MAT4010: Functional Analysis
Homework 7

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Problem 4.6-3. If a normed space X is reflexive, show that X' is reflexive.

To show X' is reflexive, we only need to show that for Canonical mapping $C' : X''' \mapsto X'$ is surjective, i.e., for any fixed $x_0''' \in X'''$, there exists $x_0' \in X'$ such that $C'x_0' = x_0'''$. We first construct a $x_0' : X' \mapsto X$ according to x_0''' by

$$\langle x_0', x \rangle_{X',X} = \langle x_0''', Cx \rangle_{X''',X''}, \quad \text{for all } x \in X$$

where $C : X'' \mapsto X$ is the Canonical mapping defined in class. Then, we need to show such x_0' is really in X'. It is linear in x, because C is linear in any elements in X and x_0''' is linear in any elements in X''. It is bounded because

$$|\langle x_0', x \rangle_{X',X}| = |\langle x_0''', Cx \rangle_{X''',X''}| \leq \|x_0'''\|_{X'''} \|Cx\|_{X''} \leq \|x_0'''\|_{X'''} \|x\|_X$$

Therefore, $x_0' \in X'$. Now we want to show such x_0' is the preimage of x_0''' under C', where by definition of C',

$$\langle C'x_0', f'' \rangle_{X''',X''} = \langle f'', x_0' \rangle_{X'',X'}, \quad \text{for all } f'' \in X''$$

Since X is reflexive, Canonical mapping C is surjective, i.e., for all $f'' \in X''$, there exists $f \in X$, such that $Cf = f''$, thus,

$$\langle f'', x_0' \rangle_{X'',X'} = \langle Cf, x_0' \rangle_{X'',X'} \triangleq \langle x_0', f \rangle_{X',X}$$

However, by definition of x_0', we have

$$\langle x_0', f \rangle_{X',X} = \langle x_0''', Cf \rangle_{X''',X''} = \langle x_0''', f'' \rangle_{X''',X''}$$

Therefore, we proved that for all $f'' \in X''$, $\langle C'x_0', f'' \rangle_{X''',X''} = \langle x_0''', f'' \rangle_{X''',X''}$, which is equivalent to $C'x_0' = x_0'''$. Therefore, X' is also reflexive.

Problem 4.6-4. Show that a Banach space X is reflexive if and only if its dual space X' is reflexive.

For "only if" part, since a Banach space X is a normed space, by Problem 4.6-3, X' must be reflexive.

For "if" part, we assume X' is reflexive. If X is not reflexive, then the image of X under Canonical mapping $C(X) \subset X''$ is a proper subset of X''. Since X is Banach, C is injective and isometry, we can conclude that $C(X)$ is closed in X'' (Problem 3.10-9). Therefore, we can apply

Hahn-Banach (Fact 5) to a point $x_0'' \in X'' \backslash C(X)$, then there exists $x_0''' \in X'''$ such that $x_0'''\Big|_{C(X)} = 0$, $\langle x_0''', x_0'' \rangle_{X''', X''} > 0$. Since X' is reflexive, Canonical map $C' : X' \mapsto X'''$ is surjective, i.e., there exists $x_0' \in X'$ such that $C'x_0' = x_0'''$. So by definition of C', we have

$$\langle x_0''', x'' \rangle_{X''', X''} = \langle C'x_0', x'' \rangle_{X''', X''} = \langle x'', x_0' \rangle_{X'', X'}, \quad \text{for all } x'' \in X''$$

Therefore, we have $\langle x_0'', x_0' \rangle_{X'', X'} > 0$. However, since $x_0'''\Big|_{C(X)} = 0$, for all $x'' \in C(X)$, there exists $x \in X$ such that $Cx = x''$, i.e.,

$$0 = \langle x_0''', Cx \rangle_{X''', X''} = \langle C'x_0', Cx \rangle_{X''', X''} = \langle Cx, x_0' \rangle_{X'', X'}, \quad \text{for all } x \in X$$

Therefore, $\langle Cx, x_0' \rangle_{X'', X'} = 0$ for all $x \in X$, which by definition of C, means $\langle Cx, x_0' \rangle_{X'', X'} = \langle x_0', x \rangle_{X', X} = 0$. This implies that x_0' is the zero element in X'. Recall x_0'' is a linear functional defined on X', so $\langle x_0'', x_0' \rangle_{X'', X'} = 0$, but this is a contradiction to $\langle x_0'', x_0' \rangle_{X'', X'} > 0$, thus x_0'' does not exist and $C(X)$ is equal to X'', which means X is reflexive.

