MAT4010: Functional Analysis
Homework 9

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Problem 4.8-1. If $x_n \in \mathcal{C}[a,b]$ and $x_n \xrightarrow{w} x \in \mathcal{C}[a,b]$, show that (x_n) is pointwise convergent on $[a,b]$, that is, $(x_n(t))$ converges for every $t \in [a,b]$.

For each fixed $t \in [a,b]$, define $f_t(x) = x(t)$ for all $x \in \mathcal{C}[a,b]$. Then, $f_t(x)$ is linear because for all scalar a, b and $y \in \mathcal{C}[a,b]$, we have

$$f_t(ax + by) = (ax + by)(t) = ax(t) + by(t) = af_t(x) + bf_t(y)$$

$f_t(x)$ is bounded because $\|x\| = \sup_{t\in[a,b]} |x(t)|$, and $\|f_t\| = \sup_{\|x\|=1} |x(t)| = 1$. Since $x_n \xrightarrow{w} x$, we have $\lim_{n\to\infty} f_t(x_n) \to f_t(x)$ for each fixed t, i.e., $\lim_{n\to\infty} x_n(t) \to x(t)$. This implies that $(x_n(t))$ converges for every $t \in [a,b]$.

Problem 4.8-8. A weak Cauchy sequence in a real or complex normed space X is a sequence (x_n) in X such that for every $f \in X'$ the sequence $(f(x_n))$ is Cauchy in $\mathbb{R}$ or $\mathbb{C}$, respectively. Show that a weak Cauchy sequence is bounded.

Since $(f(x_n))$ is Cauchy, $f(x_n)$ must be convergent, and hence bounded, i.e., $\sup_n |f(x_n)| < \infty$ for all f. Since $f(x_n) = Cx_n(f)$ for all f, where C is the Canonical map, we can see $\sup_n |Cx_n(f)| < \infty$. Since X^* is Banach space, by Banach-Steinhauss, $\sup_n \|Cx_n\| < \infty$. Since C is isometric mapping, $\sup_n \|x_n\| < \infty$, i.e., x_n is bounded.

Extra Problem 1. Let H be Hilbert.

(i) Suppose H is also separable, then $x_n \xrightarrow{w} x_\infty$ as $n \to \infty$ if and only if $\|x_n\|$ is bounded over $n \in \mathbb{N}^+$ and $\langle x_n, e_k\rangle_H \to \langle x_\infty, e_k\rangle_H$ as $n \to \infty$, for all $k \geq 1$.

For "only if" part, that weak convergence implies bounded has been proved in lecture; by Riesz representation, there exists $f_{e_k} \in H^*$ such that $\langle x_n, e_k\rangle_H = \langle f_{e_k}, x_n\rangle_{H^*,H}$ and $\langle x_\infty, e_k\rangle_H = \langle f_{e_k}, x_\infty\rangle_{H^*,H}$. By weak convergence, $\langle f_{e_k}, x_n\rangle_{H^*,H} \to \langle f_{e_k}, x_\infty\rangle_{H^*,H}$. Therefore, we can conclude that $\langle x_n, e_k\rangle_H \to \langle x_\infty, e_k\rangle_H$ for all $k \geq 1$.

For "if" part, since all $f \in H^*$ can be represented by $\langle \cdot, z\rangle_H$ for unique $z \in H$, we only need to prove for all $z \in H$, $\langle x_n, z\rangle_H \to \langle x_\infty, z\rangle_H$. Write $z = \sum_{i=1}^\infty b_i e_i$, where $b_i = \langle z, e_i\rangle_H$. This implies that

$$\langle x_\infty, z\rangle_H = \sum_{i=1}^\infty \bar{b}_i \langle x_\infty, e_i\rangle_H = \lim_{m\to\infty} \lim_{n\to\infty} \left\langle x_n, \sum_{i=1}^m b_i e_i \right\rangle_H = \lim_{m\to\infty} \lim_{n\to\infty} a_{mn}$$

Since $\langle x_n, e_k\rangle_H \to \langle x_\infty, e_k\rangle_H$ for all k, $a_{mn} \to a_m$ pointwise for any fixed n, where $a_m = \langle x_\infty, \sum_{i=1}^m b_i e_i\rangle_H$. Now consider

$$\left\langle x_n, \sum_{i=1}^m b_i e_i \right\rangle_H - \left\langle x_n, \sum_{i=1}^\infty b_i e_i \right\rangle_H = \left\langle x_n, \sum_{i=m+1}^\infty b_i e_i \right\rangle_H \le \|x_n\| \left\| \sum_{i=m+1}^\infty b_i e_i \right\|$$

