MAT4010: Functional Analysis
Homework 8

李肖鹏 (116010114)

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Problem 4.7-10. Let $y = (\eta_j)$, $\eta_j \in \mathbb{C}$, be such that $\sum \xi_j \eta_j$ converges for every $x = (\xi_j) \in c_0$, where $c_0 \subset l^\infty$ is the subspace of all complex sequences converging to zero. Show that $\sum |\eta_j| < \infty$.

Since $(c_0)' = l^1$, for any fixed x, define $f_n(x) = \sum_{j=1}^n \xi_j \eta_j$, then $\|f_n\| = \sum_{j=1}^n |\eta_j|$. Since $\sum \xi_j \eta_j$ converges, we know $f_n(x) \to f(x)$ for each fixed $x \in c_0$, where $f(x) = \sum_{j=1}^\infty \xi_j \eta_j$. This implies that $f_n(x)$ are bounded for all n, so $\sup_{n\in\mathbb{N}^+} |f_n(x)| < \infty$. By Uniform Boundedness Principle, $\sup_{n\in\mathbb{N}^+} \|f_n\| < \infty$. Since $\sup_{n\in\mathbb{N}^+} \|f_n\| = \|f\|$, we can conclude that $\sum_{j=1}^\infty |\eta_j| < \infty$.

Problem 4.7-14. If X and Y are Banach spaces and $T_n \in B(X, Y)$, $n = 1, 2, \ldots$, show that equivalent statements are:

(a) $(\|T_n\|)$ is bounded,

(b) $(\|T_n x\|)$ is bounded for all $x \in X$,

(c) $(|g(T_n x)|)$ is bounded for all $x \in X$ and all $g \in Y'$.

First, (a) implies (b) because $\|T_n x\|_Y \le \|T_n\| \|x\|_X$. Since $\|T_n\|$ is bounded, for each fixed $x \in X$, $\|x\|_X$ is also bounded, it is obvious that $\|T_n x\|_Y$ is also bounded.

Then, (b) implies (c) is also trivial, because $|g(T_n x)| \le \|g\|_{Y'} \|T_n x\|_Y$. Since $\|T_n x\|_Y$ is supposed to be bounded, and each fixed $g \in Y'$ is also bounded, it is obvious that $|g(T_n x)|$ is bounded.

Next, (c) implies (b) is a little subtle. Let $C : Y \mapsto Y''$ be Canonical map, then $g(T_n x) = C(T_n x)(g)$ for all $g \in Y'$. Since $\sup_{x\in X} |C(T_n x)(g)| < \infty$, by Uniform Boundedness Principle, $\sup_{x\in X} \|C(T_n x)\|_{Y''} < \infty$. However, C is isometric operator, so $\|C(T_n x)\|_{Y''} = \|T_n x\|_Y$. Therefore, $\|T_n x\|_Y$ is bounded.

Finally, (b) implies (a) is trivial because this is just the statement of Uniform Boundedness Principle.

Extra Problem 1. Let K be a convex subset of normed space X with $\mathring{K} \neq \varnothing$. Prove that $\overline{\mathring{K}} = \overline{K}$.

Since $\mathring{K} \subset K$, so $\overline{K}$ is a closed set containing $\mathring{K}$, but $\overline{\mathring{K}}$ is the smallest closed set containing $\mathring{K}$ by definition of closure, so $\overline{\mathring{K}} \subset \overline{K}$.

To prove $\overline{K} \subset \overline{\mathring{K}}$ we only need to prove $\partial K \subset \overline{\mathring{K}}$, i.e., any neighborhood of a boundary point of K must have non-empty intersecton with $\mathring{K}$. Take arbitrary boundary point $x_0 \in \partial K$, and pick any interior point y_0 of K. Consider the open ball satisfying $N_r(y_0) \subset \mathring{K}$. For each $t \in (0, 1)$, we claim that $z = tx_0 + (1-t)y_0$ is an interior point of K. This is because for each t, the open ball centered at z with radius $(1-t)r$ is contained in K. To see this, consider each point u in open ball $N_{(1-t)r}(z)$,

$$\|u - tx_0 - (1-t)y_0\| < (1-t)r \Longrightarrow \left\| \frac{1}{1-t}u - \frac{t}{1-t}x_0 - y_0 \right\| < r$$