Problem 4.6-8. Let M be any subset of a normed space X. Show that an $x_0 \in X$ is an element of $A = \overline{\text{span}(M)}$ if and only if $f(x_0) = 0$ for every $f \in X'$ such that $f\Big|_M = 0$.

For "only if" part, if $x_0 \in X$ implies $x_0 \in A$, then there exists a sequence $x_0^{(k)} \in \text{span}(M)$ such that $x_0^{(k)} \to x_0$. Since $x_0^{(k)} \in \text{span}(M)$, for all $f \in X'$ satisfies $f\Big|_M = 0$, by linearity we have $f(x_0^{(k)}) = 0$ for all k. Since f is linear bounded functional, it is continuous, and we have $f(x_0^{(k)}) \to f(x_0)$ as $x_0^{(k)} \to x_0$. Therefore, $f(x_0) = 0$.

For "if" part, if $x_0 \notin A$, then by Hahn-Banach (Fact 5), since $x_0 \in X \setminus A$ where A is closed subspace of X, there exists $f \in X'$ such that $f\Big|_A = 0$ but $f(x_0) > 0$. However, $f\Big|_A = 0$ implies that $f\Big|_M = 0$. This contradicts to our assumption that all $f \in X'$ satisfies $f\Big|_M = 0$ should give $f(x_0) = 0$. Therefore, $x_0 \in A$.

Problem 4.6-10. Show that if a normed space X has a linearly independent subset of n elements, so does the dual space X'.

Denote the linearly independent subset of n elements in X as $S = \{x_1, x_2, \ldots, x_n\}$. Construct n subsets of S by $S_i = S \setminus \{x_i\}$ for all $i = 1, 2, \ldots, n$. Denote $Y_i = \text{span}(S_i)$. Since Y_i is of finite dimensional, it is obvious that they are closed subspace of X. Since x_i is not in Y_i, Y_i is proper closed subspace of X, so we can apply Hahn-Banach (Fact 5) to each Y_i, then there exists $f_i \in X'$ such that $f_i\Big|_{Y_i} = 0$ and $f_i(x_i) > 0$ for all i. Then such $T = \{f_i \,|\, i = 1, 2, \ldots, n\}$ is a linear independent subset of X' with size n. To see this, consider applying $a_1 f_1 + \ldots + a_n f_n = 0$ on each x_i, we have

$$(a_1 f_1 + \ldots + a_n f_n)(x_i) = a_i f_i(x_i) = 0 \Longrightarrow a_i = 0$$

Therefore, T forms a linearly independent set in X' of size n.

Extra Problem 1. Consider $L^p(E)$, $1 \leq p < \infty$, where E is Lebesgue measurable subset of $\mathbb{R}$.

(i) Let $-\infty < a < b < \infty$. Prove that $\mathcal{C}([a, b])$ is separable.

We only need to show $\mathbb{Q}[x]$ is dense in $\mathcal{C}([a,b])$ because $\mathbb{Q}[x]$ is countable. First, by Weierstrass Approximation, for any $f \in \mathcal{C}([a,b])$, there exists a sequence of polynomial with real coefficients that uniformly converges to f. Denote this sequence as $r_n(x) \in \mathbb{R}[x]$ and its real coefficients as $a_{n,0}, a_{n,1}, \dots, a_{n,k}, \dots, a_{n,n}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $b_{n,k,j} \in \mathbb{Q}$ such that $b_{n,k,j} \to a_{n,k}$ for all n, k as $j \to \infty$. Let $q_{n,j}(x) \in \mathbb{Q}[x]$ denote the polynomials with coefficients $b_{n,k,j}$ for $k = 0, 1, \dots, n$. Then we claim that $q_{n,j}(x) \to r_n(x)$ uniformly as $j \to \infty$. This is true because for arbitrary $\epsilon > 0$, there exists $J \in \mathbb{N}$ such that $|b_{n,k,j} - a_{n,k}| \le \epsilon$ for all $j \ge J$ and $k = 0, 1, \dots n$. There also exists $x_0 \in [a,b]$ such that

$$\|q_{n,j}(x) - r_n(x)\|_\infty = |q_{n,j}(x_0) - r_n(x_0)| \le \epsilon(1 + |x_0| + |x_0|^2 + \cdots + |x_0|^n)$$