Since $\|x_n\|$ is bounded, and

$$\left\| \sum_{i=m+1}^\infty b_i e_i \right\|^2 = \sum_{i=m+1}^\infty |\langle z, e_i\rangle|^2 \to 0$$

We can conclude that $a_{mn} \to a_n$ uniformly on m. Thus, by Theorem 7.11 in Rudin's book, we can exchange the order of limit, i.e.,

$$\langle x_\infty, z\rangle_H = \lim_{m\to\infty} \lim_{n\to\infty} a_{mn} = \lim_{n\to\infty} \lim_{m\to\infty} a_{mn} = \lim_{n\to\infty} \langle x_n, z\rangle_H$$

Therefore, we proved that $\langle x_n, z\rangle_H \to \langle x_\infty, z\rangle_H$ for all z, and this implies that x_n is weakly convergent to x_∞.

(ii) $x_n \to x_\infty$ if and only if $x_n \xrightarrow{w} x_\infty$ and $\|x_n\| \to \|x_\infty\|$ as $n \to \infty$.

The "only if" part is trivial, because strong convergence implies weak convergence and norm convergence.

For the "if" part, since $x_n \xrightarrow{w} x_\infty$, by Riesz representation, $\langle x_n, z\rangle_H \to \langle x_\infty, z\rangle_H$ for all $z \in H$. Take $z = x_\infty$, then we have $\langle x_n, x_\infty\rangle_H \to \langle x_\infty, x_\infty\rangle_H$. Notice that

$$\begin{aligned}
\|x_n - x_\infty\|^2 &= |\langle x_n, x_n\rangle_H - \langle x_n, x_\infty\rangle_H - \langle x_\infty, x_n\rangle_H + \langle x_\infty, x_\infty\rangle_H| \\
&= |(\langle x_n, x_n\rangle_H - \langle x_\infty, x_\infty\rangle_H) - (\langle x_n, x_\infty\rangle_H - \langle x_\infty, x_\infty\rangle_H) - (\langle x_\infty, x_n\rangle_H - \langle x_\infty, x_\infty\rangle_H)| \\
&\le |\langle x_n, x_n\rangle_H - \langle x_\infty, x_\infty\rangle_H| + |\langle x_n, x_\infty\rangle_H - \langle x_\infty, x_\infty\rangle_H| + |\langle x_\infty, x_n\rangle_H - \langle x_\infty, x_\infty\rangle_H|
\end{aligned}$$

Since $\|x_n\| \to \|x_\infty\|$, we have $\langle x_n, x_n\rangle_H \to \langle x_\infty, x_\infty\rangle_H$. Furthermore,

$$|\langle x_n, x_\infty\rangle_H - \langle x_\infty, x_\infty\rangle_H| = |\langle x_\infty, x_n\rangle_H - \langle x_\infty, x_\infty\rangle_H|$$

Therefore, $\|x_n - x_\infty\|^2 \le |\langle x_n, x_n\rangle_H - \langle x_\infty, x_\infty\rangle_H| + 2|\langle x_n, x_\infty\rangle_H - \langle x_\infty, x_\infty\rangle_H| \to 0$. We can conclude that $\|x_n - x_\infty\| \to 0$, thus $x_n \to x_\infty$.

(iii) If $x_n \xrightarrow{w} x_\infty$, $y \to y_\infty$ as $n \to \infty$, then $\langle x_n, y_n\rangle_H \to \langle x_\infty, y_\infty\rangle_H$ as $n \to \infty$.

Again, by Riesz representation, weak convergence of x_n to x_∞ implies that $\langle x_n, y_\infty\rangle_H \to \langle x_\infty, y_\infty\rangle_H$. Notice that

$$\begin{aligned}
|\langle x_n, y_n\rangle_H - \langle x_\infty, y_\infty\rangle_H| &\le |\langle x_n, y_n\rangle_H - \langle x_n, y_\infty\rangle_H| + |\langle x_n, y_\infty\rangle_H - \langle x_\infty, y_\infty\rangle_H| \\
&\le \|x_n\|\|y_n - y_\infty\| + |\langle x_n, y_\infty\rangle_H - \langle x_\infty, y_\infty\rangle_H|
\end{aligned}$$

Since $x_n \xrightarrow{w} x_\infty$, $\|x_n\|$ is bounded, and $y_n \to y_\infty$ implies $\|y_n - y_\infty\| \to 0$, we conclude that $\langle x_n, y_n\rangle_H \to \langle x_\infty, y_\infty\rangle_H$.

