $a \in A$, $a \in C$ and for all $b \in B$, $b \in C$. For all $x \in A \cup B$, x is either in A or B, so $x \in C$. Thus, $A \cup B \subset C$, which implies that $A \cup B \le C$. This implies that $A \cup B$ is the least upper bound.

Similarly, it is easy to show that $A \cap B$ is an lower bound, because $A \cap B \subset A$ and $A \cap B \subset B$. For any lower bound D of $\{A, B\}$, we must have $D \le A$ and $D \le B$, which implies that $D \subset A$ and $D \subset B$. Therefore, for all $d \in D$, $d \in A$ and $d \in B$, so $d \in A \cap B$. This shows that $D \subset A \cap B$, so $D \le A \cap B$. By definition, $A \cap B$ is the greatest lower bound of $\{A, B\}$.

Now for any two elements A, B of $\mathcal{P}(X)$, $A \cap B$ and $A \cup B$ are both subset of X, so $A \cap B \in \mathcal{P}(X)$ and $A \cup B \in \mathcal{P}(X)$. This implies that any two elements of $\mathcal{P}(X)$ has a g.l.b. and a l.u.b., which means $\mathcal{P}(X)$ is a lattice.

Problem 4.2-3. Show that $p(x) = \overline{\lim\limits_{n\to\infty}} \xi_n$, where $x = (\xi_n) \in l^\infty$, ξ_n real, defines a sublinear functional on l^∞.

Notice that since $x \in l^\infty$, each x represents a bounded real sequence. To prove $p(x)$ is sublinear, we need to prove for all positive real number a, $p(ax) = ax$ and $p(x+y) \le p(x)+p(y)$ for all $x, y \in \mathbb{R}$. Consider for bounded real sequence ξ_n, $a > 0$, $\sup_{n\ge m}(a\xi_n) = a \sup_{n\ge m} \xi_n$. To prove this, recall that for all $n \ge m$,

$$\xi_n \le \sup_{n\ge m} \xi_n \Longrightarrow a\xi_n \le a \sup_{n\ge m} \xi_n \Longrightarrow \sup_{n\ge m}(a\xi_n) \le a \sup_{n\ge m} \xi_n$$

For all $\epsilon > 0$, there exists ξ_p such that $p \ge m$ and

$$\xi_p > \sup_{n\ge m} \xi_n - \epsilon \Longrightarrow a\xi_p > a \sup_{n\ge m} \xi_n - a\epsilon \Longrightarrow \sup_{n\ge m}(a\xi_n) \ge a\xi_p > a \sup_{n\ge m} \xi_n - a\epsilon$$

Take $\epsilon \to 0$, we obtain $\sup_{n\ge m}(a\xi_n) \ge a \sup_{n\ge m} \xi_n$. Therefore, we proved our claim.

Since $\sup_{n\geq m}(a\xi_n)$ and $\sup_{n\geq m}\xi_n$ are always decreasing and bounded, they must be convergent, so by taking $m\to\infty$ on both sides, we obtain $\overline{\lim_{n\to\infty}}(a\xi_n) = a\overline{\lim_{n\to\infty}}\xi_n$, which is equivalent to $p(ax) = ap(x)$ for $a > 0$.

To prove $p(x+y) \leq p(x) + p(y)$, where $x = (\xi_n)$ and $y = (\eta_n)$, we only need to prove

$$\limsup_{n\to\infty}(\xi_n+\eta_n) \leq \limsup_{n\to\infty}\xi_n + \limsup_{n\to\infty}\eta_n \tag{4.2}$$

To achieve this, we consider for all $n \geq m$,

$$\xi_n \leq \sup_{n\geq m}\xi_n,\ \eta_n \leq \sup_{n\geq m}\eta_n \Longrightarrow \xi_n+\eta_n \leq \sup_{n\geq m}\xi_n + \sup_{n\geq m}\eta_n \Longrightarrow \sup_{n\geq m}(\xi_n+\eta_n) \leq \sup_{n\geq m}\xi_n + \sup_{n\geq m}\eta_n$$

Since on both sides, all terms are decreasing and bounded with respect to m, they must be convergent. Take $m\to\infty$, we obtain (4.2), which yields $p(x+y) \leq p(x)+p(y)$. Therefore, $p(x)$ is a sublinear functional.

Problem 4.2-5. If p is a sublinear functional on a vector space X, show that $M = \{x \,|\, p(x) \leq \gamma, \gamma > 0 \text{ fixed }\}$, is a convex set.

Suppose $a \in [0,1]$, then we need to prove for all $x, y \in M$,

$$p(ax+(1-a)y) \leq \gamma, \quad \text{given } p(x)\leq\gamma \text{ and } p(y)\leq\gamma$$

Consider the definition of sublinearity, we have

$$p(ax+(1-a)y) \leq p(ax) + p((1-a)y) = ap(x) + (1-a)p(y) \leq a\gamma + (1-a)\gamma = \gamma$$

where the first equality and the second inequality are due to $a \geq 0$ and $1-a \geq 0$. Therefore, if $x, y \in M$ then $ax + (1-a)y \in M$ for all $a \in [0,1]$, which impiles that M is convex.

Problem 4.2-8. If a subadditive functional defined on a normed space X is nonnegative outside a sphere $\{x \,|\, \|x\| = r\}$, show that it is nonnegative for all $x \in X$.