Let $v = \frac{1}{1-t}u - \frac{t}{1-t}x_0$, then v is in $N_r(y_0)$, hence in K. Notice that $(1-t)v + tx_0 = u$, thus u is also in K by the convexity of K. This implies that $tx_0 + (1-t)y_0$ is a interior point for $t \in (0,1)$. Therefore, any neighborhood of x_0 has non-empty intersecton with $\mathring{K}$, which shows $\partial K \subset \overline{\mathring{K}}$.

Extra Problem 2. Let K be given as in the last problem. Suppose $x_0 \in \mathring{K}$ and $x_1 \in \partial K$. Define $x_2 = m(x_1 - x_0) + x_0$, where $m > 1$. Prove that $x_2 \notin K$.

Suppose $x_2 \in K$, by the claim in last problem, for $t \in (0,1)$, let $z = tx_0 + (1-t)x_2$, then z must be an interior point of K. Consider $t = 1 - \frac{1}{m} \in (0,1)$, then $(1-t) = \frac{1}{m} \in (0,1)$, and

$$z = \left(1 - \frac{1}{m}\right)x_0 + \frac{1}{m}x_2 = \left(1 - \frac{1}{m}\right)x_0 + \frac{1}{m}[m(x_1 - x_0) + x_0] = x_1$$

Thus, x_1 should be an interior point of K, but by assumption it is in ∂K, contradiction. Therefore, $x_2 \notin K$.

Extra Problem 3. Let K be a closed convex subset of normed space X. Prove that $\forall\, x \in X \setminus K$, $\exists\, f \in X^*$ such that $\|f\| = 1$ and

$$\sup_{y \in K} f(y) \le f(x) - \operatorname{dist}(x, K)$$

For each x, take $E = B(x; r)$ where $r = \operatorname{dist}(x, K) > 0$. Recall K is closed and $X \setminus K$ is open, so x is an interior point, and there exists an open neighbood $N_\delta(x)$ of x such that $N_\delta(x)$ does not intersect with K, then we will have $\operatorname{dist}(x, K) > 0$. Since K is closed convex and E has nonempty interior, by Geometric Hahn-Banach II, there exists $f \in X'$ such that $\|f\| = 1$ and $f(y) \le f(x)$ for all $y \in K$ and $x \in E$. Therefore, it is obvious that $\sup_{y \in K} f(y) \le f(x)$ for all $x \in E$. For any small $\epsilon > 0$, since $\|f\| = 1$, there exists x_0, such that $|x_0| = 1$ and $|f(x_0)| > 1 - \epsilon$. Take $c = \frac{\overline{f(x_0)}}{|f(x_0)|}(r - \epsilon)$, then we have $|cx_0| = r - \epsilon < r$. This implies that $x - cx_0 \in E$, so we have

$$\sup_{y \in K} f(y) \le f(x - cx_0) = f(x) - cf(x_0) \le f(x) - (r - \epsilon)(1 - \epsilon)$$

Take $\epsilon \to 0$, we have

$$\sup_{y \in K} f(y) \le f(x) - r = f(x) - \operatorname{dist}(x, K)$$

Extra Problem 4. Let K be given as in last problem. Prove that for normed space X, $\forall\, x \in X \setminus K$,

$$\operatorname{dist}(x, K) = \sup_{f \in X^*, \|f\| = 1} \left\{ f(x) - \sup_{z \in K} f(z) \right\}$$

while for $x \in K$, only $\ge$ sign holds.