Extra Problem 2. Let K be a closed convex subset of a real normed space X.

(i) Prove that K is weakly closed, i.e., if $\{x_n\}_{n=1}^{\infty} \subset K$ and $x_n \xrightarrow{w} x_\infty$ in X as $n \to \infty$, then $x_\infty \in K$.

Suppose $\{x_n\}_{n=1}^{\infty} \subset K$, but $x_\infty \notin K$, then since K is convex and closed, by Ascoli Theorem, there exists $f \in X^*$, such that $f(x_n) < c < f(x_\infty)$ for all n. By weak convergence, $\lim_{n\to\infty} f(x_n) = f(x_\infty)$, so we obtain $f(x_\infty) \leq c < f(x_\infty)$ by taking limit on both sides, but it is impossible that $f(x_\infty) < f(x_\infty)$, so contradiction shows that $x_\infty \in K$. Hence, K is weakly closed.

(ii) Prove that if $x_n \xrightarrow{w} x_0$, then $x_0 \in \overline{\text{span}(x_n)}$. Also show that any closed subspace Y of a normed space X contains the limits of all weakly convergent sequences of its elements.

Since $\overline{\text{span}(x_n)}$ is closed, and it is also a subspace of X (the closure of a subspace is also subspace), since any subspace of X is convex, it is closed and convex. Then by (i), if $x_n \xrightarrow{w} x_0$, $x_0 \in \overline{\text{span}(x_n)}$.

Again, since any subspace Y of X must be convex, and in addition, it is closed, so by (i), it is weakly closed, i.e., any weakly convergent sequence of Y must weakly converge to a point in Y.

(iii) If (x_n) is a weakly convergent sequence in X, say, $x_n \xrightarrow{w} x_0$, show that there is a sequence (y_m) of linear combinations of elements of (x_n) which converges strongly to x_0.

By (ii), $x_n \xrightarrow{w} x_0$ implies $x_0 \in \overline{\text{span}(x_n)}$, this implies that there exists a sequence $y_n \in \text{span}(x_n)$, and $y_n \to x_0$ strongly. However, $y_n \in \text{span}(x_n)$ just means y_n is a linear combinations of elements of (x_n), so the proof is finished.

Extra Problem 3. Let K and X be given in last problem, Assume that X is also reflexive. Prove that

(i) $\forall\, y_0 \in X \setminus K$, $\exists\, x_0 \in K$ such that $\|x_0 - y_0\| = \text{dist}(y_0, K)$.

Let $\text{dist}(y_0, K) = c$ be a constant. By definition of distance, there exists $x_n \in K$ such that $\|x_n - y_0\| \to c$. It is obvious that $\|x_n\|$ is bounded, and X is reflexive, then by Banach-Eberlein Theorem, $x_n \xrightarrow{w} x_\infty \in K$. Now we claim that $\|y_0 - x_\infty\| = c$, i.e., x_∞ is the required x_0 in the question.

Since $x_\infty \in K$, $c = \inf_{x\in K} \|y_0 - x\| \leq \|y_0 - x_\infty\|$, so we only need to show $\|y_0 - x_\infty\| \leq c$. Recall application 2 of Hahn-Banach in lecture,

$$\begin{aligned}
\|x_\infty - y_0\| &= \sup_{\|f\|=1, f\in X^*} |f(x_\infty - y_0)| = \sup_{\|f\|=1, f\in X^*} |\lim_{n\to\infty} f(x_n - y_0)| \\
&= \sup_{\|f\|=1, f\in X^*} \lim_{n\to\infty} |f(x_n - y_0)| \leq \sup_{\|f\|=1, f\in X^*} \lim_{n\to\infty} \|x_n - y_0\| \\
&= \lim_{n\to\infty} \|x_n - y_0\| = c
\end{aligned}$$

Therefore, we can conclude that $\|y_0 - x_\infty\| = c$.

(ii) Assume also that K is bounded. Prove that $\forall f \in X^*$, f attains its maximum and minimum over K.