Consider all points $x \in X$ satisfies $\|x\| \leq r$, if $x = 0$, then since $p(0) = p(0\cdot x) = 0p(x) = 0$, it is nonnegative. If $\|x\| = d \in (0, r)$, and $p(x) = k < 0$, then let $c = \frac{2r}{d} > 0$, and we will have $p(cx) = cp(x) = \frac{2rk}{d} < 0$. However, $\|cx\| = c\|x\| = 2r > r$, so cx is outside the sphere $\{x \,|\, \|x\| = r\}$, by assumption $p(cx) \geq 0$, contradiction! Thus, $p(x) \geq 0$. If $\|x\| = d$, and $p(x) = k < 0$, then since $\|2x\| = 2d > d$, so similarly, $p(2x)$ should be nonnegative by assumption. However, $p(2x) = 2p(x) = 2k < 0$, contradiction again, so $p(x) \geq 0$ for all $\|x\| = d$. In conclusion, for all $\|x\| \leq r$, $p(x) \geq 0$; together with our assumption, $p(x) \geq 0$ for all $x \in X$.

Problem 4.2-9. Let p be a sublinear functional on a real vector space X. Let f be defined on $Z = \{x \in X \,|\, x = \alpha x_0, \alpha \in \mathbb{R}\}$ by $f(x) = \alpha p(x_0)$ with fixed $x_0 \in X$. Show that f is a linear functional on Z satisfying $f(x) \leq p(x)$.

First, we need to prove for all $b \in \mathbb{R}$, $f(bx) = bf(x)$ for all $x \in Z$. Since $x \in Z$, we have $x = \alpha x_0$, and $f(bx) = f(b\alpha x_0) = b\alpha p(x_0) = bf(\alpha x_0) = bf(x)$, because $b\alpha \in \mathbb{R}$.

Then we need to show for all $x, y \in Z$, $f(x+y) = f(x) + f(y)$. Since $x = \alpha x_0$ and $y = \beta x_0$, we have

$$f(x+y) = f(\alpha x_0 + \beta x_0) = f((\alpha+\beta)x_0) = (\alpha+\beta)p(x_0) = \alpha p(x_0) + \beta p(x_0) = f(x) + f(y)$$

because $\alpha + \beta \in \mathbb{R}$.

Now we prove that $f(x)$ is linear on Z, the last thing is to prove $f(x) \leq p(x)$ for all $x \in Z$. For any $x \in Z$, we always have $x = \alpha x_0$, so if $\alpha \geq 0$, then $f(x) = \alpha p(x_0) = p(\alpha x_0) = p(x)$. However, if $\alpha < 0$, we have

$$0 = p(0) \leq p(\alpha x_0) + p(-\alpha x_0) \Longrightarrow -p(-\alpha x_0) \leq p(\alpha x_0)$$

Since $f(x) = \alpha p(x) = -p(-\alpha x_0) \leq p(\alpha x_0) = p(x)$, we can still obtain $f(x) \leq p(x)$. Therefore, we can conclude that for all $x \in Z$, $f(x) \leq p(x)$.

Problem 4.3-4. Let $p(x)$ be a real value functional defined on a vector space X and satisfy that for all $x, y \in X$ and scalar α,

$$p(x+y) \leq p(x) + p(y), \qquad p(\alpha x) = |\alpha| p(x)$$

Show that for any given $x_0 \in X$ there is a linear functional $\tilde{f}$ on X such that $\tilde{f}(x_0) = p(x_0)$ and $|\tilde{f}(x)| \leq p(x)$ for all $x \in X$.

Take $M = \text{span}(x_0)$, then M is a subspace of X. Define f on M by $f(\alpha x_0) = \alpha p(x_0)$. Then we claim that $f(x)$ is a linear functional defined on M and $|f(x)| \leq p(x)$ for all $x \in M$. If this is true, then by Hahn-Banach Theorem (complex case), there exists a linear functional $\tilde{f}$ on X such that $\tilde{f}(x) = f(x)$ for all $x \in M$ and $|\tilde{f}(x)| \leq p(x)$ for all $x \in X$.

Now we prove that f defined above is a linear functional on M and $|f(x)| \leq p(x)$ by similar argument as Problem 4.2-9. For any scalar a, b and vector $x, y \in X$, we have

$$f(ax+by) = f(a\alpha x_0 + b\beta x_0) = (a\alpha + b\beta)p(x_0) = a\alpha p(x_0) + b\beta p(x_0) = af(x) + bf(y)$$

Therefore, f is linear on M. Consider the following,

$$0 = p(0) = p(x + (-x)) \leq p(x) + p(-x) = p(x) + p(x) \Longrightarrow p(x) \geq 0, \quad \forall x \in X$$

Therefore, for all $x \in M$, $x = \alpha x_0$, and $|f(x)| = |\alpha p(x_0)| = |\alpha||p(x_0)| = ||\alpha|p(x_0)| = |p(x)| = p(x)$.

Problem 4.3-13. Show that if X is a normed space and $x_0 \neq 0$ is any element in X, then there is a bounded linear functional $\hat{f}$ on X such that $\|\hat{f}\| = \|x_0\|^{-1}$ and $\hat{f}(x_0) = 1$.

Take $M = \text{span}(x_0)$ and let $f(x_0) = 1$. Define $f(x)$ on M by $f(x) = f(\alpha x_0) = \alpha f(x_0) = \alpha$. Then $f(x)$ is linear on M. Consider

$$\|f\| = \sup_{\|x\|=1, x \in M} |f(x)| = \sup_{|\alpha| = \|x_0\|^{-1}} |\alpha| = \|x_0\|^{-1}$$

Therefore, f is a linear and bounded functional on a subspace M of X, then by Application 1 in lecture, there exists a linear and bounded functional $\hat{f}$ defined on X such that $\hat{f}\Big|_M = f$ and $\|\hat{f}\| = \|f\| = \|x_0\|^{-1}$. However, since $\hat{f}\Big|_M = f$ and $x_0 \in M$, so $\hat{f}(x_0) = f(x_0) = 1$. Therefore, $\hat{f}$ is the target functional we want to find.