Thus, we can conclude that $q_{n,j}(x)$ is uniformly convergent to $r_n(x)$. Therefore, for all $\epsilon > 0$, there exists J, N such that for $j \ge J$ and $n \ge N$,

$$\|q_{n,j} - f\|_\infty \le \|q_{n,j} - r_n\|_\infty + \|r_n - f\|_\infty < \epsilon + \epsilon = 2\epsilon$$

This implies that $\mathbb{Q}[x]$ is dense in $\mathcal{C}[a,b]$ and $\mathcal{C}[a,b]$ is separable.

(ii) Use (i) and the fact that $\mathcal{C}[a,b]$ is dense in $L^p(a,b)$ to prove that $L^p(a,b)$ is separable.

For any $f \in L^p(a,b)$, by Lusin's theorem, there exists a continuous function g on $[a,b]$ such that $m(\{x \in [a,b] \,|\, g(x) \ne f(x)\}) < \epsilon$ for arbitrary $\epsilon > 0$. Denote $A = \{x \in [a,b] \,|\, g(x) \ne f(x)\}$, and consider

$$\int_{[a,b]} |f-g|^p \, dm = \int_{[a,b]\setminus A} |f-g|^p \, dm + \int_A |f-g|^p \, dm = \int_A |f-g|^p \, dm$$

By Minkowski inequality,

$$\int_A |f-g|^p \, dm \le (\|f\|_{L^p} + \|g\|_{L^p})^p < \epsilon(\|f\|_\infty + \|f\|_\infty)^p$$

where the infinity norm is defined by essential supremum of f and g, and hence they are all finite and fixed. This further implies that for some positive constant C,

$$\|f-g\|_{L^p} = \left(\int_{[a,b]} |f-g|^p \, dm\right)^{1/p} < C\epsilon^{1/p} \to 0$$

as $\epsilon \to 0$. Thus, every $f \in L^p(a,b)$ can be approximated by $g \in \mathcal{C}[a,b]$ arbitrarily close, implying that $\mathcal{C}[a,b]$ is dense in $L^p(a,b)$.

From (i), since $\mathbb{Q}[x]$ is dense in $\mathcal{C}[a,b]$, for any $f \in \mathcal{C}[a,b]$ there exists $q_n(x)$ such that $\|q_n - f\|_\infty < \epsilon$. Now we also have for all $\epsilon > 0$, there exists $\|f_n - g\|_{L^p} < \epsilon$ where $f_n \in \mathcal{C}[a,b]$ and $g \in L^p(a,b)$. This implies that

$$\|q_n(x) - g(x)\|_{L^p} \le \|q_n - f_n\|_{L^p} + \|f_n - g\|_{L^p} < (1 + (b-a)^{1/p})\epsilon$$

Therefore, $\mathbb{Q}[x]$ (defined on $[a,b]$) is dense in $L^p(a,b)$, but since $\mathbb{Q}[x]$ is countable, $L^p(a,b)$ is separable.

(iii) Prove $L^p(\mathbb{R})$ is separable. Hint: Use $P_{\mathbb{Q}}\chi_{(-r,r)}(x)$, where $r \in \mathbb{Q}^+$, to approximate elements in $L^p(\mathbb{R})$.