Let $A = \{f(x) \,|\, x \in K\}$, then $A \subset \mathbb{R}$ and A is bounded because f is linear bounded function, so it maps bounded set to bounded set. Then A has supremum c and infimum d. Then there exists $x_n, y_n \in K$ such that $f(x_n) \to c$ and $f(y_n) \to d$. Also, x_n, y_n are bounded, by Banach-Elberlein again, $x_n \to x_\infty$ and $y_n \to y_\infty$. Since K is weakly closed, $x_\infty, y_\infty \in K$. Therefore, $f(x_n) \to f(x_\infty) = c$ and $f(y_n) \to f(y_\infty) = d$, so f attains its maximum and minimum over K.

Extra Problem 4. Prove that $x_k \xrightarrow{w} x_\infty$ in l^1 implies strong convergence $x_k \to x_\infty$ as $k \to \infty$.

Suppose $y_k = x_k - x_\infty$, then $y_k \xrightarrow{w} 0$. Assume $x_k \not\to x_\infty$, then $y_k \not\to 0$ and so there exists a subsequence of y_k, i.e., y_{k_j} such that $\|y_{k_j}\|$ is bounded away from zero, then there exists a constant $c > 0$ such that $\|y_{k_j}\| \geq c$, so let $z_j = y_{k_j}/c$, $\|z_j\| \geq 1$.

Now we obtain a sequence z_j such that $z_j \xrightarrow{w} 0$ and $\|z_j\| \geq 1$. Since $z_j \xrightarrow{w} 0$, for all $f \in (l^1)^*$, $f(z_j) \to f(0) = 0$. Consider a sequence of f_n defined by $f_n(z_j) = z_{j,n}$, where $z_{j,n}$ is the n-th entry of z_j. Such functions must be linear and bounded, so this implies that $z_{j,n} \to 0$ as $j \to \infty$ for any fixed n.

Since $z_1 \in l^1$, we can choose K_1 large enough such that $\sum_{n=K_1+1}^{\infty} |z_{1,n}| < \frac{1}{5}$. Combined with $z_{j,n} \to 0$, we have $\sum_{n=1}^{K_1} |z_{j,n}| \to 0$ as $j \to \infty$. Thus, we can find j_2 such that $\sum_{n=1}^{K_1} |z_{j_2,n}| < \frac{1}{5}$. Since $Z_{j_2} \in l^1$, we can choose $K_2 > K_1$ such that $\sum_{n=K_2+1}^{\infty} |z_{j_2,n}| < \frac{1}{5}$. Continue this process, let $j_1 = 1$, then the subsequence $z_{j_m,n}$ satisfies $\sum_{n=1}^{K_{m-1}} |z_{j_m,n}| < \frac{1}{5}$ (for all $m \geq 2$) and $\sum_{n=K_m+1}^{\infty} |z_{j_m,n}| < \frac{1}{5}$ (for all $m \in \mathbb{N}^+$).

Now we define $u = (u_1, \ldots, u_n, \ldots)$ by $u_n = \frac{|z_{j_m,n}|}{z_{j_m,n}}$ for $K_{m-1} < n \leq K_m$ ($K_0 = 0$). If $z_{j_m,n} = 0$, then let $u_n = 1$. Then $u \in l^\infty$ ($|u_n| = 1$ for all n), so u can be treated as a linear and bounded functional defined l^1. By our assumption, weak convergence implies that $\langle u, z_{j_m} \rangle_{(l^1)^*, l^1} \to 0$ as $m \to \infty$. However, for all $m \geq 2$,

$$
\begin{aligned}
\left| \sum_{n=1}^{\infty} z_{j_m,n} u_n \right| &\geq \left| \sum_{n=K_{m-1}+1}^{K_m} z_{j_m,n} u_n \right| - \left| \sum_{n=1}^{K_{m-1}} z_{j_m,n} u_n \right| - \left| \sum_{n=K_m+1}^{\infty} z_{j_m,n} u_n \right| \\
&\geq \sum_{n=K_{m-1}+1}^{K_m} |z_{j_m,n}| - \sum_{n=1}^{K_{m-1}} |z_{j_m,n}| - \sum_{n=K_m+1}^{\infty} |z_{j_m,n}| \\
&= \sum_{n=1}^{\infty} |z_{j_m,n}| - 2 \sum_{n=1}^{K_{m-1}} |z_{j_m,n}| - 2 \sum_{n=K_m+1}^{\infty} |z_{j_m,n}| > \frac{1}{5}
\end{aligned}
$$

This implies that $\langle u, z_{j_m} \rangle_{(l^1)^*, l^1} \not\to 0$, contradiction implies that our assumption is wrong, i.e., such c and z_j does not exist, and $y_k \to 0$ strongly.