From the last problem, it is obvious that for all $x \in X \setminus K$,

$$\operatorname{dist}(x, K) \leq \sup_{\|f\|=1, f \in X^*} \left\{ f(x) - \sup_{z \in K} f(z) \right\}$$

For $x \in K$, this inequality does not hold. The counter-example is that for $X = \mathbb{R}$, $x = 0$ and $K = [-1, 1]$, consder any $f \in X^*$, it must be in the form of $f(x) = ax$. Since $\|f\| = 1$, we know $a = \pm 1$. Therefore, $\sup_{z \in [-1,1]} f(x) = 1$. This shows that $f(x) - \sup_{z \in K} f(z) \equiv -1$ for all $f \in X^*$. Therefore, RHS is -1 but LHS is 0, which shows $\leq$ does not hold.

Then we show for all $x \in X$ (not necessarily in $X \setminus K$), we have

$$\operatorname{dist}(x, K) \geq \sup_{\|f\|=1, f \in X^*} \left\{ f(x) - \sup_{z \in K} f(z) \right\}$$

Consider any $f \in X^*$ such that $\|f\| = 1$, we have

$$\begin{aligned} f(x) - \sup_{z \in K} f(z) &= f(x) + \inf_{z \in K} [-f(z)] = \inf_{z \in K} (f(x) - f(z)) = \inf_{z \in K} (f(x - z)) \\ &\leq \inf_{z \in K} \|f\| \|x - z\| = \inf_{z \in K} \|x - z\| = \operatorname{dist}(x, K) \end{aligned}$$

Since this is true for all f, take supremum over f on both sides, we have

$$\sup_{\|f\|=1, f \in X^*} \left\{ f(x) - \sup_{z \in K} f(z) \right\} \leq \operatorname{dist}(x, K)$$

Extra Problem 5. Let $p \in [1, \infty]$. Suppose $f(x)$ is measurable on $(0, 1)$, satisfying that for all $g \in L^{p'}(0, 1)$, we have $fg \in L^1(0, 1)$. Prove that $f \in L^p(0, 1)$.
Hint: For all $n \geq 1$, let $E_n = \{x \in (0, 1) \,|\, |f(x)| \leq n\}$. Define $f_n(x) = f(x)\chi_{E_n}(x)$. Observe $\left| \int_0^1 f_n(x)g(x)\, dx \right| \leq \int_0^1 |fg|\, dx < \infty$.

For $1 \leq p < \infty$, if we consider T_n defined on $L^{p'}(0, 1)$, we can defined $T_n(g) = \int_0^1 f_n(x)g(x)\, dx$, where f_n is defined in hint. Since

$$\left| \int_0^1 f_n(x)g(x)\, dx \right| \leq \int_0^1 |fg|\, dx < \infty$$

we can conclude $\sup_{n \in \mathbb{N}^+} |T_n(g)| < \infty$ for all $g \in L^{p'}(0, 1)$. By Uniform Boundedness Principle, $\sup_{n \in \mathbb{N}^+} \|T_n\| < \infty$. Since $(L^{p'}(0, 1))^* = L^p(0, 1)$, $\|T_n\| = \|f_n\|_{L^p(0,1)}$. We have

$$\int_0^1 |f(x)|^p\, dx = \lim_{n \to \infty} \int_{E_n} |f(x)|^p\, dx = \sup_{n \in \mathbb{N}^+} \|f_n\|^p < \infty$$

This is true because f cannot take infinity at positive measure set because if so, take $g = 1 \in L^{p'}(0, 1)$, then $fg = f$ cannot be in $L^1(0, 1)$.

If $p = \infty$, we still have $\|T_n\| = \|f_n\|_{L^p(0,1)}$. Suppose f is unbounded under essential supremum sense, then $m\{x \,|\, |f(x)| \geq c\} > 0$ for all c. Then

$$m(\{x \,|\, |f(x)| \geq c\}) = m(\cup_{n \in \mathbb{N}^+} \{x \,|\, |f_n(x)| \geq c\}) > 0$$

This implies that there exists n, such that $f_n(x) \geq c$ for all c, but this is a contradiction to $\sup_{n \in \mathbb{N}^+} \|f_n\|_{L^p(0,1)} < \infty$. Therefore, f is essentially bounded on $(0, 1)$, so $f \in L^\infty(0, 1)$.