By MAT3006, we know that for $g \in L^p(\mathbb{R})$, we have

$$\int_{\mathbb{R}} |g|^p \, dm = \lim_{n\to\infty} \int_{(-r_n,r_n)} |g|^p \, dm$$

where $r_n \in \mathbb{Q}$ is sequence increasing to infinity. Therefore, for any $g \in L^p(\mathbb{R})$, we can find a sequence $g\chi_{(-r_n,r_n)} \to g$. For each $g\chi_{(-r_n,r_n)}$, we can regarded it as function on $L^p[-r_n, r_n]$, then apply result in (i) and (ii), we can find sequence of rational polynomial $q_k\chi_{(-r_n,r_n)}$ on $[-r_n, r_n]$ such that $q_k\chi_{(-r_n,r_n)} \to g\chi_{(-r_n,r_n)}$ as $k \to \infty$. Therefore, for all $\epsilon > 0$, for sufficiently large N and K, for all $k \geq K$ and $n \geq N$, we have

$$\|q_k\chi_{(-r_n,r_n)} - g\|_{L^p} \leq \|q_k\chi_{(-r_n,r_n)} - g\chi_{(-r_n,r_n)}\|_{L^p} + \|g\chi_{(-r_n,r_n)} - g\|_{L^p} < 2\epsilon$$

Therefore, $B = \{P_{\mathbb{Q}}\chi_{(-r,r)}(x) \,|\, r \in \mathbb{Q}^+\}$ is dense in $L^p(\mathbb{R})$. Since B is countable, $L^p(\mathbb{R})$ is separable.

(iv) Prove $L^p(E)$ is separable.

For $f \in L^p(E)$, it assumes the same value as $f\chi_E \in L^p(\mathbb{R})$. Therefore, denote $B' = \{P_{\mathbb{Q}}\chi_{(-r,r)}\chi_E(x) \,|\, r \in \mathbb{Q}^+\}$, B' is still countable. Since $f\chi_E$ can be approximated by elements in B, choose $q_k\chi_{(-r_n,r_n)}$ such that $\|q_k\chi_{(-r_n,r_n)} - f\chi_E\|_{L^p(\mathbb{R})} < \epsilon$, then we have

$$\|q_k\chi_{(-r_n,r_n)}\chi_E - f\chi_E\|_{L^p(E)} = \|q_k\chi_{(-r_n,r_n)} - f\|_{L^p(E)} = \|q_k\chi_{(-r_n,r_n)}\chi_E - f\|_{L^p(E)}$$

$$\epsilon > \|q_k\chi_{(-r_n,r_n)} - f\chi_E\|_{L^p(\mathbb{R})} = \|q_k\chi_{(-r_n,r_n)} - f\|_{L^p(\mathbb{E})} + \|q_k\chi_{(-r_n,r_n)}\|_{L^p(\mathbb{R}\setminus\mathbb{E})}$$

Therefore, we can conclude that $\|q_k\chi_{(-r_n,r_n)}\chi_E - f\|_{L^p(E)} < \epsilon$, thus B' is dense in $L^p(E)$. This shows that $L^p(E)$ is separable for any measurable set E.

Extra Problem 2. Consider $L^\infty(E)$, where E is Lebesgue measurable subset with $m(E) > 0$.

(i) For all $r > 0$, let $f(r) = m(E \cap (-r, r))$. Then $f(0) = 0$, $f(r)$ is increasing in r with $\lim_{r\to\infty} f(r) = m(E) > 0$, and f is continuous on $[0, \infty)$.

We can easily see $f(0) = m(E \cap \varnothing) = m(\varnothing) = 0$. If $r_1 > r_2$, $(-r_2, r_2) \subset (-r_1, r_1)$, thus $E\cap(-r_2, r_2) \subset E\cap(-r_1, r_1)$. By monotonicity of Lebesgue measure, we have $m(E\cap(-r_2, r_2)) \leq m(E \cap (-r_1, r_1))$, i.e., $f(r_2) \leq f(r_1)$. For all $\epsilon > 0$, fix any $r_0 \in [0, \infty)$, take $\delta = \epsilon/2$, for all $0 < r - r_0 < \delta$, we have

$$\begin{aligned}
f(r) - f(r_0) &= m(E \cap (-r, r)) - m(E \cap (-r_0, r_0)) \\
&= m(E \cap (-r_0, r_0)) + m(E \cap (-r, r_0)) + m(E \cap (r_0, r)) - m(E \cap (-r_0, r_0)) \\
&= m(E \cap (-r, r_0)) + m(E \cap (r_0, r)) \leq 2(r - r_0) < 2\delta < \epsilon
\end{aligned}$$

Combined with $f(r) - f(r_0) \geq 0$, we can conclude that $f(r)$ is right continuous at any point $r_0 \in [0, \infty)$. Similarly, we can prove that $f(r)$ is left continuous at any point $r_0 \in (0, \infty)$.

Therefore, we can conclude that $f(r)$ is continuous on $[0, \infty)$. Then by "continuity" of Lebesgue measure (c.f. MAT3006, HW3, Q9), we have

$$\lim_{r\to\infty} f(r) = \lim_{n\to\infty} m(E \cap (-n, n)) = m\left(E \cap \lim_{n\to\infty}(-n, n)\right) = m(E) > 0$$

This implies that f can take any value between 0 and $m(E)$.

(ii) Let A be the collection of maximal closed subinterval I of $[0, \infty)$, such that I has nonempty interior, $f\Big|_I$ is constant on I. Prove that A is at most countable.

Since I has nonempty interior, there exists a rational number q in I. If I_1 and I_2 are two elements in A, then if $I_1 \cap I_2 \neq \varnothing$, $I_1 = I_2$. This is because if $I_1 \cap I_2 \neq \varnothing$, take $x \in I_1 \cap I_2$, then $f(x) = f\Big|_{I_1} = f\Big|_{I_2}$. Then denote $I_3 = I_1 \cup I_2$, I_3 is also a closed subinterval of $[0, \infty)$, and $f\Big|_{I_3}$ is constant. If $I_1 \neq I_2$, I_3 will be strictly larger than I_1 and I_2, but this contradicts the maximality of I_1, I_2. Thus, different elements in A must be disjoint. For each element in A, we can pick a rational number in it as a representative of that closed interval. Since each interval are disjoint, all rational number picked are distinct. However, there only exists countably many of rational numbers in total, so the number of closed intervals in A is at most countable. Therefore, A is at most countable.

(iii) Prove that $[0, \infty) \setminus \cup_{I\in A} I$ is uncoutable.

Recall that a Lipschitz continuous function maps set with zero measure to set with zero measure. We can see in part (i), f is not only continuous but also Lipschitz continuous with Lipschitz constant 2. Therefore, suppose $U = [0, \infty) \setminus \cup_{I\in A} I$ is coutable, then $f(U)$ is measure zero set. Notice that A is at most countable, and f on each I is constant, so $f(\cup_{I\in A} I)$ is also countable. This implies that $f([0, \infty))$ is measure zero set. On the otherhand, since $f(0) = 0$ and $f(\infty) = m(E) > 0$, then by intermediate value theorem, $f([0, \infty)) = [0, m(E))$, while $[0, m(E))$ cannot be measure zero. Therefore, contradiction shows that U is uncountable.

To prove a Lipschitz continuous function maps set with zero measure to set with zero measure, simply take T to be Lipschitz continuous function with Lipschitz constant C. Then for any measure zero set Z, for all $\epsilon > 0$, by definition of Lebesgue measure, there exists a collection of open interval $\{J_n\}$, such that $E \subset \cup_{n=1}^{\infty} J_n$, and $\sum_{n=1}^{\infty} m(J_n) < \epsilon$. Therefore,

$$m(T(E)) \leq m(T(\cup_{n=1}^{\infty} J_n)) = m(\cup_{n=1}^{\infty} T(J_n)) \leq \sum_{n=1}^{\infty} m(T(J_n)) \leq C \sum_{n=1}^{\infty} m(J_n) < C\epsilon$$

Therefore, $T(E)$ is also of measure zero.

(iv) For $s \in [0, \infty) \setminus \cup_{I\in A} I$, define $\chi_s(x) = \chi_{E\cap[-s,s]}(x)$. Prove that for all $s, t \in [0, \infty) \setminus \cup_{I\in A} I$, $s \neq t$, we have $\|\chi_s - \chi_t\|_{L^\infty(E)} = 1$.

We need to first claim that there only exists one $s \in [0, \infty) \setminus \cup_{I\in A} I$ such that $E \cap [-s, s]$ is empty. Suppose there exists $s > t \geq 0$ such that $E \cap [-s, s] = E \cap [-t, t] = 0$, then there exists a function $f(x) = m(E \cap [-x, x])$ defined on $x \in [t, s]$ such that $f\Big|_{[t,s]}$ is constant. Then

$[t, s]$ must be in A, which means $s, t \notin [0, \infty) \setminus \cup_{I \in A} I$, contradiction! Therefore, by the same argument we can show that for each $s \neq t$, $m(E \cap [-s, s]) \neq m(E \cap [-t, t])$. Recall that

$$\|\chi_s - \chi_t\|_{L^\infty(E)} = \|\chi_{[-s,s]\cap E} - \chi_{[-t,t]\cap E}\|_{L^\infty(E)} = 1$$

because χ_s and χ_t can only take value 1 or 0 and they differs by 1 on a positive measure set since $m(E \cap [-s, s]) \neq m(E \cap [-t, t])$.

(v) Argue by contradiction to prove that $L^\infty(E)$ cannot be separable. Hint: consider countable orthogonal basis.

Since $\|\chi_s - \chi_t\|_{L^\infty(E)} = 1$ for any two elements $s \neq t$, we can construct a collection of open balls, i.e., $G = \{O_s \mid s \in [0, \infty) \setminus \cup_{I \in A} I\}$, where O_s is the open ball centered at χ_s with radius $1/2$ in $L^\infty(E)$ space. By this construction, $O_s \cap O_t = \varnothing$ if $s \neq t$. By part (iii), G is uncountable. Suppose $L^\infty(E)$ is separable, then there exists a countable dense subset $\{u_i\}_{i=1}^\infty$ of $L^\infty(E)$. For each distinct $O_s \in G$, since O_s is open, $O_s \cap \{u_i\}_{i=1}^\infty \neq \varnothing$. Then, we can denote it as u_s. Since each two O_s are pairwise disjoint, u_s is also distinct. However, we have uncountably many distinct s, meaning that we will obtain uncountably many distinct $u_s \in \{u_i\}_{i=1}^\infty$. This is a contradiction since $\{u_i\}_{i=1}^\infty$ is only countable. Therefore, $L^\infty(E)$ cannot be separable.

Extra Problem 3. Prove that $L^1(E)$ and L^∞ are not reflexive.

Now we have obtain the fact that $L^1(E)$ is separable and $L^\infty(E)$ is not separable. We also know that under Lebesgue measure, the dual space of $L^1(E)$, $(L^1(E))'$ can be identified with $L^\infty(E)$. Now suppose $L^1(E)$ is reflexive, since it is also separable, by Fact 4 in class, its dual must be separable, which is contradiction. Thus, $L^1(E)$ is not reflexive.

Of course, we know $L^1(E)$ is Banach space , by Problem 4.6-4, if $(L^1(E))'$ is reflexive, then $L^1(E)$ must be reflexive. However, we just derived that $L^1(E)$ is not reflexive, so contradition shows that $(L^1(E))'$ is not reflexive, i.e., $L^\infty(E)$ is not reflexive.

Extra Problem 4. Let X and Y be normed spaces and suppose that there exists a bijective linear isometry between them. Prove that X is reflexive if and only if Y is reflexive.

Given a bijective linear isometry $T : X \mapsto Y$, its adjoint map $T' : Y' \mapsto X'$ and second adjoint map $T'' : X'' \mapsto Y''$ are both bijective linear isometry.

Let $C : X \mapsto X''$ be the Canonical map, since X is reflexive, C is surjective. For any $y_0'' \in Y''$, there exists a unique $x_0'' \in X''$ such that $T''(x_0'') = y_0''$. Also, there exists $x_0 \in X$ such that $x_0'' = Cx_0$, and this further implies $T''(Cx_0) = y_0''$.

For arbitrary $y' \in Y'$, we have $y_0''(y') = T''(Cx_0)(y')$. By definition of adjoint operator,

$$T''(Cx_0)(y') = Cx_0(T'y') = T'y'(x_0) = y'(T(x_0)) = D(Tx_0)(y')$$

where $D : Y \mapsto Y''$ is the Canonical map. This implies that $y_0''(y') = D(Tx_0)(y')$ for all $y' \in Y'$, so $y_0'' = D(Tx_0)$. Therefore, for all $y_0'' \in Y''$, there exists $y_0 = Tx_0 \in Y$ such that $Dy_0 = y_0''$. This shows that D is surjective, so Y is reflexive. The converse can be proved in exactly the same way.